Bending of an Eccentric Circular Ring with a Cut-Out Portion Undergoing Forces and Bending Couples at Both Ends

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To solve the problem of bending action of piston rings or crane hooks, we considered an eccentric circular ring with a cut-out portion undergoing forces and bending couples at both ends.

Using the theory of elasticity for plane stress and applying bipolar coordinates, we obtained correct solutions for the above problem and compared the solutions with those by the usual elementary theory.

From the results of calculations, we reach the conclusion that the solution by the usual elementary theory is reliable, only when the eccentricity between the centers and the difference between the radii of two circles of the boundaries are small.

1. Introduction

Bending of an eccentric circular ring with a cut-out portion subjected to loads and bending couples at its both ends is similar to that of crane hooks or piston rings. The solution for the bending of a concentric circular ring with a cut-out portion undergoing bending couples at both ends is well-known as that for pure bending of curved bars(1), but the solution for the problem of bending of an eccentric circular ring has not been obtained.

The author has solved the problem for the bending of a curved bar with a profile surrounded by two intersecting circular arcs bent by concentrated forces and couples at both ends(2). But the profiles of crane hooks or piston rings are more similar to eccentric circular arcs than to two intersecting circular arcs. Then, to solve the bending action of crane hooks or piston rings, we consider an eccentric circular ring with the narrow rectangular cross section bent in the plane of curvature by forces W and couples M applied at the ends.

2. Stress function

Let us take curvilinear coordinates defined by the conjugate functions

\[ \alpha + i \beta = \log \frac{x + i(y + a)}{x + i(y - a)} \]

where \(x, y\) are Cartesian coordinates and \(a\) is a positive real length. Solving for \(x, y\) we have

\[ x = \frac{a \sin \beta}{\cosh \alpha - \cos \beta}, \quad y = \frac{a \sinh \alpha}{\cosh \alpha - \cos \beta} \]

\[ \cdots \cdots (1) \]

The general scheme of coordinates is shown in Fig. 1. The boundaries of the eccentric circular ring are defined by \(\alpha = \alpha_1\) for the outer edge, \(\alpha = \alpha_2\) for the inner edge, \(\beta = \pi\) and \(\beta = -\pi\) for both ends, respectively.

Let \(r_1\) and \(r_2\) denote the radii of the outer circle and the inner one, respectively, and let \(d\) and \(\varepsilon\) denote the center distance and the eccentricity, respectively. They are

![Fig. 1 An eccentric circular ring and its boundaries](image-url)
\[ r_1 = a \sinh \alpha_1, \quad r_2 = a \sinh \alpha_2, \quad e = \frac{d}{r_1 - r_2} = \frac{\sinh (\alpha_2 - \alpha_1)}{\sinh \alpha_2 - \sinh \alpha_1} \]

Then, knowing the eccentricity and the radii of the outer and inner circles, we can determine the values of \( \alpha_1, \alpha_2 \) and \( a \).

The stress function \( F \) must satisfy the equation
\[
\left( \frac{\partial^4}{\partial \alpha^4} + 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + 2 \frac{\partial^4}{\partial \alpha^4} - 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} - 2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + 1 \right) \left( \frac{F}{J} \right) = 0
\]
where
\[ J = a^2 (\cosh \alpha - \cos \beta) \]

Let \( \sigma_{\alpha} \), \( \sigma_{\beta} \) and \( \tau_{\alpha\beta} \) denote the normal stress component in the radial direction, one in the tangential direction and the shearing stress component, respectively. They are given by the following formulas:
\[
\begin{align*}
\sigma_{\alpha} &= \left( \cosh \alpha - \cos \beta \right) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \alpha \left( \frac{F}{J} \right) \\
\sigma_{\beta} &= \left( \cosh \alpha - \cos \beta \right) \frac{\partial^2}{\partial \alpha^2} - \sinh \alpha \frac{\partial}{\partial \alpha} - \sin \beta \frac{\partial}{\partial \beta} + \cosh \beta \left( \frac{F}{J} \right) \\
\tau_{\alpha\beta} &= \left( \cosh \alpha - \cos \beta \right) \frac{\partial^2}{\partial \alpha \partial \beta} \left( \frac{F}{J} \right)
\end{align*}
\]

