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Vibration of a Circular Arc Bar*

(Perpendicular to its Plane)

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The vibration of a circular arc bar which occurs perpendicularly to its plane is studied. Its Lagrangian is obtained and from the condition to minimize it, frequency equations and boundary values can be obtained. For a circular ring and a circular arc bar with both ends built-in, the frequency equations obtained by the method mentioned above are compared with the known ones. For a circular arc bar with both ends free, the frequency equations and the boundary values are obtained and some frequencies are calculated and given in graphs. Also, it is ascertained that the centre of gravity of the circular arc bar does not move during the vibration.

1. Introduction

About the vibration of a circular arc bar which occurs perpendicularly to its plane, the cases of a ring⁽¹⁾ and a circular arc bar⁽²⁾, of which both ends are built-in, have been studied. The author obtains the Lagrangian for one cycle of vibration, when a circular arc bar vibrates perpendicularly to its plane. In the Lagrangian, the effect of the rotatory inertia and the slenderness ratio is considered. It is shown, in this paper, that the frequency equations of a ring and a circular arc bar with both ends built-in, obtained by minimizing the Lagrangian, when the slenderness ratio is very large, agree with the known results. The author obtains the frequency equations and the boundary values of a circular arc bar, of which both ends are free, and which vibrates perpendicularly to its plane, and shows that the effect of the slenderness ratio is large, when it vibrates symmetrically. The nondimensional frequencies in this vibration are given with graphs. The centre of gravity does not, generally, move in the vibration of a bar with both ends free, and the author ascertains the fact theoretically for the case where a circular arc bar with both ends free vibrates perpendicularly to its plane.

2. Nomenclature

x, y, z : coordinate axes (right handed system), x and z are radial and tangential directions of arc, respectively

ω : 2ω is the central angle of circular arc

R : radius of circular arc

θ : angle measured from the radius passing the midpoint of arc (see Fig. 1)

v_0 : displacement of bar to y -axis

ψ : product of twisting angle and radius of arc, direction of twisting angle is of right handed screw to z -axis

$\gamma S_0, EI_x, C'$: weight per unit length, bending and torsional rigidities of bar, respectively

i, i' : moments of inertia per unit length of bar about z - and x -axes, respectively

L : Lagrangian

$\alpha_0, \beta_0, \beta_0', \mu$: nondimensional frequencies and ratio of both rigidities

$$\alpha_0^4 = \frac{\gamma S_0 p^2 R^4}{EI_x g}, \quad \beta_0^2 = \frac{i p^2 R^2}{C'}$$

$$\beta_0'^2 = \frac{i' p^2 R^2}{EI_x}, \quad \mu = C'/EI_x$$

p : circular frequency

g : acceleration of gravity

s_x, s_y : slenderness ratios,

$$1/s_x^2 = I_x/R^2 S_0, \quad 1/s_y^2 = I_y/R^2 S_0$$

ξ : three roots of algebraic equation [see Eq. (3)]

η : $\eta = 1/(1+\mu)(1/\mu - \beta_0^2)$

$A_0, B_0, C_0, D_0, F_0, G_0, a, b, c, a', b', c'$: boundary values [see Eqs. (6) and (6)']

Fig. 1

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3. Lagrangian of a circular arc bar vibrating perpendicularly to its plane

The Lagrangian of a circular arc bar vibrating perpendicularly to its plane for one period is as follows:

$$L = \int_{-\omega}^{\omega} \left[\frac{\gamma S_0}{2g} p^2 v_0^2 + \frac{ip^2}{2} \left(\frac{\psi}{R} \right)^2 + \frac{i'p^2}{2} \left(\frac{dv_0}{Rd\theta} \right)^2 - \frac{EI_x}{2} \frac{1}{R^4} \left(\psi - \frac{d^2 v_0}{d\theta^2} \right)^2 - \frac{C'}{2} \frac{1}{R^4} \left(\frac{dv_0}{d\theta} + \frac{d\psi}{d\theta} \right)^2 \right] R d\theta$$

The first term in the right side of the above equation denotes the kinetic energy of the element of bar vibrating perpendicularly to the plane of arc, the second term denotes the kinetic energy of the element rotating about z -axis, the third denotes the kinetic energy of the element due to the slope of bar, that is, due to its rotation about x -axis, the fourth denotes the strain energy due to the bending perpendicular to the plane of arc and the fifth term denotes the strain energy of the element twisted about z -axis⁽³⁾. Using the nondimensional frequencies α_0 , β_0 , β_0' and the ratio of both rigidities, the Lagrangian mentioned above becomes,

$$L = \frac{EI_x}{2R^3} \int_{-\omega}^{\omega} \left[\alpha_0^4 v_0^2 + \mu \beta_0^2 \psi^2 + \beta_0'^2 \left(\frac{dv_0}{d\theta} \right)^2 - \left(\psi - \frac{d^2 v_0}{d\theta^2} \right)^2 - \mu \left(\frac{dv_0}{d\theta} + \frac{d\psi}{d\theta} \right)^2 \right] d\theta \quad (1)$$

Values of v_0 and ψ must minimize the Lagrangian, and by variation calculus (Appendix 1), we obtain the following equations:

$$v_0(\theta) = \sum [P_n \sinh \xi_n \theta + P_n' \cosh \xi_n \theta], \quad \psi(\theta) = \sum M_n [P_n \sinh \xi_n \theta + P_n' \cosh \xi_n \theta] \quad (2)$$

where $\sum$ means the summing up of three terms obtained by changing the suffixes as $n \rightarrow p \rightarrow r \rightarrow n$ in turn.

