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Thermal Stress of an Infinite Elastic Region in Uniform Heat Flow and Containing Some Spherical Cavities*

(Part 1, Solution for the Case of Containing Three Spherical Cavities)

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A method of solution to determining the thermal stress of an infinite elastic region under the conditions of linear thermoelasticity and containing some spherical cavities is illustrated by two examples for the problems containing two and three spherical cavities.

The fundamental equations governing the temperature distribution of a uniform heat flow in a solid are first derived and then the elastic stresses in the solid bodies under prescribed temperature distributions are determined to satisfy the boundary conditions on the peripheries of the cavities.

The method consists of a general series form of stress function in spherical coordinates, and it can be developed for a variety of cavity numbers and sizes.

In this paper, the illustration for three cavities is given, and the numerical results are reported.

1. Introduction

In recent years we have seen a rising interest in thermoelasticity, stimulated by various engineering sciences. The study of the fundamental linear thermoelastic problems is one example of such interest being taken by many investigators. Recently Florence and Goodier have studied some problems of a steady heat flow and presented solutions for the cases of a circular hole⁽¹⁾, a spherical cavity⁽¹⁾, an ovaloid hole⁽²⁾, a spheroidal cavity⁽³⁾, etc.⁽⁴⁾

In the present paper, a similar problem of the thermal stress induced by the disturbances of a steady heat flow in an infinite elastic region containing some cavities is solved for particular cases of two and three cavities, with the numerical results obtained and considerations made.

The method of the solution consists in using a potential of thermoelastic displacement and a biharmonic stress function expressed by general spherical coordinates. The solution for the case of three cavities is treated first in Part 1 and that of two cavities will be reported later in Part 2.

2. Solution for three spherical cavities

2.1 Temperature field A steady heat flow

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with uniform temperature gradient at infinity passes three spherical cavities of equal radii of a being situated on a line with equal distances of b . Denote the cartesian and spherical coordinates of points O, O_1 by $(X, Y, Z), (X_1, Y_1, Z_1)$ and $(br, \theta, \phi), (br_1, \theta_1, \phi_1)$ respectively as shown in Fig. 1. Regarding, for convenience, the coordinates as dimensionless quantities referring to a typical length b , they are related with each other by

Fig. 1 Geometry of problem

$$\left. \begin{aligned} X/b &= x, \quad Y/b = y, \quad Z/b = z, \\ X_1/b &= x_1, \quad Y_1/b = y_1, \quad Z_1/b = z_1, \\ x &= r \sin \theta \cos \phi = x_1 = r_1 \sin \theta_1 \cos \phi_1 \\ y &= r \sin \theta \sin \phi = y_1 = r_1 \sin \theta_1 \sin \phi_1 \\ z &= r \cos \theta = z_1 - 1 = r_1 \cos \theta_1 - 1 \end{aligned} \right\} \dots\dots\dots (1)$$

Since the undisturbed temperature field may be taken as τz , the temperature in the presence of the cavities is expressed as

$$T = \tau b z + T_1 \dots\dots\dots (2)$$

where T must be an odd function of z and satisfy the equation

$$V^2 T = 0 \dots\dots\dots (3)$$

Then T_1 in Eq. (2) must also be an odd function of z and satisfy the following conditions

$$V^2 T_1 = 0, \quad \frac{\partial T_1}{\partial z} \rightarrow 0 \text{ as } z \rightarrow \infty \dots\dots\dots (4)$$

Note that V^2 denotes the Laplacian operator and in spherical coordinates

$$V^2 = \frac{1}{b^2} \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial}{\partial \mu} \right] \right\} = 0 \dots\dots\dots (5)$$

This function T_1 can consist of the following functions:

$$\begin{aligned} U_{2s} &= \frac{-1}{(2s-1)!} \frac{\partial^{2s}}{\partial z^{2s}} \frac{1}{2} \left\{ {}^0C_{2s} \log \frac{1+\mu}{1-\mu} + {}^1C_{2s} \left[\log \frac{\sqrt{r^2+2r\mu+1}+(r\mu+1)}{\sqrt{r^2+2r\mu+1}-(r\mu+1)} \right. \right. \\ &\quad \left. \left. + \log \frac{\sqrt{r^2-2r\mu+1}+(r\mu-1)}{\sqrt{r^2-2r\mu+1}-(r\mu-1)} - 4r\mu \right] \right\} \\ &= {}^0C_{2s} \frac{P_{2s-1}(\mu)}{r^{2s}} - 2^1C_{2s} \sum_{n=0}^{\infty} \left(\frac{2n+2s}{2n+1} \right) r^{2n+1} P_{2n+1}(\mu), \quad s \geq 1 \dots\dots\dots (6) \end{aligned}$$

$$\begin{aligned} V_{2s+1} &= \frac{1}{(2s)!} \frac{\partial^{2s+1}}{\partial z^{2s+1}} \frac{1}{2} {}^1C_{2s+1} \left[\log \frac{\sqrt{r^2+2r\mu+1}+(r\mu+1)}{\sqrt{r^2+2r\mu+1}-(r\mu+1)} - \log \frac{\sqrt{r^2-2r\mu+1}+(r\mu-1)}{\sqrt{r^2-2r\mu+1}-(r\mu-1)} \right] \\ &= -2^1C_{2s+1} \sum_{n=0}^{\infty} \left(\frac{2n+2s+1}{2n+1} \right) r^{2n+1} P_{2n+1}(\mu), \quad s \geq 1 \dots\dots\dots (7) \end{aligned}$$

Using these functions T is written as

$$T = \tau b r \cos \theta + \tau a \lambda^2 \sum_{s=1}^{\infty} \left[{}^0C_{2s} \frac{P_{2s-1}(\mu)}{r^{2s}} - {}^1C_{s+1} \sum_{n=0}^{\infty} 2 \left(\frac{2n+s+1}{2n+1} \right) r^{2n+1} P_{2n+1}(\mu) \right] \dots\dots\dots (8)$$

Expressing the function in terms of the spherical coordinates (r_1, θ_1, ϕ_1) , Eq. (8) takes the form

$$T = \tau b (r_1 \cos \theta_1 - 1) + \tau a \lambda^2 \sum_{s=1}^{\infty} \left\{ {}^1C_{s+1} \frac{P_s(\mu_1)}{r_1^{s+1}} - \sum_{n=0}^{\infty} \left[{}^1C_{s+1} \binom{n+s}{s} / 2^{n+s+1} + {}^0C_{2s} \binom{2s+n-1}{2s-1} \right] r_1^n P_n(\mu_1) \right\} \dots\dots\dots (9)$$

In these equations, $\mu = \cos \theta$, $\mu_1 = \cos \theta_1$.

If the cavities are insulated we must have

$$\frac{\partial T}{\partial r} \Big|_{r=\lambda} = \frac{\partial T}{\partial r_1} \Big|_{r_1=\lambda} = 0, \quad (\lambda = a/b) \dots\dots\dots (10)$$

The coefficients ${}^0C_{2n}$, ${}^1C_{n+1}$ in the function T are found such as to satisfy the condition (10). Then a formal solution by successive approximations is given as follows:

Write, for $n \geq 1$,

$${}^0C_{2n} = \sum_{r=0}^{\infty} {}^0C_{2n}^{(r)}, \quad {}^1C_{n+1} = \sum_{r=0}^{\infty} {}^1C_{n+1}^{(r)} \dots\dots\dots (11)$$

where

$$\left. \begin{aligned} {}^0C_2^{(0)} &= \frac{1}{2}, \quad {}^0C_4^{(0)} = {}^0C_6^{(0)} = \dots\dots = 0, \quad {}^1C_2^{(0)} = \frac{1}{2}, \quad {}^1C_3^{(0)} = {}^1C_4^{(0)} = \dots\dots = 0 \\ {}^0C_{2n}^{(r)} &= - \sum_{s=1}^{\infty} {}^1C_{s+1}^{(r-1)} \frac{(2n-1)}{n} \binom{2n+s-1}{2n-1} \lambda^{4n-1} \\ {}^1C_{n+1}^{(r)} &= - \sum_{s=1}^{\infty} \left[{}^1C_{s+1}^{(r-1)} \binom{n+s}{s} / 2^{n+s+1} + {}^0C_{2s}^{(r-1)} \binom{2s+n-1}{2s-1} \right] \frac{n}{n+1} \lambda^{2n+1} \end{aligned} \right\} \dots\dots\dots (12)$$

2.2 Potential of thermoelastic displacement To find the state of stress, we shall use the potential of thermoelastic displacement F . The function F is related to displacements by the equations

$$v = \partial F(r, \theta) / b \partial r, \quad w = \partial F(r, \theta) / b r \partial \theta \dots\dots\dots (13)$$

Then F is determined by

$$V^2 F = \frac{1+\nu}{1-\nu} \alpha T_1 \dots (14)$$

where ν is a Poisson's ratio and α is the coefficient of thermal expansion.