Putting the stress function satisfying Eq. (2)
\[
\frac{F}{J} = (B_0 \cosh \alpha + D_0 \sinh \alpha + D \cos \beta) \alpha + (A \cosh 2\alpha + B + C \sinh 2\alpha) \cos \beta
\]
and substituting this in Eqs. (3), we have
\[
\begin{align*}
\sigma_{\alpha} &= B_0 (\alpha + \sinh \alpha \cosh \alpha - 2 \sinh \alpha \cos \beta) + D_0 (\cosh^2 \alpha + 1 - 2 \cosh \alpha \cos \beta) + D (\alpha - \sinh \alpha \cos \beta) \\
&+ A (4 \cosh 2\alpha \cos \beta (\cosh \alpha - \cos \beta) - 2 \sinh \alpha \sinh 2\alpha \cos \beta + \cosh 2\alpha) \\
&+ B + C (4 \sinh 2\alpha \cos \beta (\cosh \alpha - \cos \beta) - 2 \sinh \alpha \cosh 2\alpha \cos \beta + \sinh 2\alpha) \cos \beta
\end{align*}
\]

3. Boundary conditions

At the boundaries, the stress function must satisfy the following conditions.

(i) Since the outer edge of the eccentric circular ring is free from external forces,
\[
\left\{ \frac{\partial}{\partial \alpha} \left( \frac{F}{J} \right) \right\}_{a=\alpha_1} = \rho, \quad \left( \frac{F}{J} \right)_{a=\alpha_1} = \rho \tanh \alpha_1 + \sigma (\cosh \alpha_1 \cos \beta - 1) + \tau \sin \beta
\]
where \( \rho, \sigma \) and \( \tau \) are constants. Then we have
\[
\begin{align*}
B_0 (\alpha_1 \sinh \alpha_1 + \cosh \alpha_1 \alpha_1 + D_0 (\cosh \alpha_1 \cos \beta + \sinh \alpha_1) + (D + 2A) \sinh 2\alpha_1 + 2C \cosh 2\alpha_1) \cos \beta &= \rho \\
B_0 \cosh \alpha_1 + D_0 \sinh \alpha_1 + D \cos \beta \alpha_1 + (A \cosh 2\alpha_1 + B + C \sinh 2\alpha_1) \cos \beta &= \rho \tanh \alpha_1 + \sigma (\cosh \alpha_1 \cos \beta - 1) + \tau \sin \beta
\end{align*}
\]
from which we can obtain the following equations:
\[
\begin{align*}
B_0 (\alpha_2 - \sinh \alpha_1 \cosh \alpha_1) - D_0 \sinh^2 \alpha_1 + D (\cosh 2\alpha_1 + 1 + 2 \cosh \alpha_1) - \alpha_1 + B + C \sinh 2\alpha_1 &= 0 \quad (5) \\
D + 2A \sinh 2\alpha_1 + 2C \cosh 2\alpha_1 &= 0 \quad (6)
\end{align*}
\]

(ii) The inner edge of the ring is free from external forces, too, and so in the same manner we find
\[
\begin{align*}
B_0 (\alpha_2 - \sinh \alpha_1 \cosh \alpha_1) - D_0 \sinh^2 \alpha_2 + D (\cosh 2\alpha_2 + 1 + 2 \cosh \alpha_2) - \alpha_2 + B + C \sinh 2\alpha_2 &= 0 \quad (7) \\
D + 2A \sinh 2\alpha_2 + 2C \cosh 2\alpha_2 &= 0 \quad (8)
\end{align*}
\]