Symbol ξ_n is three roots of the following algebraic equation (we obtain three pairs with the same absolute values and the opposite signs or of the conjugate complexes, and ones of each pair are used as ξ_n).

$$\xi^6 + (2 + \beta_0^2 + \beta_0'^2) \xi^4 + \left[-\alpha_0^4 + (\mu - \beta_0'^2) \left(\frac{1}{\mu} - \beta_0^2 \right) \right] \xi^2 + \alpha_0^4 \left(\frac{1}{\mu} - \beta_0^2 \right) = 0 \quad (3)$$

We put M_n as follows,

$$M_n = \gamma \left[\xi_n^4 + \left(2 + \frac{1}{\mu} + \beta_0'^2 \right) \xi_n^2 - \alpha_0^4 \right] \quad (4)$$

Notations β_0 and β_0' can be represented with α_0 and the slenderness ratios and

$$\beta_0^2 = \frac{\alpha_0^4}{\mu} \left(\frac{1}{s_x^2} + \frac{1}{s_y^2} \right), \quad \beta_0'^2 = \alpha_0^4 / s_x^2 \quad (5)$$

Notations P_n and P_n' in v_0 and ψ are determined from the boundary values and we put the boundary values as follows:

$$\left. \begin{aligned} v_0(-\omega) &= 2A_0, \quad \frac{dv_0}{d\theta} \Big|_{\theta=-\omega} = 2B_0, \quad \psi(-\omega) = 2F_0, \\ v_0(\omega) &= 2C_0, \quad \frac{dv_0}{d\theta} \Big|_{\theta=\omega} = 2D_0, \quad \psi(\omega) = 2G_0 \end{aligned} \right\} \quad (6)$$

For convenience, we put the combinations of them as follows:

$$\left. \begin{aligned} a &= C_0 - A_0, & b &= D_0 + B_0, & c &= G_0 - F_0, \\ a' &= C_0 + A_0, & b' &= D_0 - B_0, & c' &= G_0 + F_0 \end{aligned} \right\} \quad (6)'$$

Then we obtain P_n and P_n' as follows (hereafter, the suffixes must be changed in order as $n \rightarrow p \rightarrow r \rightarrow n$):

$$P_n = \frac{1}{N} \left[a(-M_p \xi_r \sinh \xi_p \omega \cosh \xi_r \omega + M_r \xi_p \cosh \xi_p \omega \sinh \xi_r \omega) \right. \\ \left. + b(M_p - M_r) \sinh \xi_p \omega \sinh \xi_r \omega \right. \\ \left. + c(\xi_r \sinh \xi_p \omega \cosh \xi_r \omega - \xi_p \cosh \xi_p \omega \sinh \xi_r \omega) \right] \quad (7)$$

where

$$N = \sum \xi_n (M_p - M_r) \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \quad (8)$$

Symbol N' is obtained by changing $\sinh$ and $\cosh$ with each other in N , and P_n' is obtained by changing $\sinh$, $\cosh$, a' , b' , c' and N' for $\cosh$, $\sinh$, a , b , c and N in P_n , respectively.

Substituting Eq. (2) in Eq. (1) and using Eqs. (24) and (25) in Appendix 2, we obtain

$$L = -\frac{EI_x}{2R^3} \sum [2(P_n^2 + P_n'^2) \xi_n \{ \mu(1 + M_n)^2 - \beta_0'^2 \} \sinh \xi_n \omega \cosh \xi_n \omega \\ - 4P_n P_p \{ \xi_p (M_n - \xi_n^2) \sinh \xi_n \omega \cosh \xi_p \omega + \xi_n (M_p - \xi_p^2) \cosh \xi_n \omega \sinh \xi_p \omega \} \\ - 4P_n' P_p' \{ \xi_n (M_p - \xi_p^2) \sinh \xi_n \omega \cosh \xi_p \omega + \xi_p (M_n - \xi_n^2) \cosh \xi_n \omega \sinh \xi_p \omega \}] \quad (9)$$

