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Tension of a Semi-infinite Plate with a Circular Hole Connected to the Straight Edge by a Slit*

By Kyohei MORI**

When there is a crack on the straight edge of a plate under tension, to prevent further development of the crack, a circular hole is drilled at the end of the crack. Considering the above case, the author has solved the problem of a semi-infinite plate with a circular hole connected to the straight edge by a straight and narrow slit subjected to uniform tension parallel to the straight edge.

He has obtained stress distributions on the plate and the factors of stress concentration at the bottom of the circular hole.

1. Introduction

There are many theoretical investigations on the tension of a semi-infinite plate with a circular⁽¹⁾, elliptic⁽²⁾, U-type⁽³⁾ notch, etc. From those results, it is well known that a high stress concentration is produced at the bottom of the notch and, therefore, when there is a crack on the straight edge of a plate under tension, the crack tends to spread. The spreading can be stopped by drilling a hole at the end of the crack to eliminate the sharp curvature responsible for the high stress concentration.

To solve the above case, we consider a semi-infinite plate with a circular hole connected to the straight edge by a straight and narrow slit subjected to uniform tension parallel to the straight edge as shown in Fig. 1. Using the theory of two-dimensional elasticity, we can obtain a solution which gives the stress distributions in the plate and the factor of stress concentration at the bottom of the notch, i. e., that of the circular hole.

2. Stress functions

Let us take curvilinear coordinates defined by the conjugate functions

Fig. 1

$$\alpha + i\beta = \log \frac{x + i(y+a)}{x + i(y-a)}$$

where x, y are Cartesian coordinates and a is a positive real length. Solving for x, y we have

$$x = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}, \quad y = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta} \quad \dots (1)$$

The general scheme of coordinates is shown in Fig. 2. The boundaries of the plate are defined by $\alpha=0$ for the straight edge, $\alpha=\alpha_1$ for the circular hole, $\beta=\pi$ and $\beta=-\pi$ for both sides of the slit.

The stress function F must satisfy the equation

$$\left. \begin{aligned} & \left(\frac{\partial^4}{\partial \alpha^4} + 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \frac{\partial^4}{\partial \beta^4} \right. \\ & \quad \left. - 2 \frac{\partial^2}{\partial \alpha^2} + 2 \frac{\partial^2}{\partial \beta^2} + 1 \right) \left(\frac{F}{J} \right) = 0 \end{aligned} \right\} \dots (2)$$

where

$$J = \frac{a}{\cosh \alpha - \cos \beta}$$

Fig. 2

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The stress components are

$$\left. \begin{aligned} a\sigma_{\alpha} &= \left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \beta^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha \right\} \left(\frac{F}{J} \right) \\ a\sigma_{\beta} &= \left\{ (\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cos \beta \right\} \left(\frac{F}{J} \right) \\ a\tau_{\alpha\beta} &= -(\cosh \alpha - \cos \beta) \frac{\partial^2}{\partial \alpha \partial \beta} \left(\frac{F}{J} \right) \end{aligned} \right\} \dots \dots \dots (3)$$

Putting the stress function satisfying Eq. (2), we get

$$F = F_1 + F_2$$

Let $\sigma_{\alpha 1}, \sigma_{\beta 1}, \sigma_{\alpha\beta 1}$ denote the stress components derived from Eq. (3) by substituting F_1 for F and let $\sigma_{\alpha 2}, \sigma_{\beta 2}, \tau_{\alpha\beta 2}$ denote the same by F_2 , respectively.

F_1 is the stress function which has been used in the solution for the problem of a semi-infinite plate containing a circular hole and under uniform tension T parallel to its straight edge. This is given by G. B. Jeffery⁽⁴⁾ as follows:

$$\begin{aligned} \frac{F_1}{J} &= aT \left[\frac{1}{2} \frac{\sinh^2 \alpha}{(\cosh \alpha - \cos \beta)} + B_0 \alpha (\cosh \alpha - \cos \beta) + A_1 (\cosh 2\alpha - 1) \cos \beta \right. \\ &\quad + \sum_{n=2}^{\infty} A_n \{ \cosh (n+1)\alpha - \cosh (n-1)\alpha \} \cos n\beta \\ &\quad \left. + \sum_{n=2}^{\infty} E_n \{ (n-1) \sinh (n+1)\alpha - (n+1) \sinh (n-1)\alpha \} \cos n\beta \right] \end{aligned}$$