(iii) At the ends the resultant of normal stresses is equal to the load \(-W\). Then
\[
-W = \int_{\alpha_1}^{\alpha_2} \langle \delta \beta \rangle \rho d(y) \beta = \pi
\]
From Eq. (4) we have
\[
\begin{align*}
\sigma (\delta \beta)_{\alpha = \alpha_1} &= B_0 (\alpha_1 + \sinh \alpha \cosh \alpha + 2 \sinh \alpha) + D_0 (\cosh^2 \alpha + 1 + 2 \cosh \alpha) \\
&+ D (\alpha_1 + \sinh \alpha) + A (-4 \cosh 2\alpha (\cosh \alpha + 1) + 2 \sinh \alpha \sinh 2\alpha + \cosh 2\alpha) \\
&+ B + C (-4 \sinh 2\alpha (\cosh \alpha + 1) + 2 \sinh \alpha \cosh 2\alpha + \sinh 2\alpha)
\end{align*}
\]
and from Eq. (1) we find
\[
\begin{align*}
\langle y \rangle_{\beta = \pi} &= \frac{a \sinh \alpha}{\cosh \alpha + 1}, \\
\langle d(y) \rangle_{\beta = \pi} &= \frac{a}{\cosh \alpha + 1} \, d\alpha
\end{align*}
\]
Hence, finally,
\[ B_b \left( \cosh \alpha_2 - \cosh \alpha_1 + \alpha_2 \tanh \frac{\alpha_2}{2} - \alpha_1 \tanh \frac{\alpha_1}{2} \right) + D_b (\sinh \alpha_2 - \sinh \alpha_1 + \alpha_2 - \alpha_1) \\
+ D \left( \alpha_2 \tanh \frac{\alpha_2}{2} - \alpha_1 \tanh \frac{\alpha_1}{2} \right) - A \left( \sinh 2\alpha_2 - \sinh 2\alpha_1 + 2(\sinh \alpha_2 - \sinh \alpha_1) - \left( \tanh \frac{\alpha_2}{2} - \tanh \frac{\alpha_1}{2} \right) \right) \\
+ B \left( \tanh \frac{\alpha_2}{2} - \tanh \frac{\alpha_1}{2} \right) - C (\cosh 2\alpha_2 - \cosh 2\alpha_1 + 2(\cosh \alpha_2 - \cosh \alpha_1)) = -W \tag{9} \]

(iv) The other condition at the ends is
\[ -(M + Wy_b) = \int_{\alpha_1}^{\alpha_2} (\sigma_\phi)_{\phi=\alpha} (y dy)_{\phi=\alpha} \]

where
\[ y_b = \frac{a}{2} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \]

The above equation states that there are loads \( W \) and bending couples \( M \) applied at the ends of the ring.

In the same way we obtain
\[ B_b \left[ \sinh \alpha_2 - \sinh \alpha_1 - \left( \tanh \frac{\alpha_2}{2} - \tanh \frac{\alpha_1}{2} \right) - \left( \frac{\alpha_2}{\cosh \alpha_2 + 1} - \frac{\alpha_1}{\cosh \alpha_1 + 1} \right) \right] \\
+ D_b (\cosh \alpha_2 - \cosh \alpha_1) + D \left[ - \left( \tanh \frac{\alpha_2}{2} - \tanh \frac{\alpha_1}{2} \right) + \alpha_2 - \alpha_1 - \left( \frac{\alpha_2}{\cosh \alpha_2 + 1} - \frac{\alpha_1}{\cosh \alpha_1 + 1} \right) \right] \\
- A \left( \cosh 2\alpha_2 - \cosh 2\alpha_1 - 2(\cosh \alpha_2 - \cosh \alpha_1) + \frac{1}{\cosh \alpha_2 + 1} - \frac{1}{\cosh \alpha_1 + 1} \right) \\
- B \left( \frac{1}{\cosh \alpha_2 + 1} - \frac{1}{\cosh \alpha_1 + 1} \right) - C (\sinh 2\alpha_2 - \sinh 2\alpha_1 - 2(\sinh \alpha_2 - \sinh \alpha_1)) \]

\[ = -\left( \frac{M}{a} + \frac{W}{2} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \right) \tag{10} \]