Now put

$$\Phi = -\gamma(\xi_n^2 - \xi_p^2)(\xi_p^2 - \xi_r^2)(\xi_r^2 - \xi_n^2) \dots \dots \dots (10)$$

and the Lagrangian becomes as follows, according to Appendix 3:

$$L = -\frac{EI_x}{R^3} \left[\frac{Z}{N} + \frac{Z'}{N'} \right] \dots \dots \dots (11)$$

$$\begin{aligned} Z = \Phi \sum [& a^2 M_n^2 f_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega \\ & + c^2 f_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega \\ & - 2ca M_n f_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega \\ & - 2ab \xi_n g_n \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \\ & - 2bc \xi_n k_n \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega] \\ & - b^2 \Phi \sinh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \dots \dots \dots (12) \end{aligned}$$

where

$$\left. \begin{aligned} f_n &= \xi_p \xi_r / (M_r - M_n)(M_n - M_p), \\ g_n &= (\xi_p^2 \xi_r^2 + \alpha_0^4) / (\xi_r^2 - \xi_n^2)(\xi_n^2 - \xi_p^2), \\ k_n &= \left[\xi_n^2 + \mu \left(\frac{1}{\mu} - \beta_0^2 \right) \right] / (\xi_r^2 - \xi_n^2)(\xi_n^2 - \xi_p^2) \end{aligned} \right\} \dots \dots \dots (13)$$

In the calculation of Eq. (9), we obtain g_n as the form of the first equation of Eq. (33) in Appendix 4, but using Eq. (3) and its relations between roots and coefficients, we obtain g_n in Eq. (13). And

$$\begin{aligned} Z' = \Phi \sum [& a'^2 M_n^2 f_n \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \\ & + c'^2 f_n \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \\ & - 2c'a' M_n f_n \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \\ & - 2a'b' \xi_n g_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega \\ & - 2b'c' \xi_n k_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega] \\ & - \Phi b'^2 \cosh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega \dots \dots \dots (14) \end{aligned}$$

Of course, Z' is obtained by exchanging $\sinh$, $\cosh$, a' , b' and c' for $\cosh$, $\sinh$, a , b and c in Z , respectively.

Some of the boundary values a , b , etc. become zero and are represented by other boundary values according to the boundary conditions and the conditions of continuity. The undetermined boundary values must be such quantities as to minimize the Lagrangian, and from this condition we can obtain the frequency equations and the boundary values.

4. Comparisons with the known results

The frequency and the frequency equations are already known for the case where a ring and a circular arc bar with both ends built-in vibrate

Fig. 2

perpendicularly to their planes, and they are very slender. To be very slender means, using the author's notation, that s_x and s_y are very large, that is, that β_0 and β_0' are zero according to Eq. (5).

At first, we consider a ring. At any point of a ring, we consider the conditions of continuity. At both sides of that point, the deflections, the slopes and the twisting angles are equal to each other, and we obtain the following relations of the boundary values:

$$A_0 = C_0, \quad B_0 = D_0, \quad F_0 = G_0$$

That is, $a=b'=c=0$ and L is the function of a' , b and c' . Value of ω is equal to π . From $\partial L / \partial b = 0$, we obtain

$$\sinh \xi_n \pi = 0,$$

or

$$\xi_n = ik \quad (i = \sqrt{-1}, \quad k = 0, 1, 2, \dots)$$

Substituting this ξ_n in Eq. (3) and putting $\beta_0 = \beta_0' = 0$ (for the large slenderness ratios), we obtain

$$\alpha_0 = \sqrt{\frac{k^2(k^2 - 1)^2}{k^2 + 1/\mu}}$$

Love shows the frequency of a ring with a circular section and so $1/\mu = 1 + \sigma$ (σ is Poisson's ratio).

Then, the above frequency agrees with his result.

Two equations obtained from $\partial L / \partial a'$, $c' = 0$ are same when $\sinh \xi_n \pi = 0$ and from one of them we obtain the ratio of the boundary values, $c'/a' = -M_n$.

Then, we consider a circular arc bar with both ends built-in (Fig. 3). The midpoint of the arc is B and we denote the quantities of arc AB by suffix 1 and those of arc BC by suffix 2. From the boundary condition that the bar is built-in at both points A and C, we obtain

$$A_{01} = B_{01} = F_{01} = C_{02} = D_{02} = G_{02} = 0$$

At point B, both parts of the bar are continuous, and

$$C_{01} = A_{02}, \quad D_{01} = B_{02}, \quad G_{01} = F_{02},$$

Then,

$$a_{01} = a_{01}' = -a_{02} = a_{02}',$$

$$b_{01} = b_{01}' = b_{02} = -b_{02}',$$

$$c_{01} = c_{01}' = -c_{02} = c_{02}'$$

Putting these values a , b and c , the coefficients of ab and bc in the Lagrangian become zero and we obtain the following Lagrangian:

Fig. 3

$$L = -\frac{2EI_x\Phi}{R^3} \left[a^2 \sum M_n^2 f_n \left(\frac{\sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega}{N} + \frac{\cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega}{N'} \right) \right. \\ \left. + c^2 \sum f_n \left(\frac{\sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega}{N} + \frac{\cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega}{N'} \right) \right. \\ \left. - 2ca \sum M_n f_n \left(\frac{\sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega}{N} + \frac{\cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega}{N'} \right) \right. \\ \left. - b^2 \left(\frac{\sinh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega}{N} + \frac{\cosh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega}{N'} \right) \right]$$