The solution is given by the function

$$F(r, \theta) = -\frac{1+\nu}{1-\nu} \alpha \tau a b^2 \lambda^2 \sum_{s=1}^{\infty} \left\{ {}^0C_{2s} \frac{P_{2s-1}(\mu)}{(8s-6)r^{2s-2}} + {}^1C_{s+1} \sum_{n=0}^{\infty} \left[\binom{2n+s}{s} \frac{4n-8ns-2s+2}{(4s-2)(4n+1)(2n+1)} \right. \right. \\ \left. \left. \times r^{2n+1} P_{2n+1}(\mu) + \frac{1}{4n+5} \binom{2n+s+1}{s} r^{2n+3} P_{2n+1}(\mu) \right] \right\} \dots (15)$$

Expressing the function in terms of the spherical coordinates (r_1, θ_1, ϕ_1) , we have

$$F(r_1, \theta_1) = -\frac{1+\nu}{1-\nu} \alpha \tau a b^2 \lambda^2 \sum_{s=1}^{\infty} \left\{ {}^1C_{s+1} \frac{P_s(\mu_1)}{(4s-2)r_1^{s-1}} + {}^0C_{2s} \left[\sum_{n=0}^{\infty} \binom{n+2s-1}{2s-1} \frac{1}{4n+6} r_1^{n+2} P_n(\mu_1) \right. \right. \\ \left. \left. - \sum_{n=1}^{\infty} \binom{n+2s-2}{n-1} \frac{4ns-3n-2s+1}{n(2n-1)(8s-6)} r_1^n P_n(\mu_1) \right] + {}^1C_{s+1} \left[\sum_{n=0}^{\infty} \binom{n+s}{n} \frac{1}{(4n+6)2^{n+s+1}} r_1^{n+2} P_n(\mu_1) \right. \right. \\ \left. \left. - 4 \sum_{n=1}^{\infty} \binom{n+s-1}{n-1} \frac{2ns-n-s}{n(2n-1)(4s-2)2^{n+s+1}} r_1^n P_n(\mu_1) \right] \right\} \dots (16)$$

Having the function $F(r, \theta)$ we can calculate the stresses with the relations

$$\left. \begin{aligned} \sigma_r^{(1)} &= \frac{E}{b^2(1+\nu)} \frac{\partial^2 F}{\partial r^2} - \frac{E\alpha}{1-\nu} T_1, & \sigma_\theta^{(1)} &= \frac{E}{b^2(1+\nu)} \left(\frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2} \right) - \frac{E\alpha}{1-\nu} T_1 \\ \sigma_\phi^{(1)} &= \frac{E}{b^2(1+\nu)} \left(\frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \operatorname{ctg} \theta \frac{\partial F}{\partial \theta} \right) - \frac{E\alpha}{1-\nu} T_1, & \tau_{r\theta}^{(1)} &= \frac{E}{b^2(1+\nu)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F}{\partial \theta} \right) \end{aligned} \right\} \dots (17)$$

and the stress system corresponding to the coordinates (r_1, θ, ϕ_1) can be given by similar relations.

2.3 Stress function Since the function F does not satisfy the boundary conditions on the spherical surfaces where tractions should vanish, we complete the states $\sigma_r^{(1)}$, $\sigma_{r_1}^{(1)}$ and $\tau_{r\theta}^{(1)}$, $\tau_{r_1\theta_1}^{(1)}$ by the stresses $\sigma_r^{(2)}$, $\sigma_{r_1}^{(2)}$ and $\tau_{r\theta}^{(2)}$, $\tau_{r_1\theta_1}^{(2)}$ respectively. The stress function is chosen such that

$$|\sigma_r^{(1)} + \sigma_r^{(2)}|_{r=\lambda} = 0, \quad |\tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)}|_{r=\lambda} = 0 \dots (18)$$

$$|\sigma_{r_1}^{(1)} + \sigma_{r_1}^{(2)}|_{r_1=\lambda} = 0, \quad |\tau_{r_1\theta_1}^{(1)} + \tau_{r_1\theta_1}^{(2)}|_{r_1=\lambda} = 0 \dots (19)$$

where $\sigma_r^{(2)}$, $\tau_{r\theta}^{(2)}$, $\sigma_{r_1}^{(2)}$ and $\tau_{r_1\theta_1}^{(2)}$ are calculated by the stress function which has singularities at the origins of the cavities. The function can be constructed from the following functions:

$$P_{2s+1} = \frac{1}{(2s)!} \frac{\partial^{2s+1}}{\partial z^{2s+1}} \frac{1}{2} \left[{}^0A_{2s+1} \log \frac{1+\mu}{1-\mu} + {}^1A_{2s+1} \left\{ \log \frac{\sqrt{r^2+2r\mu+1} + (r\mu+1)}{\sqrt{r^2+2r\mu+1} - (r\mu+1)} \right. \right. \\ \left. \left. + \log \frac{\sqrt{r^2-2r\mu+1} + (r\mu-1)}{\sqrt{r^2-2r\mu+1} - (r\mu-1)} - 4r\mu \right\} \right] \\ = {}^0A_{2s+1} \frac{P_{2s}(\mu)}{r^{2s+1}} + 2^1A_{2s+1} \sum_{n=0}^{\infty} \binom{2n+2s}{2n} r^{2n} P_{2n}(\mu), \quad (s \geq 0) \dots (20)$$

$$\left. \begin{aligned} Q_{2s} &= -\frac{1}{(2s-1)!} \frac{\partial^{2s}}{\partial z^{2s}} \frac{1}{2} {}^1A_{2s} \left[\log \frac{\sqrt{r^2+2r\mu+1} + (r\mu+1)}{\sqrt{r^2+2r\mu+1} - (r\mu+1)} - \log \frac{\sqrt{r^2-2r\mu+1} + (r\mu-1)}{\sqrt{r^2-2r\mu+1} - (r\mu-1)} \right] \\ &= 2^1A_{2s} \sum_{n=0}^{\infty} \binom{2n+2s-1}{2n} r^{2n} P_{2n}(\mu), \quad s > 0 \\ Q_0 &= -{}^1A_0 \sum_{n=1}^{\infty} \frac{1}{n} r^{2n} P_{2n}(\mu) \end{aligned} \right\} \dots (21)$$

$$R_{2s+1} = \frac{-(4s+3)}{(2s+2)!} {}^0B_{2s+1} \frac{\partial^{2s+1}}{\partial z^{2s+1}} \mu = {}^0B_{2s+1} \frac{1}{r^{2s+1}} \{P_{2s+2}(\mu) - P_{2s}(\mu)\} \dots (22)$$

$$S_{2s+1} = \frac{-(4s+3)}{(2s+2)!} {}^1B_{2s+1} \frac{\partial^{2s+1}}{\partial z^{2s+1}} \left\{ \frac{r\mu+1}{\sqrt{r^2+2r\mu+1}} + \frac{r\mu-1}{\sqrt{r^2-2r\mu+1}} \right\} \\ = 2^1B_{2s+1} \sum_{n=1}^{\infty} \frac{(4s+3)}{4n-1} \binom{2n+2s}{2n-2} r^{2n} \{P_{2n}(\mu) - P_{2n-2}(\mu)\} \dots (23)$$

$$T_{2s} = \frac{(4s+1)}{(2s+1)!} {}^1B_{2s} \frac{\partial^{2s}}{\partial z^{2s}} \left\{ \frac{r\mu+1}{\sqrt{r^2+2r\mu+1}} - \frac{r\mu-1}{\sqrt{r^2-2r\mu+1}} - 2 \right\} \\ = 2^1B_{2s} \sum_{n=1}^{\infty} \frac{4s+1}{4n-1} \binom{2n+2s-1}{2n-2} r^{2n} \{P_{2n}(\mu) - P_{2n-2}(\mu)\} \dots (24)$$