All the conditions of Eqs. (5), (6), (7), (8), (9) and (10) can be satisfied by taking the constants \( B_b, D_b, D, A, B, C \) as follows:
\[ B_b = - \frac{M}{a} (k_1 \sinh^2 \alpha_m + k_2 \cosh^2 \alpha_m) + \frac{W}{2} (k_1 \sinh \alpha_m (\cosh \alpha_2/2 + \cosh \alpha_1/2) \\
+ k_2 \cosh \alpha_m (\sinh \alpha_2/2 + \sinh \alpha_1/2)) \]
\[ D_b = \frac{M}{a} (k_1 + k_2) \sinh (\alpha_1 + \alpha_2) - \frac{W}{2} (k_1 \cosh \alpha_m (\cosh \alpha_2/2 + \cosh \alpha_1/2) \\
+ k_2 \sinh \alpha_m (\sinh \alpha_2/2 + \sinh \alpha_1/2)) \]
\[ D = -k \cosh 2\alpha_0 \left( \frac{M}{a} + \frac{W}{2} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \right) \]
\[ A = -\frac{k}{2} \sinh 2\alpha_0 \left( \frac{M}{a} + \frac{W}{2} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \right) \]
\[ C = -\frac{k}{2} \cosh 2\alpha_0 \left( \frac{M}{a} + \frac{W}{2} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \right) \]
\[ B = -\left( \frac{M}{a} \right)^2 \left[ k_2 \cosh^2 \alpha_0 - k_1 \sinh^2 \alpha_0 \right] + \alpha_m \left[ k \cosh 2\alpha_0 + k_1 \sinh^2 \alpha_m + k_2 \cosh^2 \alpha_m \right] \]
\[ + \frac{W}{2} (k_2 \cosh^2 \alpha_0 \sinh \alpha_m (\sinh \alpha_2/2 + \sinh \alpha_1/2) - k_1 \sinh^2 \alpha_0 \cosh \alpha_m (\cosh \alpha_2/2 + \cosh \alpha_1/2) \\
- \alpha_m (k_1 \cosh \alpha_m (\cosh \alpha_2/2 + \cosh \alpha_1/2)) + k_2 \sinh \alpha_m (\cosh \alpha_2/2 + \cosh \alpha_1/2) \]
\[ - k \cosh 2\alpha_0 \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \]

where
\[ \alpha_0 = \frac{1}{2} (\alpha_2 - \alpha_1), \quad \alpha_m = \frac{1}{2} (\alpha_1 + \alpha_2), \quad k_1 = \frac{1}{\sinh 2\alpha_0 + 2\alpha_0}, \quad k = \frac{1}{2 \alpha_0 \cosh 2\alpha_0 - \sinh 2\alpha_0} \]}
4. Stress distributions

Substituting the values of the constants in Eqs. (3), we can obtain the stress distributions in the ring. The bending stress in the central cross section given by Eq. (4) is

$$
\langle \sigma_\theta \rangle_{\theta=0} = k_1 \{ -\frac{M}{a^2} \sinh \alpha_m + \frac{W}{2a} \left( \frac{\cosh \alpha_m/2}{\cosh \alpha_m/2} + \frac{\cosh \alpha_i/2}{\cosh \alpha_i/2} \right) \} \sinh \alpha_m \]

$$

$$
- \cosh (\alpha-\alpha_m)(\sinh \alpha-2) - \sinh^2 \alpha_m \cosh \alpha_m + k_2 \left( -\frac{M}{a^2} \cosh \alpha_m + \frac{W}{2a} \left( \frac{\cosh \alpha_i/2}{\cosh \alpha_i/2} \right) \right)
$$

