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The Lateral Vibration of a Leaf Spring with an Added Mass and an Added Moment of Inertia*

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This paper deals with the natural frequencies and modes of a simple supported beam which is equivalent to a leaf spring with an added mass and an added moment of inertia.

The results may be summarized as follows:

- (1) If the added mass is at the center of a uniform beam, there is no coupling between the vertical and angular vibrations of the mass.
- (2) Neglecting the width of the added mass, the higher the vibration order is, the more the natural frequency deviates from the accurate value.
- (3) According to the numerical examples in this paper, the natural modes have a little difference between a uniform beam and three laminated beams; the natural frequencies have, however, little between them.

1. Introduction

One of the problems which have been overlooked and are now awaiting its solution with the recent progress of the vehicle performance, is the wind-up vibration of the leaf springs for suspension. This vibration in a system consisting of a rigid axle and leaf springs created by the torque change during one cycle of an engine, causes sound and body oscillation, so that the counter-measures for it are now being researched in many quarters.

This paper deals with the natural frequencies and modes of a simple supported beam which is equivalent to a leaf spring with an added mass and an added moment of inertia.

There are two methods of the tip contact method and the spreading method for calculation of the load-deflection characteristics of a leaf spring. The tip contact method is based on the assumption that the forces are transmitted from a leaf to the adjoining one only at the tip of the leaf. The spreading method is based on the assumption that the adjoining leaves contact over the whole surface and the curvature of leaves at the same position is constant, accordingly a leaf spring is equivalent to a beam consisting of the leaves which are dissected and each half laid side by side in one plane as shown in Fig. 1.

The latter assumption is adopted in this paper.

2. Nomenclature

Fig. 1 shows a beam having n steps but it does not always mean that the number of leaves is n . For example, if there are two full length leaves, b_1 of Fig. 1 is twice as large as width of the main leaf and the total number of the laminated leaves is $n+1$.

In the following explanation, if the symbols need the distinction of left and right beam of the added mass in Fig. 1, an apostrophe is given on the shoulder of the symbol corresponding to the right beam.

x, y : coordinates

t : time

n : total number of steps

b_i : width of beam of the i th step counted

Fig. 1

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from the supporting point in Fig. 1

h : thickness of beam (assumed that it is constant through the steps)

$J_i = bh^3/12$: moment of inertia of area of the beam cross section of the i th step

$\kappa_{i-1} = J_{i-1}/J_i = b_{i-1}/b_i$

ρ : mass of the beam per unit of volume

$S_i = b_i h$: area of the cross section of the i th step beam

E : modulus of elasticity of the beam material

$a^2 = EJ_i/\rho S_i = Eh^2/12\rho = \text{constant}$

ω_j : natural circular frequency of the j th order

$\beta_j^4 = \omega_j^2/a^2$

l : total length of the leaf spring

l_i : the distance of the i th beam from the simply supported end

l_m : width of the added mass

l_a, l_b : length to left and right end of the beam from the center of the added mass, respectively

W : weight of the added mass

I : moment of inertia of body of the added mass with respect to its center axis

W_b : weight of the beam

I_b : moment of inertia of body of the beam with respect to the center axis of the added mass

$\lambda_i = l_i/l$

3. Frequency equation

For convenience the beam is divided into two parts at the center of the added mass, and x, y coordinates and x', y' coordinates are taken as shown in Fig. 2.

Disregarding the effect of the shearing force and the rotatory inertia of the beam, the differential equation of the vibrating beam may be written as

$$a^2 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0 \quad (1)$$

Assuming that the beam thickness h is constant, a is constant over the beam, too.

$$a^2 = \frac{EJ_i}{\rho S_i} = \frac{Eh^2}{12\rho} \quad (2)$$

Let

$$y_i(x, t) \text{ for } l_{i-1} < x < l_i \quad (i = 1, 2, 3, \dots, n)$$

be the solution of Eq. (1) for the i th step of the left hand of the added mass in Fig. 1, where $l_0 = 0$. Furthermore, let

$$\Theta_{i-1} \begin{pmatrix} A_i \\ B_i \\ C_i \\ D_i \end{pmatrix} = \begin{pmatrix} \sin \beta l_{i-1} & \cos \beta l_{i-1} \\ \cos \beta l_{i-1} & -\sin \beta l_{i-1} \\ -\kappa_{i-1} \sin \beta l_{i-1} & -\kappa_{i-1} \cos \beta l_{i-1} \\ -\kappa_{i-1} \cos \beta l_{i-1} & \kappa_{i-1} \sin \beta l_{i-1} \end{pmatrix}$$