$\sigma_{\beta 1}$, one of the stress components derived from the stress function F_1 , is

$$\begin{aligned} \sigma_{\beta 1} &= T [1 + B_0 \sinh \alpha (\cosh \alpha - \cos \beta) + A_1 \{ 4 \cosh 2\alpha \cos \beta (\cosh \alpha - \cos \beta) - 2 \sinh \alpha \sinh 2\alpha \cos \beta \\ &\quad + \cosh 2\alpha - 1 \} + \sum_{n=2}^{\infty} A_n \{ (n+1)^2 (\cosh \alpha - \cos \beta) \cosh (n+1)\alpha \cos n\beta \\ &\quad - (n+1) \sinh \alpha \sinh (n+1)\alpha \cos n\beta + n \sin \beta \cosh (n+1)\alpha \sin n\beta \\ &\quad + \cos \beta \sinh (n+1)\alpha \cos n\beta - (n-1)^2 (\cosh \alpha - \cos \beta) \cosh (n-1)\alpha \cos n\beta \\ &\quad + (n-1) \sinh \alpha \sinh (n-1)\alpha \cos n\beta - n \sin \beta \cosh (n-1)\alpha \sin n\beta - \cos \beta \cosh (n-1)\alpha \cos n\beta \} \\ &\quad + \sum_{n=2}^{\infty} E_n \{ (n-1)(n+1)^2 (\cosh \alpha - \cos \beta) \sinh (n+1)\alpha \cos n\beta \\ &\quad - (n-1)(n+1) \sinh \alpha \cosh (n+1)\alpha \cos n\beta + n(n-1) \sin \beta \sinh (n+1)\alpha \sin n\beta \\ &\quad + (n-1) \cos \beta \sinh (n+1)\alpha \cos n\beta - (n+1)(n-1)^2 (\cosh \alpha - \cos \beta) \sinh (n-1)\alpha \cos n\beta \\ &\quad + (n+1)(n-1) \sinh \alpha \cosh (n-1)\alpha \cos n\beta - n(n+1) \sin \beta \sinh (n-1)\alpha \sin n\beta \\ &\quad - (n+1) \cos \beta \sinh (n-1)\alpha \cos n\beta \}] \dots \dots \dots (4) \end{aligned}$$

where

$$\left. \begin{aligned} A_1 &= \frac{1}{2} e^{-2\alpha_1} \operatorname{sech} 2\alpha_1, \quad B_0 = \operatorname{sech} 2\alpha_1, \quad A_n = - \frac{n^2 \sinh^2 \alpha_1 - n \sinh \alpha_1 \cosh \alpha_1 + e^{-n\alpha_1} \sinh n\alpha_1}{2(\sinh^2 n\alpha_1 - n^2 \sinh^2 \alpha_1)} \\ E_n &= \frac{n \sinh^2 \alpha_1}{2(\sinh^2 n\alpha_1 - n^2 \sinh^2 \alpha_1)} \end{aligned} \right\} \dots \dots \dots (5)$$

The stress components derived from F_1 satisfies boundary conditions that $(\sigma_{\alpha 1})_{\alpha=0} = (\tau_{\alpha\beta 1})_{\alpha=0} = 0$ for the straight edge, $(\sigma_{\alpha 1})_{\alpha=\alpha_1} = (\tau_{\alpha\beta 1})_{\alpha=\alpha_1} = 0$ for the circular hole and $(\sigma_{\beta 1})_{\alpha=0} = T$ for the point at infinity. But, for the slit, the boundary condition is not satisfied, as $(\sigma_{\beta 1})_{\beta=\pi} = (\sigma_{\beta 1})_{\beta=-\pi} \neq 0$ and some resultant force and couple are left on the side OA.