When $\beta_0 = \beta_0' = 0$, from two equations of $\partial L / \partial a, c = 0$, we obtain the frequency equation as follows:

$$\xi_1(M_2 - M_3) \sinh 2\xi_1 \omega \cosh 2\xi_2 \omega \cosh 2\xi_3 \omega \\ + \xi_2(M_3 - M_1) \cosh 2\xi_1 \omega \sinh 2\xi_2 \omega \cosh 2\xi_3 \omega \\ + \xi_3(M_1 - M_2) \cosh 2\xi_1 \omega \cosh 2\xi_2 \omega \sinh 2\xi_3 \omega = 0$$

The equation in which $\sinh$ and $\cosh$ are exchanged for each other in the above equation, is also obtained from $\partial L / \partial b = 0$. Both equations are the frequency equations. Notation e_n in the paper of Federhofer becomes, according to the author's notation,

$$e_n = \left(1 + \frac{1}{\mu} \right) (M_p - M_r)$$

and so both frequency equations mentioned above agree with those of Federhofer.

5. Vibration of a circular arc bar with both ends free

We consider the case where a circular arc bar, of which both ends are free and the central angle is 2ω , vibrates perpendicularly to its plane. All boundary values are free and the vibration occurs in such manner as to minimize the Lagrangian, that is,

$$\partial L / \partial a, b, c = 0, \quad \partial L / \partial a', b', c' = 0$$

From $\partial L / \partial a, b, c = 0$, we obtain three linear homogeneous equations. If any of a, b and c is not zero, the determinant made of the coefficients of a, b and c must be zero, that is,

$$\sum \xi_n^2 f_n \sinh \xi_n \omega \cosh^2 \xi_p \omega \sinh^2 \xi_r \omega \cosh \xi_p \omega \sinh^2 \xi_r \omega \cosh \xi_r \omega \\ + \sum f_n \sinh \xi_n \omega \cosh \xi_p \omega \cosh \xi_r \omega [\sum (\xi_n g_n + \xi_n M_n k_n) \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega]^2 = 0$$

Since the left side of the above equation can be factorized with N^2 , we obtain the following frequency equation, when $N \neq 0$. In this calculation, Eqs. (30) and (31) in Appendix 4 are used.

$$\frac{\xi_2 \xi_3}{M_2 - M_3} (g_1 + M_1 k_1 + 1)^2 \sinh \xi_1 \omega \cosh \xi_2 \omega \cosh \xi_3 \omega \\ + \frac{\xi_3 \xi_1}{M_3 - M_1} (g_2 + M_2 k_2 + 1)^2 \cosh \xi_1 \omega \sinh \xi_2 \omega \cosh \xi_3 \omega \\ + \frac{\xi_1 \xi_2}{M_1 - M_2} (g_3 + M_3 k_3 + 1)^2 \cosh \xi_1 \omega \cosh \xi_2 \omega \sinh \xi_3 \omega = 0 \dots \dots \dots (\text{F. E. 1})$$

When F. E. 1 holds good, $a' = b' = c' = 0$ and a circular arc bar vibrates antisymmetrically about its midpoint. Similarly we obtain the following frequency equation from $\partial L / \partial a', b', c' = 0$:

$$\frac{\xi_2 \xi_3}{M_2 - M_3} (g_1 + M_1 k_1 + 1)^2 \cosh \xi_1 \omega \sinh \xi_2 \omega \sinh \xi_3 \omega \\ + \frac{\xi_3 \xi_1}{M_3 - M_1} (g_2 + M_2 k_2 + 1)^2 \sinh \xi_1 \omega \cosh \xi_2 \omega \sinh \xi_3 \omega \\ + \frac{\xi_1 \xi_2}{M_1 - M_2} (g_3 + M_3 k_3 + 1)^2 \sinh \xi_1 \omega \sinh \xi_2 \omega \cosh \xi_3 \omega = 0 \dots \dots \dots (\text{F. E. 2})$$

When F. E. 2 holds good, $a = b = c = 0$ and a circular arc bar vibrates symmetrically about its midpoint. Moreover

$$\frac{\xi_p \xi_r}{M_p - M_r} (g_n + M_n k_n + 1)^2 = C \frac{\xi_n^3 (\xi_p^2 - \xi_r^2) (\xi_n^2 + 1 + \beta_0^2)}{\xi_n^2 - \frac{1}{\mu} + \beta_0^2} \dots \dots \dots (15)$$

where C is a constant independent of n, p and r . This equation is obtained by using Eq. (32) in Appendix 4, and in the numerical calculation it is convenient to use the right side of Eq. (15).