where

Fig. 2

$$y_i(x, t) = Y_i(x) \sin \omega t \quad (3)$$

for the normal function $Y_i(x)$ of the solution $y_i(x, t)$. Substitution of Eq. (3) in Eq. (1) leads to the following equation

$$\frac{d^4 Y_i(x)}{dx^4} - \beta^4 Y_i(x) = 0 \quad (4)$$

where β is the so-called eigen value:

$$\beta^4 = \frac{\omega^2}{a^2} \quad (5)$$

The general solution of Eq. (4) is

$$Y_i(x) = A_i \sin \beta x + B_i \cos \beta x + C_i \sinh \beta x + D_i \cosh \beta x \quad (6)$$

where A_i, B_i, C_i , and D_i are arbitrary constants of which values are determined from the boundary conditions.

In a similar manner, the normal function $Y_i'(x)$ of the right beam of the added mass in Fig. 2 may be written as

$$Y_i'(x) = A_i' \sin \beta x' + B_i' \cos \beta x' + C_i' \sinh \beta x' + D_i' \cosh \beta x' \quad (7)$$

The following descriptions are about the normal function $Y_i(x)$ of the left beam.

First the boundary conditions of $Y_1(x)$ are

$$Y_1(0) = \frac{d^2}{dx^2} Y_1(0) = 0 \quad (8)$$

From Eqs. (6) and (8)

$$B_1 = D_1 = 0 \quad (9)$$

Since $Y_{i-1}(x)$ and $Y_i(x)$ must be continuous at $x = l_{i-1}$ ($i = 2, 3, \dots, n$), their boundary conditions are

$$\left. \begin{aligned} Y_{i-1}(l_{i-1}) &= Y_i(l_{i-1}), \\ \frac{d}{dx} Y_{i-1}(l_{i-1}) &= \frac{d}{dx} Y_i(l_{i-1}), \\ \kappa_{i-1} \frac{d^2}{dx^2} Y_{i-1}(l_{i-1}) &= \frac{d^2}{dx^2} Y_i(l_{i-1}), \\ \kappa_{i-1} \frac{d^3}{dx^3} Y_{i-1}(l_{i-1}) &= \frac{d^3}{dx^3} Y_i(l_{i-1}) \end{aligned} \right\} \quad (10)$$

where

$$\kappa_{i-1} = \frac{J_{i-1}}{J_i} = \frac{b_{i-1}}{b_i} \quad (11)$$

Substituting Eq. (6) into Eq. (10), we obtain

$$\begin{pmatrix} \sinh \beta l_{i-1} & \cosh \beta l_{i-1} \\ \cosh \beta l_{i-1} & \sinh \beta l_{i-1} \\ \kappa_{i-1} \sinh \beta l_{i-1} & \kappa_{i-1} \cosh \beta l_{i-1} \\ \kappa_{i-1} \cosh \beta l_{i-1} & \kappa_{i-1} \sinh \beta l_{i-1} \end{pmatrix} \begin{pmatrix} A_{i-1} \\ B_{i-1} \\ C_{i-1} \\ D_{i-1} \end{pmatrix} \quad (12)$$

$$\lambda_{i-1} = \frac{l_{i-1}}{l} \dots\dots\dots (13)$$

and, Θ_i is a square matrix with the following form:

$$\Theta_i = \begin{pmatrix} \sin \beta l \lambda_i & \cos \beta l \lambda_i & \sinh \beta l \lambda_i & \cosh \beta l \lambda_i \\ \cos \beta l \lambda_i & -\sin \beta l \lambda_i & \cosh \beta l \lambda_i & \sinh \beta l \lambda_i \\ -\sin \beta l \lambda_i & -\cos \beta l \lambda_i & \sinh \beta l \lambda_i & \cosh \beta l \lambda_i \\ -\cos \beta l \lambda_i & \sin \beta l \lambda_i & \cosh \beta l \lambda_i & \sinh \beta l \lambda_i \end{pmatrix} \dots\dots\dots (14)$$

Multiplying the inverse matrix of Θ_{i-1} by both sides of Eq. (12) leads to

$$\begin{pmatrix} A_i \\ B_i \\ C_i \\ D_i \end{pmatrix} = \frac{1}{2} A_{i-1} \begin{pmatrix} A_{i-1} \\ B_{i-1} \\ C_{i-1} \\ D_{i-1} \end{pmatrix} \quad (i=2, 3, \dots, n) \dots\dots\dots (15)$$

in which A_i is a transfer matrix with the following form:

$$A_i = \begin{pmatrix} (1+\kappa_i) & 0 & (1-\kappa_i)\zeta_1(\beta l \lambda_i) & (1-\kappa_i)\zeta_3(\beta l \lambda_i) \\ 0 & (1+\kappa_i) & -(1-\kappa_i)\zeta_4(\beta l \lambda_i) & -(1-\kappa_i)\zeta_2(\beta l \lambda_i) \\ -(1-\kappa_i)\zeta_2(\beta l \lambda_i) & -(1-\kappa_i)\zeta_3(\beta l \lambda_i) & (1+\kappa_i) & 0 \\ (1-\kappa_i)\zeta_4(\beta l \lambda_i) & (1-\kappa_i)\zeta_1(\beta l \lambda_i) & 0 & (1+\kappa_i) \end{pmatrix} \dots\dots\dots (16)$$

where

$$\left. \begin{aligned} \zeta_1(\beta l \lambda_i) &= \sin \beta l \lambda_i \sinh \beta l \lambda_i + \cos \beta l \lambda_i \cosh \beta l \lambda_i \\ \zeta_2(\beta l \lambda_i) &= \sin \beta l \lambda_i \sinh \beta l \lambda_i - \cos \beta l \lambda_i \cosh \beta l \lambda_i \\ \zeta_3(\beta l \lambda_i) &= \sin \beta l \lambda_i \cosh \beta l \lambda_i + \cos \beta l \lambda_i \sinh \beta l \lambda_i \\ \zeta_4(\beta l \lambda_i) &= \sin \beta l \lambda_i \cosh \beta l \lambda_i - \cos \beta l \lambda_i \sinh \beta l \lambda_i \end{aligned} \right\} \dots\dots\dots (17)$$

Since $B_1 = D_1 = 0$, putting

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \dots\dots\dots (18)$$

we get

$$\begin{pmatrix} A_i \\ B_i \\ C_i \\ D_i \end{pmatrix} = A_0 \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \dots\dots\dots (19)$$

The arbitrary constants A_i, B_i, C_i , and D_i are obtained successively as the functions of A_1 and B_1 from Eqs. (15) and (19), and they can finally be written as

$$\begin{pmatrix} A_i \\ B_i \\ C_i \\ D_i \end{pmatrix} = \frac{1}{2^{i-1}} \Gamma_i \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \dots\dots\dots (20)$$

Γ_i is also a transfer matrix which has four rows and two columns, and is expressed by

$$\Gamma_i = A_{i-1} A_{i-2} \dots A_1 A_0 \dots\dots\dots (21)$$

For the case of $n=1, 2$, and 3 , the concrete forms of Γ_i are given as follows respectively:

$$\Gamma_1 = A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \dots\dots\dots (22)$$

$$\begin{aligned} \Gamma_2 &= A_1 \Gamma_1 \\ &= \begin{pmatrix} (1+\kappa_1) & (1-\kappa_1)\zeta_1(\beta l \lambda_1) \\ 0 & -(1-\kappa_1)\zeta_4(\beta l \lambda_1) \\ -(1-\kappa_1)\zeta_2(\beta l \lambda_1) & (1+\kappa_1) \\ (1-\kappa_1)\zeta_4(\beta l \lambda_1) & 0 \end{pmatrix} \dots\dots\dots (23) \end{aligned}$$

$$\Gamma = A_2 \Gamma_2 = \begin{pmatrix} (1+\kappa_1)(1+\kappa_2) + (1-\kappa_1)(1-\kappa_2) & (1-\kappa_1)(1+\kappa_2)\zeta_1(\beta l \lambda_1) \\ \times \{\zeta_4(\beta l \lambda_1)\zeta_3(\beta l \lambda_2) - \zeta_2(\beta l \lambda_1)\zeta_1(\beta l \lambda_2)\} & + (1+\kappa_1)(1-\kappa_2)\zeta_1(\beta l \lambda_2) \\ (1-\kappa_1)(1-\kappa_2)\{\zeta_2(\beta l \lambda_1)\zeta_4(\beta l \lambda_2) & - (1-\kappa_1)(1+\kappa_2)\zeta_4(\beta l \lambda_1) \\ - \zeta_4(\beta l \lambda_1)\zeta_2(\beta l \lambda_2)\} & - (1+\kappa_1)(1-\kappa_2)\zeta_4(\beta l \lambda_2) \\ - (1-\kappa_1)(1+\kappa_2)\zeta_2(\beta l \lambda_1) - (1+\kappa_1) & (1+\kappa_1)(1+\kappa_2) + (1-\kappa_1)(1-\kappa_2) \\ \times (1-\kappa_2)\zeta_2(\beta l \lambda_2) & \times \{\zeta_4(\beta l \lambda_1)\zeta_3(\beta l \lambda_2) - \zeta_1(\beta l \lambda_1)\zeta_2(\beta l \lambda_2)\} \\ (1-\kappa_1)(1+\kappa_2)\zeta_4(\beta l \lambda_1) + (1+\kappa_1) & (1-\kappa_1)(1-\kappa_2) \\ \times (1-\kappa_2)\zeta_4(\beta l \lambda_2) & \{\zeta_1(\beta l \lambda_1)\zeta_4(\beta l \lambda_2) - \zeta_4(\beta l \lambda_1)\zeta_1(\beta l \lambda_2)\} \end{pmatrix} \dots (24)$$