Putting W and M_o for the resultant force and resultant moment about the origin, respectively, we have

$$W = \int_0^{\alpha_1} (\sigma_{\beta 1} dy)_{\beta=\pi}, \quad M_o = \int_0^{\alpha_1} (\sigma_{\beta 1} y dy)_{\beta=\pi}$$

Substituting the value of $\sigma_{\beta 1}$ and y of Eq. (1) into the above formula, we have

$$\begin{aligned} W &= aT \left[\tanh \frac{\alpha_1}{2} + B_0 (\cosh \alpha_1 - 1) - A_1 (\sinh 2\alpha_1 + 2 \sinh \alpha_1) + 2 \sum_{n=2}^{\infty} A_n (-1)^n \{ n \sinh \alpha_1 \cosh n\alpha_1 + \sinh n\alpha_1 \} \right. \\ &\quad \left. + 2 \sum_{n=2}^{\infty} E_n (-1)^n \left\{ n^2 \sinh \alpha_1 \sinh n\alpha_1 - n (\cosh \alpha_1 - 1) \cosh n\alpha_1 - \tanh \frac{\alpha_1}{2} \sinh n\alpha_1 \right\} \right] \\ M_o &= a^2 T \left[\frac{1}{2} \tanh^2 \frac{\alpha_1}{2} + B_0 (\sinh \alpha_1 - \alpha_1) - A_1 (\cosh 2\alpha_1 - 2 \cosh \alpha_1 + 1) + 2 \sum_{n=2}^{\infty} A_n (-1)^n \frac{n \sinh^2 \alpha_1 \cosh n\alpha_1}{\cosh \alpha_1 + 1} \right. \\ &\quad \left. + 2 \sum_{n=2}^{\infty} E_n (-1)^n \frac{1}{\cosh \alpha_1 + 1} (n^2 \sinh^2 \alpha_1 \sinh n\alpha_1 - n \sinh \alpha_1 \cosh \alpha_1 \cosh n\alpha_1 + \sinh n\alpha_1) \right] \end{aligned}$$

Substituting the value of Eq. (5) into the above, we find

$$W = aT \left(1 - \frac{3}{2} \operatorname{sech} 2\alpha_1 \right), \quad M_o = a^2 T \frac{1}{2} \operatorname{sech} 2\alpha_1 (\sinh 2\alpha_1 - 2\alpha_1) \quad (6)$$

The other stress function F_2 satisfies boundary conditions that there is no stress on the straight edge, on the periphery of the circular hole and at infinity, but the resultant force and moment on the sides of the slit are $-W$ and $-M_o$, respectively. Then the stress function $F = F_1 + F_2$ satisfies the condition that the resultant force and couple on the sides of the slit will vanish. Therefore it may be considered that the slit has free boundaries. Then putting

$$\frac{F_2}{J} = aT [B\alpha \cosh \alpha + D\alpha \sinh \alpha + D_1\alpha \cos \beta + \{A(\cosh 2\alpha - 1) + C \sinh 2\alpha\} \cos \beta]$$

from Eq. (3) we obtain one of the stress components by F_2 ,

$$\begin{aligned} \sigma_{\beta_2} = T [& B(\alpha + \sinh \alpha \cosh \alpha - 2 \sinh \alpha \cos \beta) + D(\cosh^2 \alpha + 1 - 2 \cosh \alpha \cos \beta) + D_1(\alpha - \sinh \alpha \cos \beta) \\ & + A \{ 4 \cosh 2\alpha \cos \beta (\cosh \alpha - \cos \beta) - 2 \sinh \alpha \sinh 2\alpha \cos \beta + \cosh 2\alpha - 1 \} \\ & + C \{ 4 \sinh 2\alpha \cos \beta (\cosh \alpha - \cos \beta) - 2 \sinh \alpha \cosh 2\alpha \cos \beta + \sinh 2\alpha \}] \quad (7) \end{aligned}$$

When the straight edge and the periphery of the circular hole are free from stress, the boundary conditions assume a very simple form as follows:

When $\alpha = \alpha_1$

$$\frac{\partial}{\partial \alpha} \left(\frac{F_2}{J} \right) = \rho_1, \quad \frac{F_2}{J} = \rho_1 \tanh \alpha_1 + \sigma_1 (\cosh \alpha_1 \cos \beta - 1)$$

and when $\alpha = 0$

$$\frac{\partial}{\partial \alpha} \left(\frac{F_2}{J} \right) = \rho_0, \quad \frac{F_2}{J} = \sigma_0 (\cos \beta - 1)$$

where ρ_1, σ_1, ρ_0 and σ_0 are constants.