When α_0 is large, both F. E. 1 and 2 become the frequency equations of a straight bar with uniform section, that is, (see Appendix 6)

$$\text{F. E. 1} \longrightarrow \sin \alpha_0 \omega \cosh \alpha_0 \omega - \cos \alpha_0 \omega \sinh \alpha_0 \omega = 0 \dots \dots \dots (16)$$

$$\text{F. E. 2} \longrightarrow \sin \alpha_0 \omega \cosh \alpha_0 \omega + \cos \alpha_0 \omega \sinh \alpha_0 \omega = 0 \dots \dots \dots (17)$$

Eq. (16) is the frequency equation of the antisymmetric vibration of a straight bar and Eq. (17) is one of the symmetric vibration.

In Fig. 4, α_0 of F. E. 1 (antisymmetric vibration) is shown, for the case where $1/\mu=1.3$ {for example, the section of a circular arc bar is circular (solid or hollow) and the Poisson's ratio is 0.3}. In this case, α_0 in the case where the slenderness ratios are very large is almost equal to α_0 in the case where $s_x=s_y=100$ (when the central angles 2ω are equal to each other, α_0 in the case where $s_x=s_y=100$ is a little smaller than α_0 in the case where s_x and s_y are infinite). In Fig. 4, these full curves denoted with B. S. 2 and B. S. 4 show α_0 in Eq. (16) and correspond to the second and the fourth modes of the lateral vibration of a straight bar. When $\alpha_0 > 10$, α_0 in F. E. 1 is nearly equal to α_0 in Eq. (16). Moreover, for the case where $1/\mu=0.65$ (see Appendix 7) and the slenderness ratios are very large, α_0 is calculated. The results are as follows: (i) when $\alpha_0 > 6$, 2ω in the case where $1/\mu=0.65$ is almost equal to 2ω in the case where $1/\mu=1.3$ for the same α_0 . (ii) when $\alpha_0=3$, 2ω in the case where $1/\mu=0.65$ is a little less than 4% larger than 2ω in the case where $1/\mu=1.3$. So the effect of $1/\mu$ on the nondimensional frequency α_0 is small.

In Fig. 5, α_0 in the symmetric vibration (F. E. 2) is given against the central angle 2ω , when $1/\mu=1.3$, and the slenderness ratios are very large and $s_x=s_y=100$. In this case, the effect of the slenderness ratios is remarkable. When the slenderness ratios are infinite and α_0 is large, α_0 obtained from

$$\left. \begin{aligned} a &= -K \sum \frac{\tilde{\xi}_n k_n^2}{M_p - M_r} \cosh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega \\ b &= \frac{K}{(M_n - M_p)(M_p - M_r)(M_r - M_n)} \sum \tilde{\xi}_p \tilde{\xi}_r (g_n + M_n k_n + 1) \sinh \tilde{\xi}_n \omega \cosh \tilde{\xi}_p \omega \cosh \tilde{\xi}_r \omega, \\ c &= K \sum \frac{\tilde{\xi}_n g_n k_n}{M_p - M_r} \cosh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega \end{aligned} \right\} \dots \dots \dots (19)$$

Fig. 4

F. E. 2 is equal to the nondimensional frequency of the first mode of a straight bar. When α_0 becomes smaller, α_0 obtained from F. E. 2 comes nearer to the nondimensional frequency of the third mode of a straight bar. With smaller slenderness ratios, α_0 moves from the curve of the first mode to that of the third mode of a straight bar at smaller α_0 . The same is true for higher modes. B. S. 1, 3 and 5 in Fig. 5 are α_0 of Eq. (17) and they are the frequencies of the first, the third and the fifth modes, respectively. In both Figs. 4 and 5, the spots (black and white) are the calculated points.

6. Boundary values of a circular arc bar with both ends free

We consider the boundary values of a circular arc bar of which both ends are free and which vibrates perpendicularly to its plane. In the case of the antisymmetric vibration,

$$a' = b' = c' = 0 \dots \dots \dots (18)$$

For convenience, we put as follows:

$$W_n = f_n \sinh \tilde{\xi}_n \omega \cosh \tilde{\xi}_p \omega \cosh \tilde{\xi}_r \omega$$

$$X = \sum \tilde{\xi}_n g_n \cosh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega$$

$$Y = \sum \tilde{\xi}_n k_n \cosh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega$$

From any two equations of $\partial L / \partial a = \partial L / \partial b = \partial L / \partial c = 0$, we obtain a , b and c as follows, where K' is constant:

$$a = -K' (Y^2 + \sum W_n \cdot \sinh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega)$$

$$b = K' (X \sum W_n + Y \sum M_n W_n)$$

$$c = K' (XY - \sum M_n W_n \cdot \sinh \tilde{\xi}_n \omega \sinh \tilde{\xi}_p \omega \sinh \tilde{\xi}_r \omega)$$

Each equation is factorized with N and we put $K'N$ as K , so we obtain

Fig. 5

Values of a and c in the above equations are obtained by using Eqs. (35) to (37) in Appendix 5 and b is obtained by using Eqs. (30) and (31) in Appendix 4. From Eqs. (18), (19) and (6) it is easy to obtain the boundary values $A_0, B_0, \dots, G_0$. In the case of the symmetric vibration, $a=b=c=0$ and a', b' and c' are obtained by exchanging $\sinh$ and $\cosh$ for each other in Eq. (19).