Consequently we have

$$\begin{aligned} \chi_1 = & b^3 \sum_{n=1}^{\infty} \left\{ {}^0A_1 \frac{1}{r} + {}^0A_{2n+1} \frac{1}{r^{2n+1}} + \left[(2^1A_1 - {}^1A_0/n) + \sum_{s=2}^{\infty} 2^1A_s \binom{2n+s-1}{2n} \right] r^{2n} \right\} \\ & \times P_{2n}(\mu) + b^3 \sum_{n=1}^{\infty} \left\{ {}^0B_{2n-1} \frac{1}{r^{2n-1}} + 2 \left[{}^1B_0 \frac{2n-1}{4n-1} + \sum_{s=1}^{\infty} {}^1B_s \frac{2s+1}{4n-1} \binom{2n+s-1}{2n-2} \right] r^{2n} \right\} \\ & \times [P_{2n}(\mu) - P_{2n-2}(\mu)] \dots\dots\dots (25) \end{aligned}$$

χ_1 is expressed in the spherical coordinates (r_1, θ, ϕ_1) in the form

$$\begin{aligned} \chi_1 = & b^3 \left\{ {}^1A_0 \frac{1}{2} \log \frac{1+\mu_1}{1-\mu_1} + {}^1A_1 \frac{1}{r_1} + \sum_{s=1}^{\infty} {}^1A_{s+1} \frac{P_s(\mu_1)}{r_1^{s+1}} + \sum_{n=1}^{\infty} \left[{}^0A_1 + \sum_{s=1}^{\infty} {}^0A_{2s+1} \binom{n+2s}{n} - {}^1A_0 \frac{1}{n2^n} + {}^1A_1 \frac{1}{2^{n+1}} \right. \right. \\ & \left. \left. + \sum_{s=2}^{\infty} {}^0A_s \frac{1}{2^{n+s}} \binom{n+s-1}{n} \right] r_1^n P_n(\mu_1) \right\} + b^3 \left\{ {}^1B_0 \mu_1 + \sum_{s=0}^{\infty} {}^1B_{s+1} \frac{P_{s+2}(\mu_1) - P_s(\mu_1)}{r_1^{s+1}} \right. \\ & \left. + \sum_{n=2}^{\infty} \left[\sum_{s=0}^{\infty} {}^0B_{2s+1} \frac{4s+3}{2n-1} \binom{n+2s}{n-2} + {}^1B_0 \frac{1}{2^n} \frac{n-1}{2n-1} \right. \right. \\ & \left. \left. + \sum_{s=1}^{\infty} {}^1B_s \frac{2s+1}{2n-1} \binom{n+s-1}{n-2} \right] \frac{1}{2^{n+s}} \right\} r_1^n [P_n(\mu_1) - P_{n-2}(\mu_1)] \dots\dots\dots (26) \end{aligned}$$

Having the function we can calculate the stresses from the ordinary formulae. The normal and tangential stresses in Eqs. (18) and (19) are determined as follows:

$$\begin{aligned} \sigma_r^{(1)} + \sigma_r^{(2)} = & -\frac{E\alpha}{1-\nu} \tau a \lambda^2 \sum_{s=1}^{\infty} \left\{ \frac{(2s^2+s-2)}{(4s-3)} \frac{P_{2s-1}(\mu)}{r^{2s}} {}^0C_{2s} + {}^1C_{s+1} \left[\sum_{n=1}^{\infty} \binom{2n+s}{s} \frac{2n(4n-8ns-2s+2)}{(4s-2)(4n+1)} r^{2n-1} \right. \right. \\ & \left. \left. + \sum_{n=0}^{\infty} \binom{2n+s+1}{2n+1} \frac{(4n^2+2n-4)}{(4n+5)} r^{2n+1} \right] P_{2n+1}(\mu) \right\} + \left\{ \frac{6P_1(\mu)}{r^4} {}^0A_1 + \sum_{s=1}^{\infty} \frac{(2s+3)(2s+2)(2s+1)}{r^{2s+4}} \right. \\ & \times P_{2s+1}(\mu) {}^0A_{2s+1} - \sum_{n=2}^{\infty} \left[(2^1A_1 - {}^1A_0/n) 2n(2n-1)(2n-2) + \sum_{s=2}^{\infty} 2^1A_s \binom{2n+s-1}{2n} \right. \\ & \left. \times 2n(2n-1)(2n-2) \right] r^{2n-3} P_{2n-1}(\mu) \left\{ \right. \\ & \left. + \sum_{s=0}^{\infty} \left\{ \frac{(4s+3)(2s+3)[2(2s^2+9s+9-\nu)P_{2s+3}(\mu) - (2s+2)(2s+4\nu-1)P_{2s+1}(\mu)]}{(4s+5)r^{2s+4}} {}^0B_{2s+1} \right. \right. \\ & \left. \left. - \sum_{n=1}^{\infty} 8n(2s+1) \binom{2n+s+1}{2n} \frac{[(2n+1)(n-2\nu+2)P_{2n+1}(\mu) - (2n^2-3n-\nu)P_{2n-1}(\mu)]}{(4n+1)} r^{2n-1} B_s \right\} \right\} \dots\dots\dots (27) \end{aligned}$$

$$\begin{aligned} \tau_{r\theta}^{(1)} + \tau_{r\theta}^{(2)} = & -\frac{E\alpha}{1-\nu} \tau a \lambda^2 \sqrt{1-\mu^2} \sum_{s=1}^{\infty} \left\{ \frac{(2s-1)}{(8s-6)} \frac{P'_{2s-1}(\mu)}{r^{2s}} {}^0C_{2s} - \sum_{n=1}^{\infty} {}^1C_{s+1} \left[\binom{2n+s}{s} \right. \right. \\ & \times \frac{2n(4n-8ns-2s+2)}{(4s-2)(4n+1)(2n+1)} r^{2n-1} + \frac{(2n+2)}{(4n+5)} \binom{2n+s+1}{s} r^{2n+1} \left. \right] P'_{2n+1}(\mu) \left\{ + \sqrt{1-\mu^2} \right. \\ & \times \left[\frac{3P'_1(\mu)}{r^4} {}^0A_1 + \sum_{s=1}^{\infty} \frac{(2s+1)(2s+3)P'_{2s+1}(\mu)}{r^{2s+4}} {}^0A_{2s+1} + \sum_{n=1}^{\infty} (2^1A_1 - {}^1A_0/n) 2n(2n-2) r^{2n-3} P'_{2n-1}(\mu) \right. \\ & \left. \left. + \sum_{s=2}^{\infty} \sum_{n=2}^{\infty} 2^1A_s \binom{2n+s-1}{2n} 2n(2n-2) r^{2n-3} P'_{2n-1}(\mu) \right] - \sum_{s=0}^{\infty} \frac{(4s+3)}{(4s+5)r^{2s+4}} [(2s+3)(2s-1+4\nu) \right. \\ & \times P'_{2s+1}(\mu) - (4s^2+12s+7+2\nu)P'_{2s+3}(\mu)] {}^0B_{2s+1} + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} 2 \frac{(2s+1)}{(4n+1)} \binom{2n+s+1}{2n} \\ & \left. \times [4n(n+2-2\nu)P'_{2n+1}(\mu) - 2(2n^2-1+\nu)P'_{2n-1}(\mu)] r^{2n-1} B_s \right\} \dots\dots\dots (28) \end{aligned}$$