From the above conditions, we have

$$\begin{aligned} B(\alpha_1 \sinh \alpha_1 + \cosh \alpha_1) + D(\alpha_1 \cosh \alpha_1 + \sinh \alpha_1) &= \frac{\rho_1}{aT}, \quad D_1 + A \cdot 2 \sinh 2\alpha_1 + C \cdot 2 \cosh 2\alpha_1 = 0, \\ B\alpha_1 \cosh \alpha_1 + D\alpha_1 \sinh \alpha_1 &= \frac{1}{aT} (\rho_1 \tanh \alpha_1 - \sigma_1), \quad D_1\alpha_1 + A(\cosh 2\alpha_1 - 1) + C \sinh 2\alpha_1 = \frac{\sigma_1 \cosh \alpha_1}{aT}, \\ D_1 + 2C &= 0 \end{aligned}$$

and eliminating ρ_1 and σ_1 , we obtain the following three equations:

$$\begin{aligned} D_1 &= -2C, \quad A = -C \tanh \alpha_1 \\ B(\alpha_1 - \sinh \alpha_1 \cosh \alpha_1) - D \sinh^2 \alpha_1 + 2C \operatorname{sech} \alpha_1 (-\alpha_1 \cosh \alpha_1 + \sinh \alpha_1) &= 0 \end{aligned} \quad (a)$$

On the other hand

$$\int_0^{\alpha_1} (\sigma_{\beta_2} dy)_{\beta=\pi} = -W, \quad \int_0^{\alpha_1} (\sigma_{\beta_2} y dy)_{\beta=\pi} = -M_o$$

from which we have

$$aT \left\{ B \frac{\sinh \alpha_1 (\alpha_1 + \sinh \alpha_1)}{\cosh \alpha_1 + 1} + D(\alpha_1 + \sinh \alpha_1) - C \cdot 2 \sinh \alpha_1 \frac{(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)}{\cosh \alpha_1 + 1} \right\} = -W \quad (b)$$

$$a^2 T \left\{ B \frac{\sinh \alpha_1 \cosh \alpha_1 - \alpha_1}{\cosh \alpha_1 + 1} + D_1(\cosh \alpha_1 - 1) - C \cdot 2 \frac{(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)}{\cosh \alpha_1 + 1} \right\} = -M_o \quad (c)$$

From Eqs. (a), (b) and (c), we find the following values for the coefficients:

$$\begin{aligned} B &= \frac{W}{aT} \frac{\sinh^2 \alpha_1}{(\sinh^2 \alpha_1 - \alpha_1^2)} - \frac{M_o}{a^2 T} \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{(\sinh^2 \alpha_1 - \alpha_1^2)}, \\ D &= -\frac{W}{aT} \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{(\sinh^2 \alpha_1 - \alpha_1^2)} + \frac{M_o}{a^2 T} \frac{\sinh^2 \alpha_1}{(\sinh^2 \alpha_1 - \alpha_1^2)}, \\ D_1 &= -\frac{M_o}{a^2 T} \frac{\cosh \alpha_1}{(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)}, \quad A = -\frac{M_o}{a^2 T} \frac{\sinh \alpha_1}{2(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)}, \\ C &= \frac{M_o}{a^2 T} \frac{\cosh \alpha_1}{2(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)} \end{aligned}$$

Hence, substituting the values of W and M_o of Eq. (6), we have

$$\left. \begin{aligned} B &= \left\{ \left(1 - \frac{3}{2} \operatorname{sech} 2\alpha_1 \right) \sinh^2 \alpha_1 - (\sinh^2 \alpha_1 \cosh^2 \alpha_1 - \alpha_1^2) \operatorname{sech} 2\alpha_1 \right\} \frac{1}{(\sinh^2 \alpha_1 - \alpha_1^2)} \\ D &= - \left\{ \left(1 - \frac{3}{2} \operatorname{sech} 2\alpha_1 \right) - \sinh^2 \alpha_1 \operatorname{sech} 2\alpha_1 \right\} \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{(\sinh^2 \alpha_1 - \alpha_1^2)} \\ D_1 &= - \cosh \alpha_1 \operatorname{sech} 2\alpha_1 \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)} \\ A &= - \sinh \alpha_1 \operatorname{sech} 2\alpha_1 \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{2(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)} \\ C &= \cosh \alpha_1 \operatorname{sech} 2\alpha_1 \frac{(\sinh \alpha_1 \cosh \alpha_1 - \alpha_1)}{2(\alpha_1 \cosh \alpha_1 - \sinh \alpha_1)} \end{aligned} \right\} \dots \dots \dots (8)$$

3. Stress distributions

Knowing the values of the constants A_1, B_0, A_n, E_n and B_1, D, D_1, A, C for F_1 and F_2 , respectively, we obtain stress distributions in the plate.