The displacement (perpendicular to the arc) of the centre of gravity of a circular arc bar in its vibration is given as follows:

$$\delta = \int_{-\omega}^{\omega} v_0(\theta) \gamma S_0 \cdot R d\theta = 2\gamma S_0 R \sum \frac{P_n'}{\xi_n} \sinh \xi_n \omega$$

In the antisymmetric vibration, $a'=b'=c'=0$ and so

$P_n'=0$. Then, the centre of gravity does not move. In the case of the symmetric vibration, a', b' and c' which are obtained by exchanging $\sinh$ and $\cosh$ for each other in Eq. (19), are substituted for P_n' , and we obtain the following equation by using Eqs. (39) to (41) in Appendix 5:

$$\delta = -2\gamma S_0 R K \frac{\mu}{\alpha_0^4} \sum \frac{\xi_p \xi_r (g_n + M_n k_n + 1)^2}{M_p - M_r} \times \cosh \xi_n \omega \sinh \xi_p \omega \sinh \xi_r \omega \dots \dots \dots (20)$$

The left side of Eq. (20) is equal to F. E. 2 and so in the symmetric vibration, $\delta=0$. Thus, in the free vibration of a circular arc bar with free ends, its centre of gravity does not move, as other free-free bars.

Appendix 1

In the following equation,

$$L = \int_{-\omega}^{\omega} f(v_0, \psi, v_0', \psi', v_0'') d\theta \quad (v_0' = dv_0/d\theta)$$

v_0 and ψ which minimize L are given by variation calculus⁽⁴⁾ as follows:

$$\frac{\partial f}{\partial v_0} - \frac{d}{d\theta} \left(\frac{\partial f}{\partial v_0'} \right) + \frac{a^2}{d\theta^2} \left(\frac{\partial f}{\partial v_0''} \right) = 0, \quad \frac{\partial f}{\partial \psi} - \frac{d}{d\theta} \left(\frac{\partial f}{\partial \psi'} \right) = 0$$

Calculating like the above, we obtain from Eq. (1)

$$\frac{d^4 v_0}{d\theta^4} - (\mu - \beta_0'^2) \frac{d^2 v_0}{d\theta^2} - \alpha_0^4 v_0 = (1 + \mu) \frac{d^2 \psi}{d\theta^2}, \quad -(1 + \mu) \frac{d^2 v_0}{d\theta^2} = \mu \frac{d^2 \psi}{d\theta^2} - (1 - \mu \beta_0'^2) \psi \dots \dots \dots (21)$$

Eq. (21) becomes

$$\left. \begin{aligned} \psi(\theta) = & \eta \left[\frac{d^4 v_0}{d\theta^4} + \left(2 + \frac{1}{\mu} + \beta_0'^2 \right) \frac{d^2 v_0}{d\theta^2} - \alpha_0^4 v_0(\theta) \right] \\ & \frac{d^6 v_0}{d\theta^6} + (2 + \beta_0'^2 + \beta_0'^2) \frac{d^4 v_0}{d\theta^4} + \left[-\alpha_0^4 + (\mu - \beta_0'^2) \left(\frac{1}{\mu} - \beta_0'^2 \right) \right] \frac{d^2 v_0}{d\theta^2} + \alpha_0^4 \left(\frac{1}{\mu} - \beta_0'^2 \right) v_0(\theta) = 0 \end{aligned} \right\} \dots \dots \dots (22)$$

From Eq. (22), Eqs (2), (3) and (4) are obtained and the following equations are obtained by substituting Eq. (2) in Eq. (21):

$$\xi_n^4 - (\mu - \beta_0'^2) \xi_n^2 - \alpha_0^4 = (1 + \mu) M_n \xi_n^2, \quad -(1 + \mu) \xi_n^2 = \mu M_n \left(\xi_n^2 - \frac{1}{\mu} + \beta_0'^2 \right) \dots \dots \dots (23)$$

All the equations which appear in Appendix 2, 3 etc. are obtained from the following: Eq. (23), Eq. (3) and the relations between roots and coefficients, and the definitions of M_n, g_n etc. All the equations in Appendixes hold good, when the suffixes are changed as $n \rightarrow p \rightarrow r \rightarrow n$ in turn.

Since Love⁽¹⁾ has obtained the equation of vibration of a circular arc bar with a circular section, the equation obtained by eliminating N', G and H from his Eqs. (18), (19) and (20) are equal to Eq. (21) given above for a circular section.