$$
+ \sinh \alpha_i/2 \left[ (\alpha-\alpha_m) \cosh \alpha_m + \sinh (\alpha-\alpha_m)(\sinh \alpha-2) + \sinh^2 \alpha_m \sinh \alpha_m \right]
$$

$$
- k \left( \frac{M}{a^2} + \frac{W}{2a} \left( \tanh \frac{\alpha_1}{2} + \tanh \frac{\alpha_2}{2} \right) \right) \] \left[ (\alpha-\alpha_m) \cosh 2\alpha_m - \sinh 2(\alpha-\alpha_m) \cosh \alpha_m - \frac{3}{2} \right]
$$

$$
- \left( \cosh 2\alpha_m - \cosh 2(\alpha-\alpha_m) \right) \sinh \alpha_m \]
$$

Putting \( \alpha = \alpha_1 \) and \( \alpha = \alpha_2 \), we find the normal stress at the outer edge and one at the inner edge in the central cross section. They are, respectively,

$$
\sigma_1 = \langle \sigma_\theta \rangle_{\theta=0} = \frac{M}{a^2} 2 \sinh^3 \frac{\alpha_1}{2} \left\{ k_2 (\sinh \alpha_2 - \sinh \alpha_1) + k_1 (\sinh \alpha_1 + \sinh \alpha_2) - k-2 \sinh 2\alpha_m \right\}
$$

$$
- \frac{W}{a} 2 \sinh^2 \frac{\alpha_1}{2} \left\{ k_2 \sinh \alpha_3 \left( \frac{\sinh \alpha_1/2}{\cosh \alpha_1/2} + \frac{\sinh \alpha_3/2}{\cosh \alpha_3/2} \right) + k_1 \cosh \alpha_0 \left( \frac{\cosh \alpha_1/2}{\cosh \alpha_1/2} + \frac{\cosh \alpha_2/2}{\cosh \alpha_2/2} \right) + k \sinh 2\alpha_m \tanh \frac{\alpha_1}{2} \tanh \frac{\alpha_3}{2} \right\} \]
$$

$$
\sigma_2 = \langle \sigma_\theta \rangle_{\theta=0} = -\frac{M}{a^2} 2 \sinh^3 \frac{\alpha_2}{2} \left\{ k_2 (\sinh \alpha_2 - \sinh \alpha_1) - k_1 (\sinh \alpha_1 + \sinh \alpha_2) - k-2 \sinh 2\alpha_m \right\}
$$

$$
+ \frac{W}{a} 2 \sinh^2 \frac{\alpha_2}{2} \left\{ k_2 \sinh \alpha_3 \left( \frac{\sinh \alpha_1/2}{\cosh \alpha_1/2} + \frac{\sinh \alpha_3/2}{\cosh \alpha_3/2} \right) - k_1 \cosh \alpha_0 \left( \frac{\cosh \alpha_1/2}{\cosh \alpha_1/2} + \frac{\cosh \alpha_2/2}{\cosh \alpha_2/2} \right) + k \sinh 2\alpha_m \tanh \frac{\alpha_1}{2} \tanh \frac{\alpha_3}{2} \right\} \]
$$

Stress distributions in a concentric circular ring can be obtained by putting the center distance \( d=0 \), i.e., making the magnitudes of \( \alpha_1 \) and \( \alpha_2 \) large enough. Using the fact that \( \sinh \alpha_1 = a/r_1 \) and \( \sinh \alpha_2 = a/r_2 \), we find

$$
\varepsilon_{x_1} = 2 \frac{a}{r_1}, \quad \varepsilon_{x_2} = 2 \frac{a}{r_2}, \quad 2\alpha_m = \log \left( \frac{r_1}{r_2} \right)
$$