and

$$\begin{aligned} \sigma_{r_1}^{(1)} + \sigma_{r_1}^{(2)} = & -\frac{E\alpha}{1-\nu} \tau a \lambda^2 \sum_{s=1}^{\infty} \left\{ \frac{(s^2+3s-2)}{4s-2} \frac{P_s(\mu_1)}{r_1^{s+1}} {}^1C_{s+1} + {}^0C_{2s} \left[\sum_{n=0}^{\infty} \binom{n+2s-1}{2s-1} \frac{(n^2-n-4)}{4n+6} r_1^n \right. \right. \\ & \left. \left. - \sum_{n=2}^{\infty} \binom{n+2s-2}{n-1} \frac{(4ns-3n-2s+1)(n-1)}{(2n-1)(8s-6)} r_1^{n-2} \right] P_n(\mu_1) + {}^1C_{s+1} \left[\sum_{n=0}^{\infty} \binom{n+s}{s} \frac{(n^2-n-4)}{(4n+6)2^{n+s+1}} r_1^n \right. \right. \\ & \left. \left. - \sum_{n=2}^{\infty} 4 \binom{n+s-1}{n-1} \frac{(2ns-n-s)(n-1)}{(2n-1)(4s-2)2^{n+s+1}} r_1^{n-2} \right] P_n(\mu_1) \right\} - \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} {}^0A_{2s+1} \binom{n+2s+3}{n+3} \\ & \times (n+3)(n+2)(n+1) r_1^n P_{n+2}(\mu_1) - \sum_{n=0}^{\infty} {}^1A_0 \left\{ \frac{2}{r_1^3} - \frac{1}{2^{n+3}} (n+2)(n+1) r_1^n P_{n+2}(\mu_1) \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{s=1}^{\infty} {}^1A_s \left\{ \frac{-s(s+1)(s+2)P_s(\mu_1)}{r_1^{s+3}} + \sum_{n=0}^{\infty} \binom{n+s+2}{n+3} \frac{(n+3)(n+2)(n+1)}{2^{n+s+3}} r_1^n P_{n+2}(\mu_1) \right\} \\
& - \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} {}^0B_{2s+1} \binom{n+2s+3}{n+1} \frac{(4s+3)(n+1)}{(2n+3)} r_1^n \{ (n+2)(n+5-4\nu)P_{n+2}(\mu_1) \\
& - (n^2-n-2-2\nu)P_n(\mu_1) \} - \sum_{s=0}^{\infty} {}^1B_s \left\{ \frac{(2s+1)(s+2)}{(2s+3)r_1^{s+3}} [(s+1)(s-2+4\nu)P_s(\mu_1) \right. \\
& - (s^2+7s+10-2\nu)P_{s+2}(\mu_1)] + \sum_{n=0}^{\infty} \binom{n+s+2}{n+1} \frac{(2s+1)(n+1)}{(2n+3)} \frac{r_1^n}{2^{n+s+3}} \\
& \times [(n+2)(n+5-4\nu)P_{n+2}(\mu_1) - (n^2-n-2-2\nu)P_n(\mu_1)] \} \dots\dots\dots (29) \\
\tau_{r_1}^{(1)} + \tau_{r_1}^{(2)} = & - \frac{E\alpha}{1-\nu} \tau a \lambda^2 \sqrt{1-\mu^2} \sum_{s=1}^{\infty} \left\{ \frac{s}{(4s-2)} {}^1C_{s+1} \frac{P'_s(\mu_1)}{r_1^{s+1}} - {}^0C_{2s} \left[\sum_{n=0}^{\infty} \binom{n+2s-1}{2s-1} \frac{(n+1)}{4n+6} r_1^n \right. \right. \\
& - \sum_{n=2}^{\infty} \binom{n+2s-2}{n-1} \frac{(4ns-3n-2s+1)(n-1)}{n(2n-1)(8s-6)} r_1^{n-2} \left. \right] P'_n(\mu_1) \\
& - {}^1C_{s+1} \left[\sum_{n=0}^{\infty} \binom{n+s}{n} \frac{(n+1)}{(4n+6)2^{n+s+1}} r_1^n - \sum_{n=2}^{\infty} 4 \binom{n+s-1}{n-1} \frac{(2ns-n-s)(n-1)}{n(2n-1)(4s-2)2^{n+s+1}} r_1^{n-2} \right] \\
& \times P'_n(\mu_1) \} + \sqrt{1-\mu^2} \left\{ \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} {}^0A_{2s+1} \binom{n+2s+3}{n+3} (n+3)(n+1) r_1^n P'_{n+2}(\mu_1) \right. \\
& + {}^1A_0 \sum_{n=0}^{\infty} \frac{-1}{2^{n+3}} (n+1) r_1^n P'_{n+2}(\mu_1) + \sum_{s=1}^{\infty} \sum_{n=0}^{\infty} {}^1A_s \left[s(s+2) \frac{P'_s(\mu_1)}{r_1^{s+3}} \right. \\
& + \frac{1}{2^{n+s+3}} \binom{n+s+2}{n+3} (n+1)(n+3) r_1^n P'_{n+2}(\mu_1) \left. \right] + \sum_{s=0}^{\infty} \sum_{n=0}^{\infty} {}^0B_{2s+1} (4s+3) \binom{n+2s+3}{n+1} \frac{r_1^n}{(2n+3)} \\
& \times [(n+1)(n+5-4\nu)P'_{n+2}(\mu_1) - (n^2+2n-1+2\nu)P'_n(\mu_1)] + \sum_{s=0}^{\infty} {}^1B_s \left[\frac{(2s+1)}{(2s+3)r_1^{s+3}} \{ (s+2) \right. \\
& \times (2-s-4\nu)P'_s(\mu_1) + (s^2+4\nu+2+2\nu)P'_{s+2}(\mu_1) \} + \sum_{n=0}^{\infty} (2s+1) \binom{n+s+2}{n+1} \frac{r_1^n}{(2n+3)2^{n+s+3}} \\
& \times \{ (n+1)(n-4\nu+5)P'_{n+2}(\mu_1) - (n^2+2n-1+2\nu)P'_n(\mu_1) \} \left. \right] \} \dots\dots\dots (30)
\end{aligned}$$

Then the solution will be completed by so determining the coefficients ${}^0A_{2n+1}$, ${}^0B_{2n+1}$, 1A_n and 1B_n as to satisfy the conditions (18) and (19) which give a system of linear equations obtained by equating separately the coefficient of each Legendre function or its derivative to zero. The system of equations will be omitted for brevity, but a consequent formal solution by successive approximations is as follows: Write, for $n \geq 0$,

$${}^0A_{2n+1} = \sum_{r=0}^{\infty} {}^0A_{2n+1}^{(r)}, \quad {}^0B_{2n+1} = \sum_{r=0}^{\infty} {}^0B_{2n+1}^{(r)}, \quad {}^1A_n = \sum_{r=0}^{\infty} {}^1A_n^{(r)}, \quad {}^1B_n = \sum_{r=0}^{\infty} {}^1B_n^{(r)} \dots\dots\dots (31)$$

where

$$\begin{aligned}
{}^0A_1^{(0)} &= \frac{E\alpha}{6(1-\nu)} \tau a \left\{ {}^0C_2 \lambda^4 - \sum_{s=1}^{\infty} \frac{4}{5} (s+1) {}^1C_{s+1} \lambda^7 \right\} - \frac{3(1-4\nu)}{5} {}^0B_1^{(0)} \\
{}^0A_1^{(r)} &= - \frac{3(1-4\nu)}{5} {}^0B_1^{(r)} + \sum_{s=0}^{\infty} \frac{4}{15} \binom{s+3}{2} (2s+1)(1+\nu) {}^1B_s^{(r-1)} \lambda^5 \left. \right\} \dots\dots\dots (32)
\end{aligned}$$