3.1 Stress along the straight edge

It is given by

$$(\sigma_\beta)_{\alpha=0} = (\sigma_{\beta_1})_{\alpha=0} + (\sigma_{\beta_2})_{\alpha=0}$$

substituting the following values $(\sigma_{\beta_1})_{\alpha=0}$ and $(\sigma_{\beta_2})_{\alpha=0}$ from Eqs. (4) and (7)

$$(\sigma_{\beta_1})_{\alpha=0} = T \{ 1 + 4A_1(1 - \cos \beta) \cos \beta + 4 \sum_{n=2}^{\infty} n A_n \cos n\beta (1 - \cos \beta) \}$$

$$(\sigma_{\beta_2})_{\alpha=0} = T \{ 2D(1 - \cos \beta) + 4A(1 - \cos \beta) \cos \beta \}$$

3.2 Stress along the edge of the circular hole

It is in this case

$$(\sigma_\beta)_{\alpha=\alpha_1} = (\sigma_{\beta_1})_{\alpha=\alpha_1} + (\sigma_{\beta_2})_{\alpha=\alpha_1}$$

where each term is expressed in accordance with Eqs. (4), (5) and (7):

$$\begin{aligned} (\sigma_{\beta_1})_{\alpha=\alpha_1} &= T(\cosh \alpha_1 - \cos \beta) \left[2 \sinh \alpha_1 \operatorname{sech} 2\alpha_1 \right. \\ &\quad \left. + \sum_{n=2}^{\infty} \frac{n \{ (n-1) \sinh (n+1)\alpha_1 - (n+1) \sinh (n-1)\alpha_1 \}}{(\sinh^2 n\alpha_1 - n^2 \sinh^2 \alpha_1)} \cos n\beta \right] \\ (\sigma_{\beta_2})_{\alpha=\alpha_1} &= T[B(\alpha_1 + \sinh \alpha_1 \cosh \alpha_1 - 2 \sinh \alpha_1 \cos \beta) + D(\cosh^2 \alpha_1 + 1 - 2 \cosh \alpha_1 \cos \beta) \\ &\quad + D_1(\alpha_1 - \sinh \alpha_1 \cos \beta) + A \{ 4 \cosh 2\alpha_1 \cos \beta (\cosh \alpha_1 - \cos \beta) \\ &\quad - 2 \sinh \alpha_1 \sinh 2\alpha_1 \cos \beta + \cosh 2\alpha_1 - 1 \} + C \{ 4 \sinh 2\alpha_1 \cos \beta (\cosh \alpha_1 - \cos \beta) \\ &\quad - 2 \sinh \alpha_1 \cosh 2\alpha_1 \cos \beta + \sinh 2\alpha_1 \}] \end{aligned}$$

The latter of the above two formulas is given by substituting the constants of Eq. (8).

3.3 Stress on the symmetrical axis

$$(\sigma_\beta)_{\beta=0} = (\sigma_{\beta_1})_{\beta=0} + (\sigma_{\beta_2})_{\beta=0}$$

where Eqs. (4) and (7) give

$$\begin{aligned} (\sigma_{\beta_1})_{\beta=0} &= T[1 + B_0 \sinh \alpha (\cosh \alpha - 1) + 2A_1(\cosh \alpha - 1)(\cosh 2\alpha - \cosh \alpha) \\ &\quad + \sum_{n=2}^{\infty} 2A_n n (\cosh \alpha - 1) \{ n \sinh \alpha \sinh n\alpha + (\cosh \alpha - 1) \cosh n\alpha \} \\ &\quad + \sum_{n=2}^{\infty} 2E_n \{ n^3 (\cosh \alpha - 1) \sinh \alpha \cosh n\alpha - n^2 (\cosh \alpha - 1) \sinh n\alpha \\ &\quad - n(\cosh \alpha - 2) \sinh \alpha \cosh n\alpha - \sinh n\alpha \}] \\ (\sigma_{\beta_2})_{\beta=0} &= T[B(\alpha + \sinh \alpha \cosh \alpha - 2 \sinh \alpha) + D(\cosh \alpha - 1)^2 + D_1(\alpha - \sinh \alpha) \\ &\quad + 2A(\cosh \alpha - 1)(\cosh 2\alpha - \cosh \alpha) + 2C(\cosh \alpha - 1)(\sinh 2\alpha - \sinh \alpha)] \end{aligned}$$