Substituting these into Eqs. (11), we have the stress distributions in a concentric circular ring with a cut-out portion bent by loads and couples at its both ends (see Fig. 2):

$$
\langle \sigma_1 \rangle_{x=0} = \frac{2M}{r_1} \left\{ \frac{r_1-r_2}{(r_1^2-r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} + \frac{r_1+r_2}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \right\}
$$

$$
- \frac{W}{r_1} \left\{ \frac{r_1-r_2}{(r_1^2-r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} + \frac{r_1+r_2}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \right\}
$$

$$
+ \frac{2(r_1^2+r_2^2) \log \left( \frac{r_1}{r_2} \right)}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \]
$$

$$
\langle \sigma_2 \rangle_{x=0} = -\frac{2M}{r_2} \left\{ \frac{r_1-r_2}{(r_1^2-r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} - \frac{r_1+r_2}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \right\}
$$

$$
+ \frac{W}{r_2} \left\{ \frac{r_1-r_2}{(r_1^2-r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} - \frac{r_1+r_2}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \right\}
$$

$$
+ \frac{2(r_1^2+r_2^2) \log \left( \frac{r_1}{r_2} \right)}{(r_1^2+r_2^2)2r_1r_2 \log \left( \frac{r_1}{r_2} \right)} \]
$$

In these expressions, the first terms coincide with the values usually derived in the problem of pure bending of a circular ring (11).

When the center distance \( d \) is maximum, i.e., \( e=1 \), stress distributions are obtained by bringing \( \alpha_1 \) and \( \alpha_2 \) close to zero. By substituting \( \sinh \alpha_1 = a/r_1 \) and \( \sinh \alpha_2 = a/r_2 \) in Eqs. (11) and making \( a \) infinitesimal, we find
\[
\begin{align*}
\sigma_1 &= \frac{M}{(r_1 - r_2)^3} \left( \frac{2}{5} + \frac{6}{5} \frac{r_2}{r_1} - \frac{1}{10} \frac{r_2^2}{r_1^2} \right) - \frac{W r_2}{(r_1 - r_2)^3} \left( \frac{2 + r_2}{r_1} \right) \\
\sigma_2 &= \frac{M}{(r_1 - r_2)^2} \left( \frac{2}{5} + \frac{6}{5} \frac{r_2}{r_1} - \frac{1}{10} \frac{r_2^2}{r_1^2} \right) + \frac{W r_1}{(r_1 - r_2)^2} \left( \frac{2 + r_1}{r_2} \right)
\end{align*}
\]

These formulas coincide with the expressions obtained in the problem for a curved beam with a profile surrounded by two intersecting circular arcs bent by concentrated forces and couples at both ends if the distance between its both ends is zero \(^{13}\) (see Fig. 3).

5. Deflections

The radial component of displacement of any point in the ring \(u_\alpha\) and the tangential one \(u_\beta\) are given as follows, respectively:

\[
\begin{align*}
2G u_\alpha &= \frac{m - 1}{m + 1} \frac{1}{J} \frac{\partial F}{\partial \alpha} - \frac{m + 1}{m + 1} \frac{1}{J} \frac{\partial Q}{\partial \beta}, \\
2G u_\beta &= \frac{m - 1}{m + 1} \frac{1}{J} \frac{\partial F}{\partial \beta} + \frac{m + 1}{m + 1} \frac{1}{J} \frac{\partial Q}{\partial \alpha}
\end{align*}
\]

in which

\[
\begin{align*}
\frac{Q}{J} &= \int \left( \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} - 1 \right) \left( \frac{F}{J} \right) d\alpha d\beta, \\
\left( \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} - 1 \right) \left( \frac{Q}{J} \right) + 4 \frac{\partial^2}{\partial \alpha \partial \beta} \left( \frac{F}{J} \right) &= 0
\end{align*}
\]

Substituting the value of \(F/J\) in the above equations, we have

\[
\frac{Q}{J} = \left( B_0 \cosh \alpha + D_0 \sinh \alpha + D \cos \beta \right) 2\beta + \left( A \sinh 2\alpha + C \cosh 2\alpha \right) 2 \sin \beta
\]