for $n > 1$

$$\begin{aligned}
{}^0A_{2n+1}^{(0)} &= \frac{1}{(2n+1)(2n+3)} \left\{ \frac{E\alpha\tau a}{1-\nu} \left[\frac{(2n+1)}{8n+2} {}^0C_{2n+2} \lambda^4 - \sum_{s=1}^{\infty} \binom{2n+s}{s} \frac{2n(4n-8ns-2s+2)}{(4s-2)(4n+1)(2n+1)} {}^1C_{s+1} \lambda^{4n+5} \right. \right. \\
& - \sum_{s=1}^{\infty} \binom{2n+s+1}{s} \frac{(2n+2)}{(4n+5)} {}^1C_{s+1} \lambda^{4n+7} \left. \right] - \frac{(4n-1)}{(4n+1)} (4n^2+4n-1+2\nu) {}^0B_{2n-1}^{(0)} \lambda^2 \\
& + \frac{(4n+3)}{(4n+5)} (2n+3)(2n-1+4\nu) {}^0B_{2n+1}^{(0)} \left. \right\} \\
{}^0A_{2n+1}^{(r)} &= \frac{1}{(2n+1)(2n+3)} \left\{ - \frac{(4n-1)}{4n+1} (4n^2+4n-1+2\nu) {}^0B_{2n-1}^{(r)} \lambda^2 + \frac{(4n+3)}{4n+5} (2n+3)(2n-1+4\nu) {}^0B_{2n+1}^{(r)} \right. \\
& - 2n(2n+2) [2 {}^1A_1^{(r-1)} - {}^1A_0^{(r-1)} / (n+1)] \lambda^{4n+3} - 4 \binom{2n+s+1}{2n+2} (n+1) 2n {}^1A_s^{(r-1)} \lambda^{4n+3} \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& - \sum_{s=0}^{\infty} 2 \binom{2n+s+1}{2n} \frac{(2s+1)4n(n+2-2\nu)}{(4n+1)} {}_1B_s^{(r-1)} \lambda^{4n+3} \\
& + \sum_{s=0}^{\infty} 4 \binom{2n+s+3}{2n+2} \frac{(2s+1)(2n^2+4n+1+\nu)}{(4n+5)} {}_1B_s^{(r-1)} \lambda^{4n+5} \}
\end{aligned} \tag{33}$$

for $n > 1$

$$\begin{aligned}
{}_0B_{2n-1}^{(0)} &= \frac{E\alpha\tau a}{1-\nu} \frac{1}{(4n-1)(8n^2+12n+6-8n\nu-6\nu)} \left\{ 2n^0 C_{2n+2} \lambda^2 \right. \\
& + \sum_{s=1}^{\infty} \binom{2n+s}{s} \frac{2n(4n+3)(4n-8ns-2s+2)}{(4s-2)(2n+1)} {}_1C_{s+1} \lambda^{4n+3} \\
& - \sum_{s=1}^{\infty} \binom{2n+s+1}{2n+1} \frac{2(4n+1)(3n+4)}{(4n+5)} {}_1C_{s+1} \lambda^{4n+5} \} \\
{}_0B_{2n-1}^{(r)} &= \frac{1}{(4n-1)(8n^2+12n+6-8n\nu-6\nu)} \left\{ 2n(2n+2)(4n+1)(4n+3) [2 {}^1A_1^{(r-1)} - {}^1A_0^{(r-1)} / (n+1)] \lambda^{4n+1} \right. \\
& + \sum_{s=2}^{\infty} 4 \binom{2n+s+1}{2n+2} 2n(n+1)(4n+1)(4n+3) {}^1A_s^{(r-1)} \lambda^{4n+1} \\
& + \sum_{s=0}^{\infty} \binom{2n+s+1}{2n} 8n(2s+1)(n+2-2\nu)(4n+3) {}^1B_s^{(r-1)} \lambda^{4n+1} \\
& - \sum_{s=0}^{\infty} \binom{2n+s+3}{2n+2} (2s+1)8n(n+1)(4n+1) {}^1B_s^{(r-1)} \lambda^{4n+3} \}
\end{aligned} \tag{34}$$

$$\begin{aligned}
{}^1A_0^{(0)} &= \frac{E\alpha\tau a}{3(1-\nu)} \sum_{s=1}^{\infty} ({}^0C_{2s} + {}^1C_{s+1}/2^{s+1}) \lambda^5 + \left[\frac{2(1-2\nu)}{3} - \frac{(1+\nu)}{12} \lambda^3 \right] {}^1B_0^{(0)} \\
{}^1A_0^{(r)} &= \left[\frac{2(1-2\nu)}{3} - \frac{(1+\nu)}{12} \lambda^3 \right] {}^1B_0^{(r)} - \sum_{s=0}^{\infty} \frac{(2s+3)(4s+3)(1+\nu)}{3} {}^0B_{2s+1}^{(r-1)} \lambda^3 \\
& - \sum_{s=1}^{\infty} \frac{(2s+1)(s+2)(1+\nu)}{3 \times 2^{s+3}} {}^1B_s^{(r-1)} \lambda^3 \\
{}^1A_1^{(0)} &= \frac{E\alpha\tau a}{1-\nu} \left[{}^1C_2 \lambda^4 / 6 - \sum_{s=1}^{\infty} {}^0C_{2s} \frac{2}{15} s \lambda^7 - \sum_{s=1}^{\infty} {}^1C_{s+1} \frac{(s+1)}{15 \times 2^{s+2}} \lambda^7 \right] - \frac{3}{5} (1-4\nu) {}^1B_1^{(0)} \\
{}^1A_1^{(r)} &= -\frac{3}{5} (1-4\nu) {}^1B_1^{(r)} + \sum_{s=0}^{\infty} \binom{2s+4}{2} \frac{2}{15} (4s+3)(1+\nu) {}^0B_{2s+1}^{(r-1)} \lambda^5 + \sum_{s=0}^{\infty} \binom{s+3}{2} (2s+1) \frac{(1+\nu)}{15 \times 2^{s+3}} {}^1B_s^{(r-1)} \lambda^5
\end{aligned} \tag{35}$$