Appendix 2

From Eq. (23) and Eq. (3), and the relations between roots and coefficients of Eq. (3), the following equations are obtained:

$$\alpha_0^4 + \mu \beta_0'^2 M_n^2 - \beta_0'^2 \xi_n^2 - (M_n - \xi_n^2)^2 + \mu \xi_n^2 (1 + M_n)^2 = 0 \dots \dots \dots (24)$$

$$\left. \begin{aligned} \alpha_0^4 + \mu \beta_0'^2 M_n M_p - \beta_0'^2 \xi_n^2 - (M_n - \xi_n^2)(M_p - \xi_p^2) + \mu \xi_n^2 (1 + M_n)(1 + M_p) \\ = -(\xi_n^2 - \xi_p^2)(M_n - \xi_n^2) \\ \alpha_0^4 + \mu \beta_0'^2 M_n M_p - \beta_0'^2 \xi_p^2 - (M_n - \xi_n^2)(M_p - \xi_p^2) + \mu \xi_p^2 (1 + M_n)(1 + M_p) \\ = (\xi_n^2 - \xi_p^2)(M_p - \xi_p^2) \end{aligned} \right\} \dots \dots \dots (25)$$

They are obtained in the following manner: Rearranging the left side of Eq. (24) as the decreasing powers of M_n ; M_n^2 is changed to M_n by the second equation of Eq. (23) and the left side becomes zero according to the first equation of Eq. (23). The left side of the first equation of Eq. (25) separates into the term containing M_p and the term not containing M_p . The former becomes zero according to the first equation of Eq. (23). The latter becomes the right side of Eq. (25) by substitution of α_0^4 obtained from the first equation of Eq. (23).

The second equation of Eq. (25) can be proved in the same manner.

Appendix 3

From the difference of two equations of Eq. (25), we obtain

$$\beta_0'^2 = \mu(1+M_n)(1+M_p) + (M_n - \xi_n^2) + (M_p - \xi_p^2) \dots\dots\dots (26)$$

Subtracting the equation which is obtained by changing the suffixes of Eq. (26) from Eq. (26), we obtain

$$\mu(1+M_n)(M_p - M_r) + (M_p - M_r) = \xi_p^2 - \xi_r^2 \dots\dots\dots (27)$$

Using the relations between roots and coefficients in $M_n - M_p$, we obtain

$$M_n - M_p = -\gamma(\xi_n^2 - \xi_p^2) \left(\xi_r^2 - \frac{1}{\mu} + \beta_0'^2 \right) \dots\dots\dots (28)$$

From Eqs. (26), (27) and (28), we obtain the following relation:

$$\emptyset = -\gamma(\xi_n^2 - \xi_p^2)(\xi_p^2 - \xi_r^2)(\xi_r^2 - \xi_n^2) \dots\dots\dots (29.1)$$

$$= -\mu(M_n - M_p)(M_p - M_r)(M_r - M_n) \dots\dots\dots (29.2)$$

$$= M_n(\xi_p^2 - \xi_r^2) + M_p(\xi_r^2 - \xi_n^2) + M_r(\xi_n^2 - \xi_p^2) \dots\dots\dots (29.3)$$

$$= (M_p - M_r)[\mu(1+M_n)^2 - \beta_0'^2 + 2(M_n - \xi_n^2)] \dots\dots\dots (29.4)$$

Rearranging Eq. (9) for which P_n and P_n' are substituted from Eq. (29), we obtain Eqs. (11) to (14). Each equation of Eq. (29) can be obtained as follows: we obtain Eq. (29.1) by substituting Eq. (28) in Eq. (29.2) and using the relations between roots and coefficients; we obtain Eq. (29.3) = Eq. (29.2) by using Eq. (27). It is clear by using Eqs. (26) and (27) that equations obtained by changing suffixes in Eq. (29.4) are equal to each other, and we obtain Eq. (29.4) = $\emptyset$ by summing up three equations obtained by changing their suffixes of Eq. (29.4) and by using Eqs. (29.2) and (29.3). In the calculation of the Lagrangian, we obtain the equations in form of Eq. (29.4).

Appendix 4

Between g_n and k_n , there are the following relations:

$$\frac{M_p - M_r}{M_r - M_n}(g_p + M_n k_p) - (g_n + M_n k_n) = \frac{M_p - M_r}{M_n - M_p}(g_r + M_n k_r) - (g_n + M_n k_n) = 1 \dots\dots\dots (30)$$

$$\sum (g_n + M_n k_n) = -2 \dots\dots\dots (31)$$

Moreover, there is the relation as follows:

$$g_n + M_n k_n + 1 = -\frac{\xi_n^2(\xi_n^2 + 1 + \beta_0'^2)}{(\xi_r^2 - \xi_n^2)(\xi_n^2 - \xi_p^2)} \dots\dots\dots (32)$$