Thus, we find

\[
\begin{align*}
2G u_\alpha &= \frac{m - 1}{m + 1} \frac{1}{(\cosh \alpha - \cos \beta)} \left[ B_0 (\cosh^4 \alpha - (\cosh \alpha + \sinh \alpha \sin \beta) \cos \beta) + D_0 (\alpha + \sinh \alpha \cosh \alpha \\
&- (\sinh \alpha + \cosh \alpha \cos \beta) \cos \beta) + D (\cosh \alpha - \alpha \sinh \alpha \cos \beta) \cosh \beta \\
&- B \sinh \alpha \cosh \beta + A (\sinh 2\alpha \cosh \alpha + \sinh \alpha - 2 \sinh 2\alpha \cos \beta) \cosh \beta \\
&+ C (\cosh 2\alpha \cosh \alpha + \cosh \alpha - 2 \cosh 2\alpha \cos \beta) \cos \beta \right]
\end{align*}
\]

\[
\begin{align*}
2G u_\beta &= -\frac{m - 1}{m + 1} \frac{1}{(\cosh \alpha - \cos \beta)} \left[ B_0 \cosh \alpha (\cosh \alpha - \cos \beta - \sin \beta) \\
&+ D_0 \sinh \alpha (\sinh \alpha - \cosh \beta - \sin \beta) + D (\cosh \alpha - \cosh \beta - \cosh \alpha \sin \beta) \\
&+ A \sin 2\alpha (\cosh \alpha \cos \beta - 1) + C \cosh 2\alpha (\cosh \alpha \cos \beta - 1) \right]
\end{align*}
\]

Let \(2\delta\) denote the magnitude of opening at the ends by the loads and the bending couples. Then

\[
2\delta = -2(u_\beta)_{\beta=\alpha} = \frac{2m \pi}{G(m + 1)} \left( B_0 \tanh \frac{\alpha}{2} + D_0 + D \tan \frac{\alpha}{2} \right) = \frac{4\pi}{E} \left( B_0 \tanh \frac{\alpha}{2} + D_0 + D \tan \frac{\alpha}{2} \right)
\]

from which we have the opening at the outer edge and the inner edge in the cut-out cross section. They are, respectively,
\[ 2\delta_1 = 2(\delta)_{a=a_1} = -\frac{M}{a} \frac{4\pi}{E} \left[ k_2 \cosh \alpha_m \sinh \alpha_1/2 \frac{\cosh \alpha_1/2}{\cosh \alpha_1} + k_1 \sinh \alpha_n \frac{\cosh \alpha_1/2}{\cosh \alpha_1} - k \cosh 2\alpha_0 \tanh \frac{\alpha_1}{2} + W \frac{2\pi}{E} \left\{ k_2 \frac{\sinh \alpha_2/2}{\cosh \alpha_2} + \frac{\sinh \alpha_2/2}{\cosh \alpha_1/2} \frac{\sinh \alpha_2/2}{\cosh \alpha_2} \right\} + k \left( \frac{\sinh \alpha_1}{2} + \frac{\sinh \alpha_2}{2} \right) \cosh 2\alpha_0 \tanh \frac{\alpha_1}{2} \right] \]

\[ 2\delta_2 = 2(\delta)_{a=a_2} = -\frac{M}{a} \frac{4\pi}{E} \left[ k_2 \cosh \alpha_m \sinh \alpha_1/2 \frac{\cosh \alpha_1/2}{\cosh \alpha_1} + k_1 \sinh \alpha_n \frac{\cosh \alpha_1/2}{\cosh \alpha_1} - k \cosh 2\alpha_0 \tanh \frac{\alpha_2}{2} + W \frac{2\pi}{E} \left\{ k_2 \frac{\sinh \alpha_2/2}{\cosh \alpha_2} + \frac{\sinh \alpha_2/2}{\cosh \alpha_1/2} \frac{\sinh \alpha_2/2}{\cosh \alpha_2} \right\} + k \left( \frac{\sinh \alpha_1}{2} + \frac{\sinh \alpha_2}{2} \right) \cosh 2\alpha_0 \tanh \frac{\alpha_2}{2} \right] \]