and for $n \geq 2$

$$\begin{aligned}
{}^1A_n^{(0)} &= \frac{1}{n(n+2)} \left\{ \frac{E\alpha\tau a}{1-\nu} \left[\frac{n}{4n-2} {}^1C_{n+1} \lambda^4 - \sum_{s=1}^{\infty} \binom{n+2s-1}{2s-1} \frac{(n+1)}{(4n+6)} {}^0C_{2s} \lambda^{2n+5} \right. \right. \\
& + \sum_{s=1}^{\infty} \binom{n+2s-2}{n-1} \frac{(4ns-3n-2s+1)(n-1)}{n(2n-1)(8s-6)} {}^0C_{2s} \lambda^{2n+3} \\
& - \sum_{s=1}^{\infty} \binom{n+s}{n} \frac{(n+1)}{(4n+6)2^{n+s+1}} {}^1C_{s+1} \lambda^{2n+5} + \sum_{s=1}^{\infty} 4 \binom{n+s-1}{n-1} \frac{(2ns-n-s)(n-1)}{n(2n-1)(2s-1)2^{n+s+2}} {}^1C_{s+1} \lambda^{2n+3} \left. \right] \\
& - \frac{(2n+1)(n+2)(2-n-4\nu)}{(2n+3)} {}^1B_n^{(0)} - \frac{(2n-3)}{(2n-1)} (n^2-2+2\nu) {}^1B_{n-2}^{(0)} \lambda^2 \} \\
{}^1A_n^{(r)} &= \frac{1}{n(n+2)} \left\{ -\frac{(2n+1)(n+2)(2-n-4\nu)}{(2n+3)} {}^1B_n^{(r)} - \frac{(2n-3)}{(2n-1)} (n^2-2+2\nu) {}^1B_{n-2}^{(r)} \lambda^2 \right. \\
& - \sum_{s=0}^{\infty} \binom{n+2s+1}{n+1} (n^2-1) {}^0A_{2s+1}^{(r-1)} \lambda^{2n+1} + \frac{(n-1)}{2^{n+1}} {}^1A_0^{(r-1)} \lambda^{2n+1} - \sum_{s=1}^{\infty} \binom{n+s}{n+1} \frac{(n^2-1)}{2^{n+s+1}} {}^1A_s^{(r-1)} \lambda^{2n+1} \\
& - \sum_{s=0}^{\infty} \binom{n+2s+1}{n-1} \frac{(4s+3)(n-1)(n+3-4\nu)}{(2n-1)} {}^0B_{2s+1}^{(r-1)} \lambda^{2n+1} \\
& + \sum_{s=0}^{\infty} \binom{n+2s+3}{n+1} \frac{(4s+3)(n^2+2n-1+2\nu)}{(2n+3)} {}^0B_{2s+1}^{(r-1)} \lambda^{n+3} \\
& - \sum_{s=0}^{\infty} \binom{n+s}{n-1} \frac{(2s+1)(n-1)(n-4\nu+3)}{(2n-1)2^{n+s+1}} {}^1B_s^{(r-1)} \lambda^{2n+1} \\
& + \sum_{s=0}^{\infty} \binom{n+s+2}{n+1} \frac{(2s+1)(n^2+2n-1+2\nu)}{(2n+3)2^{n+s+3}} {}^1B_s^{(r-1)} \lambda^{2n+3} \}
\end{aligned} \tag{36}$$

$$\begin{aligned}
{}^1B_{n-2}^{(0)} = & \frac{1}{(2n-3)(2n^2+2n+2-4n\nu-2\nu)} \left\{ \frac{E\alpha r a}{(1-\nu)} \left[(n-1) {}^1C_{n+1} \lambda^2 + \sum_{s=1}^{\infty} \binom{n+2s-1}{2s-1} \frac{(n-1)(2n-1)}{2} {}^0C_{2s} \lambda^{2n+3} \right] \right. \\
& - \sum_{s=1}^{\infty} \binom{n+2s-2}{n-1} \frac{(4ns-3n-2s+1)(n-1)(2n+1)}{(8s-6)n} {}^0C_{2s} \lambda^{2n+1} \\
& + \sum_{s=1}^{\infty} \binom{n+s}{s} \frac{(n-1)(2n-1)}{2^{n+s+2}} {}^1C_{s+1} \lambda^{2n+3} - \sum_{s=1}^{\infty} 4 \binom{n+s-1}{n-1} \frac{(2ns-n-s)(n-1)(2n+1)}{n(4s-2)2^{n+s+1}} {}^1C_{s+1} \lambda^{2n+1} \Big] \\
& + \sum_{s=0}^{\infty} \binom{n+2s+1}{n+1} (n^2-1)(4n^2-1) {}^0A_{2s+1}^{(0)} \lambda^{2n-1} + \sum_{s=0}^{\infty} \binom{n+2s+1}{n-1} (4s+3)(n-1)(2n+1)(n+3-4\nu) \\
& \times {}^0B_{2s+1}^{(0)} \lambda^{2n-1} - \sum_{s=0}^{\infty} \binom{n+2s+3}{n+1} (4s+3)(n^2-1)(2n-1) {}^0B_{2s+1}^{(0)} \lambda^{2n+1} \Big\} \\
{}^1B_{n-2}^{(r)} = & \frac{1}{(2n-3)(2n^2+2n+2-4n\nu-2\nu)} \left\{ \sum_{s=0}^{\infty} \binom{n+2s+1}{n+1} (n^2-1)(4n^2-1) {}^0A_{2s+1}^{(r)} \lambda^{2n-1} \right. \\
& + \sum_{s=0}^{\infty} \binom{n+2s+1}{n-1} (4s+3)(n-1)(2n+1)(n+3-4\nu) {}^0B_{2s+1}^{(r)} \lambda^{2n-1} \\
& - \sum_{s=0}^{\infty} \binom{n+2s+3}{n+1} (4s+3)(n^2-1)(2n-1) {}^0B_{2s+1}^{(r)} \lambda^{2n+1} - \frac{(n-1)(4n^2-1)}{2^{n+1}} {}^1A_0^{(r-1)} \lambda^{2n-1} \\
& + \sum_{s=1}^{\infty} \binom{n+s}{n+1} \frac{(n^2-1)(4n^2-1)}{2^{n+s+1}} {}^1A_s^{(r-1)} \lambda^{2n-1} + \sum_{s=0}^{\infty} \binom{n+s}{n-1} \frac{(2s+1)(n-1)(2n+1)(n+3-4\nu)}{2^{n+s+1}} {}^1B_s^{(r-1)} \lambda^{2n-1} \\
& \left. - \sum_{s=0}^{\infty} \binom{n+s+2}{n+1} \frac{(2s+1)(n^2-1)(2n-1)}{2^{n+s+3}} {}^1B_s^{(r-1)} \lambda^{2n+1} \right\}
\end{aligned}$$

.....(37)

The foregoing method of solving the linear equations is valid as long as the four series in Eqs. (31) are convergent. No analytic proof of convergence will be given for the sake of brevity, while the solution will be illustrated by numerical examples.

3. Numerical examples

Numerical examples will be given for the cases $\lambda = a/b = 0.05, 0.10, 0.15, 0.20$ taking Poisson's ratio ν equal to 0.25. It may be possible to compute the stresses at any point in the elastic region. The most interesting one, however, is the stress distributions occurring at the surfaces of the central and outside cavities. This is given by, for central cavity

$$\begin{aligned}
& | \sigma_{\theta}^{(1)} + \sigma_{\theta}^{(2)} |_{r=\lambda} \\
& = - \frac{E\alpha r a}{1-\nu} \lambda^2 \sum_{s=1}^{\infty} \left[\frac{{}^0C_{2s}}{\lambda^{2s}} \left[\frac{P'_{2s}(\mu) - (4s^2 - 6s + 4)P_{2s-1}(\mu)}{2(4s-3)} \right] + {}^1C_{s+1} \left[\sum_{n=1}^{\infty} \binom{2n+s}{s} \right. \right. \\
& \quad \times \frac{4n-8ns-2s+2}{(4s-2)(4n+1)(2n+1)} \lambda^{2n-1} \{ P'_{2n+2}(\mu) - (4n^2+6n+3)P_{2n+1}(\mu) \} \\
& \quad \left. \left. + \sum_{n=0}^{\infty} \binom{2n+s+1}{s} \lambda^{2n+1} \left\{ \frac{P'_{2n+2}(\mu) - (4n^2+14n+11)P_{2n+1}(\mu)}{4n+5} \right\} \right] \right] \\
& \quad - {}^0A_1 \frac{3}{\lambda^4} P_1(\mu) + \sum_{s=1}^{\infty} {}^0A_{2s+1} \frac{2s+1}{\lambda^{2s+4}} [P'_{2s+2}(\mu) - (2s+2)(2s+3)P_{2s+1}(\mu)] \\
& \quad + \sum_{n=1}^{\infty} \left[2^1A_1 - {}^1A_0 \frac{1}{(n+1)} + \sum_{s=2}^{\infty} 2^1A_s \binom{2n+s+1}{2n+2} \right] (2n+2) \lambda^{2n-1} [(4n^2+6n+3)P_{2n+1}(\mu) - P'_{2n+2}(\mu)] \\
& \quad + \sum_{s=0}^{\infty} {}^0B_{2s+1} \frac{(2s+3)(4s+3)}{(4s+5)\lambda^{2s+4}} [(2s+2)(2s-1+4\nu)P_{2s+1}(\mu) - (4s^2+6s+3-2\nu)P_{2s+3}(\mu)] \\
& \quad + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} {}^1B_s \binom{2n+s+1}{2n} 4n(2s+1) \lambda^{2n-1} \left[\frac{(2n+1)(2n+4-4\nu)P_{2n+1}(\mu) - (4n^2+6n+3-2\nu)P_{2n-1}(\mu)}{4n+1} \right]
\end{aligned}$$