In the numerical calculation, it is convenient to use the form in Eq. (32). Notations g_n and k_n are also represented as follows:

$$g_n = \frac{\xi_n^4 + (2 + \beta_0'^2 + \beta_0'^2)\xi_n^2 + (\mu - \beta_0'^2)\left(\frac{1}{\mu} - \beta_0'^2\right)}{(\xi_r^2 - \xi_n^2)(\xi_n^2 - \xi_p^2)} = \frac{\alpha_0^4}{\emptyset} \frac{M_p - M_r}{\xi_n^2} \dots\dots\dots (33)$$

$$k_n = -\frac{\mu}{\emptyset}(1 + M_n)(M_p - M_r) \dots\dots\dots (34)$$

We can prove Eqs. (30) to (34) as follows. After we represent M , g and k in all equations above with ξ , we calculate both sides of them by Eq. (3) and its relations between roots and coefficients. We obtain the frequency equations by using Eq. (31) after the terms containing the forms of $(g_p + M_n k_p)$ are changed into the forms of $(g_n + M_n k_n + 1)$ by Eq. (30). In the numerical calculation of a , b etc. in the section 6, it is convenient to use Eqs. (32), (33) and (34). These relations, Eqs. (31) and (35) to (37) are useful for calculation of the numerical values of g , k etc.

Appendix 5

Combining the equations in Appendix 4, we obtain the following equations:

$$k_n(M_r - M_n) - k_p(M_p - M_r) = 1 \dots\dots\dots (35)$$

$$g_n(M_r - M_n) - g_p(M_p - M_r) = -M_r \dots\dots\dots (36)$$

$$\sum k_n = 0, \quad \sum M_n k_n = -1, \quad \sum g_n = -1 \dots\dots\dots (37)$$

From Eq. (4) and the relations between roots and coefficients of Eq. (3), we obtain

$$M_n(M_p - M_r) = \frac{1}{\mu\left(\frac{1}{\mu} - \beta_0'^2\right)} \xi_n^2(\xi_p^2 - \xi_r^2) \dots\dots\dots (38)$$

From Eqs. (30) and (38), we obtain

$$\frac{k_p(\xi_p^2 - \xi_r^2)}{\xi_r(M_r - M_n)}(g_p + M_n k_p) - \frac{\mu}{\emptyset} \xi_r(M_r - M_n)(g_n + M_n k_n + 1) = -\frac{\mu}{\alpha_0^4} \xi_p^2 \xi_r \frac{M_r - M_n}{M_p - M_r} (g_n + M_n k_n + 1)^2 \dots (39)$$

$$\frac{k_n(\xi_r^2 - \xi_n^2)}{\xi_r(M_p - M_r)}(g_n + M_p k_n) - \frac{\mu}{\emptyset} \xi_r(M_p - M_r)(g_p + M_p k_p + 1) = -\frac{\mu}{\alpha_0^4} \xi_r \xi_n^2 \frac{M_p - M_r}{M_r - M_n} (g_p + M_p k_p + 1)^2 \dots (40)$$

From Eq. (37), we obtain

$$\begin{aligned} \sum \left[\frac{\xi_n k_n (\xi_p^2 - \xi_r^2)}{\xi_p \xi_r (M_p - M_r)} (g_n + M_n k_n) - \frac{\mu}{\emptyset} \frac{\xi_p \xi_r}{\xi_n} (M_p - M_r) (g_n + M_n k_n + 1) \right] \\ = -\frac{\mu}{\alpha_0^4} \xi_n \xi_p \xi_r \sum (g_n + M_n k_n + 1)^2 \dots (41) \end{aligned}$$

In the calculation of δ , we obtain the left sides of Eqs. (39), (40) and (41) and representing them with the right sides of these equations, we obtain Eq. (20).

Appendix 6

The frequency equations for the lateral vibration of a straight bar with the length l are

symmetric vibration

$$\sin(\alpha/2) \cosh(\alpha/2) + \cos(\alpha/2) \sinh(\alpha/2) = 0$$

antisymmetric vibration

$$\sin(\alpha/2) \cosh(\alpha/2) - \cos(\alpha/2) \sinh(\alpha/2) = 0$$

where

$$\alpha^4 = \gamma S p^2 l^4 / EI_x g.$$

When α_0 becomes large, $\xi_1 \rightarrow \sqrt{1/\mu}$, $\xi_2 \rightarrow \alpha_0$, $\xi_3 \rightarrow i\alpha_0$ (for example, when $1/\mu = 1.3$ and s_x and s_y are very large, in the case where $\alpha_0 = 10$, $\xi_1^2 = 1.3006885$, $\xi_2 = 9.91649168$, $\xi_3 = i \cdot 10.0815423$ and they are nearly equal to $\sqrt{1/\mu}$, α_0 and $i\alpha_0$, respectively) and neglecting ξ_1 in F.E. 1 and 2, we obtain the

following frequency equations (they are Eqs. (16) and (17)):

$$\sin \alpha_0 \omega \cosh \alpha_0 \omega \pm \cos \alpha_0 \omega \sinh \alpha_0 \omega = 0$$

where

$$(2\alpha_0 \omega)^4 = \gamma S_0 p^2 l'^4 / EI_x g \quad (l' \text{ is the arc length}).$$

Appendix 7

Ratio of rigidity $1/\mu = 0.65$ is such a case where the section of a circular arc bar is rectangular and its dimension in x -axis is sufficiently larger than that in y -axis.

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