For the case $\alpha_1 = 2.0$, the distributions of $(\sigma_\beta)_{\alpha=\alpha_1}$ and $(\sigma_\beta)_{\beta=0}$ are shown in Fig. 3.

3.4 The maximum stress

The maximum stress is produced at point B, the bottom of the circular

hole. Then we have

$$\sigma_{\max} = (\sigma_\beta)_{\beta=0} = (\sigma_{\beta_1})_{\beta=0} + (\sigma_{\beta_2})_{\beta=0}$$

in which

$$\begin{aligned} (\sigma_{\beta_1})_{\beta=0} &= T(\cosh \alpha_1 - 1) \left[2 \sinh \alpha_1 \operatorname{sech} 2\alpha_1 + \sum_{n=2}^{\infty} \frac{n \{ (n-1) \sinh (n+1)\alpha_1 - (n+1) \sinh (n-1)\alpha_1 \}}{\sinh^2 n\alpha_1 - n^2 \sinh^2 \alpha_1} \right] \\ (\sigma_{\beta_2})_{\beta=0} &= T[B(\alpha_1 + \sinh \alpha_1 \cosh \alpha_1 - 2 \sinh \alpha_1) + D(\cosh \alpha_1 - 1)^2 + D_1(\alpha_1 - \sinh \alpha_1) \\ &\quad + 2A(\cosh \alpha_1 - 1)(\cosh 2\alpha_1 - \cosh \alpha_1) + 2C(\cosh \alpha_1 - 1)(\sinh 2\alpha_1 - \sinh \alpha_1)] \end{aligned}$$

We varied the value of α_1 in order to vary the ratio of diameter d of the hole to the depth t of the notch and

the corresponding values of the factor of stress concentration, $\sigma_{\max}/T$ are shown in Table 1.

We obtain the magnitude of d/t from

$$\frac{d}{t} = 1 - \tanh^2 \frac{\alpha_1}{2}$$

and the relation between d/t and the factor of stress concentration $\sigma_{\max}/T$ is shown by a full line in Fig. 4.

For comparison, as shown in Fig. 5, we take an infinite plate containing an elliptic hole, whose sharpest curvature is equal to that of the above circular hole, and being subjected to uniform tension perpendicular to the long axis of the ellipse. Denoting by b the length of the short axis of the ellipse and by σ_e the maximum stress which is developed at the ends of the long axis of the elliptic hole, we find the factor of stress concentration⁽⁵⁾

$$\frac{\sigma_e}{T} = \left(1 + 2 \frac{t}{b}\right)$$

Radius of the sharpest curvature of the elliptic hole is equal to $d/2$, and so

$$\frac{t}{b} = \sqrt{\frac{2t}{d}}$$

from which we have

$$\frac{\sigma_e}{T} = 1 + 2 \sqrt{\frac{2t}{d}}$$

The relation between d/t and σ_e/T is shown by a dotted line in Fig. 4. We obtain the following conclusion that the difference between the factors of stress concentration in a circular and an elliptic hole is not significant.

3.5 Surface force on the edge of the straight slit Although there is no resultant force and couple on the boundaries of the straight slit, some surface forces are distributed there. The distribution of the surface forces is expressed as follows:

$$(\sigma_{\beta_1})_{\beta=\pi} = (\sigma_{\beta_1})_{\beta=-\pi} + (\sigma_{\beta_2})_{\beta=\pi}$$

and Eqs. (4) and (7) give

Table 1

α_1	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6
$(\sigma_{\beta_1})/T$ $\beta=0$ $\alpha=\alpha_1$	3.36	3.27	3.20	3.15	3.12	3.09	3.07	3.05	3.04	3.02	3.02	3.01	3.01	3.01	3.00	3.00
$(\sigma_{\beta_2})/T$ $\beta=0$ $\alpha=\alpha_1$	0.11	0.36	0.66	0.99	1.34	1.72	2.12	2.55	3.02	3.53	4.10	4.74	5.45	6.25	7.16	8.17
$\sigma_{\max}/T$	3.47	3.63	3.86	4.14	4.46	4.80	5.18	5.60	6.05	6.56	7.12	7.75	8.46	9.26	10.16	11.17