.....(38)

for outside cavity

$$\begin{aligned}
& | (\sigma_{\theta_1}^{(1)} + \sigma_{\theta_1}^{(2)}) + (\sigma_{r_1}^{(1)} + \sigma_{r_1}^{(2)}) |_{r_1=\lambda} \\
& = - \frac{E\alpha r a}{1-\nu} \lambda^2 \sum_{s=1}^{\infty} \left\{ \frac{{}^1C_{s+1}}{(4s-2)\lambda^{s+1}} [P'_{s+1}(\mu_1) + (4s-4)P_s(\mu_1)] + \sum_{n=0}^{\infty} [{}^0C_{2s} \binom{n+2s-1}{2s-1} \right.
\end{aligned}$$

Table 1 Values of ${}^0C_{2n}$, ${}^1C_{n+1}$

	$\lambda=0.05$	$\lambda=0.10$	$\lambda=0.15$	$\lambda=0.20$
0C_2	$4.998\ 75 \times 10^{-1}$	$4.990\ 01 \times 10^{-1}$	$4.966\ 39 \times 10^{-1}$	$4.920\ 80 \times 10^{-1}$
0C_4	$-2.343\ 42 \times 10^{-9}$	$-2.996\ 47 \times 10^{-7}$	$-5.104\ 25 \times 10^{-6}$	$-3.798\ 72 \times 10^{-5}$
0C_6	$-2.441\ 06 \times 10^{-14}$	$-4.994\ 00 \times 10^{-11}$	$-4.305\ 91 \times 10^{-9}$	$-1.012\ 10 \times 10^{-7}$
0C_8	$-2.135\ 92 \times 10^{-19}$	$-6.991\ 44 \times 10^{-15}$	$-3.051\ 22 \times 10^{-12}$	$-2.264\ 96 \times 10^{-10}$
${}^0C_{10}$	—	—	$-1.985\ 62 \times 10^{-15}$	$-4.654\ 56 \times 10^{-13}$
${}^0C_{12}$	—	—	—	$-4.546\ 05 \times 10^{-15}$
1C_2	$4.999\ 30 \times 10^{-1}$	$4.994\ 39 \times 10^{-1}$	$4.981\ 14 \times 10^{-1}$	$4.955\ 69 \times 10^{-1}$
1C_3	$-3.319\ 50 \times 10^{-7}$	$-1.060\ 43 \times 10^{-5}$	$-8.015\ 18 \times 10^{-5}$	$-3.346\ 58 \times 10^{-4}$
1C_4	$-1.208\ 20 \times 10^{-9}$	$-1.543\ 82 \times 10^{-7}$	$-2.625\ 30 \times 10^{-6}$	$-1.948\ 27 \times 10^{-5}$
1C_5	$-3.966\ 30 \times 10^{-12}$	$-2.027\ 21 \times 10^{-9}$	$-7.797\ 73 \times 10^{-8}$	$-1.023\ 11 \times 10^{-6}$
1C_6	$-1.229\ 93 \times 10^{-14}$	$-2.514\ 50 \times 10^{-11}$	$-2.164\ 52 \times 10^{-9}$	$-5.074\ 82 \times 10^{-8}$
1C_7	—	$-3.005\ 69 \times 10^{-13}$	$-5.821\ 33 \times 10^{-11}$	$-2.425\ 90 \times 10^{-9}$
1C_8	—	—	$-1.525\ 07 \times 10^{-12}$	$-1.129\ 60 \times 10^{-10}$
1C_9	—	—	$-3.917\ 64 \times 10^{-14}$	$-5.157\ 44 \times 10^{-12}$
${}^1C_{10}$	—	—	—	$-2.318\ 17 \times 10^{-13}$

Table 2 Values of ${}^0A_{2n+1}$, ${}^0B_{2n+1}$, 1A_n , ${}^1B_n/E\alpha a$

	$\lambda=0.05$	$\lambda=0.10$	$\lambda=0.15$	$\lambda=0.20$
0A_1	$6.941\ 32 \times 10^{-7}$	$1.107\ 11 \times 10^{-5}$	$5.556\ 82 \times 10^{-5}$	$1.726\ 89 \times 10^{-4}$
0A_3	$2.288\ 02 \times 10^{-13}$	$1.133\ 09 \times 10^{-10}$	$4.133\ 71 \times 10^{-9}$	$5.134\ 79 \times 10^{-8}$
0A_5	$1.868\ 52 \times 10^{-18}$	$1.834\ 91 \times 10^{-14}$	$5.419\ 81 \times 10^{-13}$	$1.278\ 02 \times 10^{-10}$
0A_7	—	$2.115\ 85 \times 10^{-18}$	$1.936\ 52 \times 10^{-15}$	$2.335\ 12 \times 10^{-13}$
0A_9	—	—	$1.102\ 44 \times 10^{-18}$	$3.906\ 80 \times 10^{-16}$
0B_1	$-2.152\ 05 \times 10^{-10}$	$-2.728\ 46 \times 10^{-8}$	$-4.606\ 56 \times 10^{-7}$	$-3.412\ 21 \times 10^{-6}$
0B_3	$-1.410\ 14 \times 10^{-15}$	$-2.869\ 29 \times 10^{-12}$	$-2.392\ 09 \times 10^{-10}$	$-5.410\ 88 \times 10^{-9}$
0B_5	$-9.260\ 10 \times 10^{-21}$	$-2.928\ 27 \times 10^{-16}$	$-1.213\ 99 \times 10^{-13}$	$-8.462\ 50 \times 10^{-12}$
0B_7	—	$-2.944\ 45 \times 10^{-20}$	$-6.071\ 82 \times 10^{-17}$	$-1.308\ 49 \times 10^{-14}$
0B_9	—	—	$-3.017\ 91 \times 10^{-20}$	$-2.051\ 99 \times 10^{-17}$
1A_0	$7.006\ 75 \times 10^{-8}$	$2.190\ 61 \times 10^{-6}$	$1.674\ 71 \times 10^{-5}$	$7.104\ 36 \times 10^{-5}$
1A_1	$6.942\ 69 \times 10^{-7}$	$1.108\ 87 \times 10^{-5}$	$5.586\ 79 \times 10^{-5}$	$1.749\ 38 \times 10^{-4}$
1A_2	$3.278\ 31 \times 10^{-11}$	$3.342\ 65 \times 10^{-9}$	$5.118\ 76 \times 10^{-8}$	$3.225\ 63 \times 10^{-7}$
1A_3	$4.903\ 71 \times 10^{-14}$	$6.242\ 67 \times 10^{-11}$	$2.187\ 92 \times 10^{-9}$	$2.508\ 07 \times 10^{-8}$
1A_4	$5.142\ 24 \times 10^{-16}$	$7.807\ 37 \times 10^{-13}$	$5.874\ 53 \times 10^{-11}$	$1.074\ 16 \times 10^{-9}$
1A_5	—	$7.308\ 00 \times 10^{-15}$	$8.514\ 08 \times 10^{-13}$	$2.280\ 68 \times 10^{-12}$
1A_6	—	—	$-8.849\ 35 \times 10^{-14}$	$-1.036\ 36 \times 10^{-11}$
1B_0	$-5.642\ 89 \times 10^{-8}$	$-1.748\ 42 \times 10^{-6}$	$-1.268\ 18 \times 10^{-5}$	$-5.009\ 34 \times 10^{-5}$
1B_1	$-1.120\ 02 \times 10^{-11}$	$-1.491\ 98 \times 10^{-8}$	$-2.414\ 60 \times 10^{-7}$	$-1.675\ 26 \times 10^{-6}$
1B_2	$-2.959\ 27 \times 10^{-13}$	$-1.457\ 28 \times 10^{-10}$	$-5.253\ 92 \times 10^{-9}$	$-6.394\ 59 \times 10^{-8}$
1B_3	$-7.367\ 35 \times 10^{-16}$	$-1.442\ 41 \times 10^{-12}$	$-1.162\ 95 \times 10^{-10}$	$-2.482\ 97 \times 10^{-9}$
1B_4	$-1.848\ 29 \times 10^{-18}$	$-1.443\ 06 \times 10^{-14}$	$-2.588\ 09 \times 10^{-12}$	$-9.704\ 12 \times 10^{-11}$
1B_5	—	$-1.442\ 98 \times 10^{-16}$	$-5.766\ 85 \times 10^{-14}$	$-3.802\ 23 \times 10^{-12}$
1B_6	—	—	$-1.286\ 21 \times 10^{-15}$	$-1.490\ 70 \times 10^{-13}$