By making \( \alpha_1 \) and \( \alpha_2 \) large enough, we obtain the openings in the case of a concentric circular ring:

\[ 2(\delta)_{a=a} = -\frac{8\pi r_1 M}{E} \left\{ \frac{1}{r_1^2+r_2^2-2r_1r_2 \log(r_1/r_2)} + \frac{1}{r_1^2-r_2^2+2r_1r_2 \log(r_1/r_2)} \right\} \]

\[ 2(\delta)_{a=a} = -\frac{8\pi r_2 M}{E} \left\{ \frac{1}{r_2^2-r_1^2+2r_2r_1 \log(r_1/r_2)} + \frac{1}{r_2^2+r_1^2-2r_2r_1 \log(r_1/r_2)} \right\} \]

6. Numerical calculations for stresses

Let us calculate the values of stresses at the outer edge and the inner edge in the central cross section. Taking as a standard the stress of a straight beam which is bent by bending moment of the same magnitude as our curved beam, we will express all such magnitudes of the stresses obtained by our correct solutions as follows:

\[ \sigma_{1,2} = \mu_w \frac{M}{h^2/6} + \mu_w \frac{W}{h^2/6} \]

where \( h \) is the width of the central cross section (see Fig. 1) and given by

\[ h = a \left( \coth \frac{\alpha_1}{2} - \coth \frac{\alpha_2}{2} \right) \]

We can determine \( \alpha_1 \) and \( \alpha_2 \) from

\[ \frac{r_2}{r_1} = \sinh \alpha_1, \quad \frac{\sinh \alpha_2}{\sinh \alpha_1} = \frac{\sinh (\alpha_2-\alpha_1)}{\sinh \alpha_2} \]

and obtain \( \sigma_1 \) and \( \sigma_2 \) from Eqs. (11) [Eqs. (12) for \( e=0 \) and Eqs. (13) for \( e=1 \)]. By substituting these in Eq. (16), we find the magnitudes of the coefficients \( \mu_w \) and \( \mu_{1,2} \) which are shown in Fig. 4 and Fig. 5, respectively.

In accordance with the results, we arrive at the following conclusion that as far as \( e \) is small and \( r_2/r_1 \) is close to 1, i.e., the center distance between the two circles and the difference between their radii are small, the usual elementary theory is satisfactory, while it becomes unreliable gradually with an increase of the center distance and the difference.

--- at the outer edge  --- at the inner edge

Fig. 4 Stresses by \( M \)
7. Numerical calculations for deflections

Taking as a standard the magnitude of deflection at the end of a concentric circular ring undergoing loads $W$ and bending couples $M$ obtained by the usual elementary theory, we will express the deflections $\delta_1$ and $\delta_2$ as follows:

$$
\delta_{1,2} = C_\alpha \frac{3\pi (r_1 + r_2)^2 M}{E(r_1 - r_2)^3} + C_w \frac{9\pi (r_1 + r_2)^3 W}{4E(r_1 - r_2)^3} \cdots (17)
$$

Determining the magnitudes of $\alpha_1$ and $\alpha_2$ from the values of $r_2/r_1$ and $e$, we have $\delta_1$ and $\delta_2$ from Eqs. (14) [Eqs. (15) for $e = 0$]. By substituting these in Eq. (17), we obtain the magnitudes of the coefficients $C_\alpha$ and $C_w$ which are shown in Fig. 6 and Fig. 7, respectively. On the other hand, when $e$ approaches to 1, $\delta_1$ and $\delta_2$ become theoretically nearer to $\infty$, because the distributed loads and couples on the cut-out cross sections become concentrated forces and couples at the ends.

References