Fig. 3

Fig. 4

Fig. 5

$$\begin{aligned}
(\sigma_{\beta_1})_{\beta=\pi} &= T[1+B_0 \sinh \alpha (\cosh \alpha + 1) - 2A_1 (\cosh \alpha + 1) (\cosh 2\alpha + \cosh \alpha) \\
&\quad + \sum_{n=2}^{\infty} 2A_n (-1)^n n (\cosh \alpha + 1) \{n \sinh \alpha \sinh n\alpha + (\cosh \alpha + 1) \cosh n\alpha\} \\
&\quad + \sum_{n=2}^{\infty} 2E_n (-1)^n \{n^3 (\cosh \alpha + 1) \sinh \alpha \cosh n\alpha + n^2 (\cosh \alpha + 1) \sinh n\alpha \\
&\quad - n (\cosh \alpha + 2) \sinh \alpha \cosh n\alpha - \sinh n\alpha\}] \\
(\sigma_{\beta_2})_{\beta=\pi} &= T[B(\alpha + \sinh \alpha \cosh \alpha + 2 \sinh \alpha) + D(\cosh \alpha + 1)^2 + D_1(\alpha + \sinh \alpha) \\
&\quad - 2A(\cosh \alpha + 1) (\cosh 2\alpha + \cosh \alpha) - 2C(\cosh \alpha + 1) (\sinh 2\alpha + \sinh \alpha)]
\end{aligned}$$

For the case $\alpha_1 = 2.0$, the values of $(\sigma_{\beta})_{\beta=\pi}$ given by the above equations are shown in Fig. 3. (The scale of $(\sigma_{\beta})_{\beta=\pi}$ is taken as 20 times $(\sigma_{\beta})_{\alpha=\alpha_1}$ or $(\sigma_{\beta})_{\alpha=0}$.) These are very small and equal to $-0.050T$ and $-0.062T$ at both ends, and $0.036T$ at the center of the edge of the slit, respectively.

Moreover, considering that the resultant force and couple are zero, the edges of the slit may be regarded as free boundaries. Of course, if the surface force is not negligible, from Saint-Venant's principle, the bottom of the circular hole which is

not in the neighbourhood of the slit is affected in the same manner as the slit having free boundaries, because there is no resultant force and couple.

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On the Steady-State Creep of Rotating Solid Disks*

By Masaru SAKATA**

Here is described an analytical method on the steady-state creep of a rotating disk with variable thickness and nonuniform temperature distribution. The numerical computation by this method is easy.

Creep tests of rotating solid disks of forged 18-8 Mo stainless steel were carried out in a hot spin tester prepared by the author and his associates. At the same time, specimens were cut out from the various parts of the disks and put to creep tests and other material tests.

The analytical results were compared with experimental ones. Mises criterion has shown better agreement with the test data.

1. Introduction

A number of studies on the creep analysis of rotating disks have been developed^{(1)~(4)} because the creep deformation and stress distribution in a turbine disk have practical importance for gas turbine designers. Most of the studies are on the steady-state creep deformations. It is apparent that accurate prediction of the stress or strain distribution is impossible without considering that the strains consist of the elastic and plastic components as well as the creep component and that the stress distribu-

tion is not always steady. However, the analyses of nonsteady-state creep are somewhat cumbersome and it is reported that the difference between the transient creep and the steady-state creep in rotating disks becomes negligible in a short time⁽⁵⁾. Therefore the analyses of the steady-state creep are of much importance for engineering practice.

Few experimental data have been published on the creep of rotating disks. Wahl and his collaborators made long time creep tests on chrome steel disks with a concentric hole⁽⁶⁾. Comparing the test results with the analyses, better coincidence was obtained using the Tresca criterion. Работнов and Рабинович reported on the test results obtained from disks of steels or heat resisting alloys⁽⁷⁾.

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