$$\begin{aligned}
& + \frac{{}^1C_{s+1}}{2^{n+s+1}} \binom{n+s}{n} \left[\frac{\lambda_n}{4n+6} [P'_{n+1}(\mu_1) - 3(2n+3)P_n(\mu_1)] - \sum_{n=2}^{\infty} [{}^0C_{2s} \binom{n+2s-2}{n-1} \frac{(4ns-3n-2s+1)}{(8s-6)} \right. \\
& + {}^1C_{s+1} \binom{n+s-1}{n-1} \frac{(2ns-n-s)}{(2s-1)2^{n+s}} \left. \frac{\lambda^{n-2}}{n(2n-1)} [P'_{n+1}(\mu_1) - (2n+1)P_n(\mu_1)] \right] \\
& - \frac{{}^1A_0}{\lambda^3} + \sum_{s=1}^{\infty} {}^1A_s \frac{s}{\lambda^{s+3}} P'_{s+1}(\mu_1) + \sum_{s=0}^{\infty} {}^1B_s \frac{3(2s+1)(s+2)}{\lambda^{s+3}} P_{s+2}(\mu_1) \\
& + \sum_{n=1}^{\infty} \left[-\frac{{}^1A_0}{2^{n+2}} + \sum_{s=1}^{\infty} {}^1A_s \binom{n+s+1}{n+2} \frac{n+2}{2^{n+s+2}} + \sum_{s=0}^{\infty} {}^0A_{2s+1} \binom{n+2s+2}{n+2} (n+2) \right] \lambda^{n-1} \\
& \times [(2n+3)P_{n+1}(\mu_1) - P'_{n+2}(\mu_1)] - \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} [{}^1B_s \binom{n+s+2}{n+1} \frac{(2s+1)}{2^{n+s+3}} \\
& + {}^0B_{2s+1} \binom{n+2s+3}{n+1} (4s+3)] 3(n+1) \lambda^n P_n(\mu_1) \dots \dots \dots (39)
\end{aligned}$$

where the equation is being simplified by introducing $\sigma_{r_1}^{(1)} + \sigma_{r_1}^{(2)}$, since as $r_1 = \lambda$ the value should vanish.

The coefficients ${}^0C_{2n}$, ${}^1C_{n+1}$, ${}^0A_{2n+1}$, ${}^0B_{2n+1}$, 1A_n , 1B_n are first computed and shown in Table 1 and Table 2. The results are obtained and shown in Tables 3, 4 and Fig. 2. It may be interesting to note that the effect of the presence of three cavities is shown clearly on the stress distribution along the periphery of each cavity. The feature of the reduction that is produced by the shielding effect of some cavities will be better clarified when the results are compared with those of the other case.

Table 3 Values of $\sigma_\theta(\lambda)/E\alpha\tau a$

θ	$\lambda=0.05$	$\lambda=0.10$	$\lambda=0.15$	$\lambda=0.20$
0°	-6.666×10^{-1}	-6.662×10^{-1}	-6.607×10^{-1}	-6.581×10^{-1}
15°	-6.437×10^{-1}	-6.427×10^{-1}	-6.355×10^{-1}	-6.279×10^{-1}
30°	-5.769×10^{-1}	-5.742×10^{-1}	-5.648×10^{-1}	-5.463×10^{-1}
45°	-4.708×10^{-1}	-4.667×10^{-1}	-4.545×10^{-1}	-4.313×10^{-1}
60°	-3.327×10^{-1}	-3.286×10^{-1}	-3.165×10^{-1}	-2.972×10^{-1}
75°	-1.721×10^{-1}	-1.696×10^{-1}	-1.619×10^{-1}	-1.515×10^{-1}
90°	0	0	0	0
105°	1.721×10^{-1}	1.696×10^{-1}	1.619×10^{-1}	1.515×10^{-1}
120°	3.327×10^{-1}	3.286×10^{-1}	3.165×10^{-1}	2.972×10^{-1}
135°	4.708×10^{-1}	4.667×10^{-1}	4.545×10^{-1}	4.313×10^{-1}
150°	5.769×10^{-1}	5.742×10^{-1}	5.648×10^{-1}	5.463×10^{-1}
165°	6.437×10^{-1}	6.427×10^{-1}	6.355×10^{-1}	6.279×10^{-1}
180°	6.666×10^{-1}	6.662×10^{-1}	6.607×10^{-1}	6.581×10^{-1}

Table 4 Values of $\sigma_{\theta_1}(\lambda)/E\alpha\tau a$

θ_1	$\lambda=0.05$	$\lambda=0.10$	$\lambda=0.15$	$\lambda=0.20$
0°	-6.658×10^{-1}	-6.624×10^{-1}	-6.568×10^{-1}	-6.488×10^{-1}
15°	-6.428×10^{-1}	-6.384×10^{-1}	-6.301×10^{-1}	-6.161×10^{-1}
30°	-5.755×10^{-1}	-5.685×10^{-1}	-5.542×10^{-1}	-5.288×10^{-1}
45°	-4.688×10^{-1}	-4.590×10^{-1}	-4.396×10^{-1}	-4.075×10^{-1}
60°	-3.300×10^{-1}	-3.186×10^{-1}	-2.832×10^{-1}	-2.668×10^{-1}
75°	-1.688×10^{-1}	-1.573×10^{-1}	-1.381×10^{-1}	-1.129×10^{-1}
90°	3.783×10^{-2}	1.397×10^{-2}	2.975×10^{-2}	4.868×10^{-2}
105°	1.760×10^{-1}	1.843×10^{-1}	1.962×10^{-1}	2.097×10^{-1}
120°	3.361×10^{-1}	3.425×10^{-1}	3.511×10^{-1}	3.606×10^{-1}
135°	4.734×10^{-1}	4.782×10^{-1}	4.845×10^{-1}	4.913×10^{-1}
150°	5.787×10^{-1}	5.822×10^{-1}	5.872×10^{-1}	5.927×10^{-1}
165°	6.448×10^{-1}	6.475×10^{-1}	6.520×10^{-1}	6.572×10^{-1}
180°	6.673×10^{-1}	6.698×10^{-1}	6.742×10^{-1}	6.794×10^{-1}

(a) Stress distribution along periphery of central cavity

(b) Stress distribution along periphery of outside cavity

Fig. 2

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References

- (1) A.L. Florence and J.N. Goodier: *Trans. ASME*, Ser. E, Vol. 26, No. 2 (1959), p. 293.
- (2) A.L. Florence and J.N. Goodier: *Trans. ASME*, Ser. E, Vol. 27, No. 3 (1960), p. 635.
- (3) A.L. Florence and J.N. Goodier: *Proc. 4th U.S. Nat. Congr. Appl. Mech.*, (1962), p. 595.
- (4) J.N. Goodier and A.L. Florence: *Quart. Jour. Mech. Appl. Math.*, Vol. 16, No. 3 (1963), p. 273.