Now from Eq. (16)

$$\begin{pmatrix} Y_n(l_n) \\ \frac{1}{\beta} \frac{d}{dx} Y_n(l_n) \\ \frac{1}{\beta^2} \frac{d^2}{dx^2} Y_n(l_n) \\ \frac{1}{\beta^3} \frac{d^3}{dx^3} Y_n(l_n) \end{pmatrix} = \Theta_n \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} \dots (25)$$

in which Θ_n is the square matrix obtained by taking $i=n$ in Eq. (14).

Substituting Eq. (20) with $i=n$ into Eq. (25), we get

$$\begin{pmatrix} Y_n(l_n) \\ \frac{1}{\beta} \frac{d}{dx} Y_n(l_n) \\ \frac{1}{\beta^2} \frac{d^2}{dx^2} Y_n(l_n) \\ \frac{1}{\beta^3} \frac{d^3}{dx^3} Y_n(l_n) \end{pmatrix} = \frac{1}{2^{n-1}} \begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \\ M_3 & N_3 \\ M_4 & N_4 \end{pmatrix} \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} \dots (26)$$

where

$$\begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \\ M_3 & N_3 \\ M_4 & N_4 \end{pmatrix} = \Theta_n \Gamma_n = \Theta_n (A_{n-1} A_{n-2} \dots A_1 A_0) \dots (27)$$

The above equations are used for the left beam of the added mass in Fig. 2, and the same equations as these are obtained for the right beam only by adding an apostrophe on the right shoulder of $Y_i(x)$, A_i , λ_i , Θ_i , A_i , Γ_i , M_i , N_i etc. in Eqs. (8) to (27).

The boundary conditions at the attached points of the added mass ($x=l_n$ on the left beam, $x=l_n'$ on the right beam) are as follows :

$$\left. \begin{aligned} Y_n(l_n) + l_n \frac{d}{dx} Y_n(l_n) - Y_n'(l_n') &= 0, & \frac{d}{dx} Y_n(l_n) + \frac{d}{dx} Y_n'(l_n') &= 0 \\ \frac{d^2}{dx^2} Y_n(l_n) - \frac{d^2}{dx^2} Y_n'(l_n') - \frac{I\omega^2}{EJ_n} \frac{d}{dx} Y_n(l_n) &= 0, \\ \frac{d^3}{dx^3} Y_n(l_n) + \frac{d^3}{dx^3} Y_n'(l_n') + \frac{W\omega^2}{gEJ_n} \frac{1}{2} \{Y_n(l_n) + Y_n'(l_n')\} &= 0 \end{aligned} \right\} \dots (28)$$

Substituting Eq. (26) and the equation of the right beam corresponding to Eq. (26) into Eq. (28), we get

$$\begin{pmatrix} M_1 + \beta l \lambda_m M_2 & N_1 + \beta l \lambda_m N_2 & -M_1' & -N_1' \\ M_2 & N_2 & M_2' & N_2' \\ M_3 - R_I(I/I_b)(\beta l)^3 M_2 & N_3 - R_I(I/I_b)(\beta l)^3 N_2 & -M_3' & -N_3' \\ M_4 + R_W(W/W_b) & N_4 + R_W(W/W_b) & M_4' + R_W(W/W_b) & N_4' + R_W(W/W_b) \\ \times (\beta l) M_1/2 & \times (\beta l) N_1/2 & \times (\beta l) M_1'/2 & \times (\beta l) N_1'/2 \end{pmatrix} \begin{pmatrix} A_1 \\ C_1 \\ A_1' \\ C_1' \end{pmatrix} = 0 \dots (29)$$

where

$$\lambda_m = l_m/l \dots (30)$$

$$\left. \begin{aligned} R_W &= \frac{1}{b_n l} \left\{ \sum_{i=1}^n b_i (l_i - l_{i-1}) + \sum_{i=1}^n b_i' (l_i' - l_{i-1}') \right\} \\ R_I &= \frac{1}{3b_n l^3} \left\{ \sum_{i=1}^n (b_i - b_{i-1}) (l_i - l_{i-1})^3 + \sum_{i=1}^n (b_i' - b_{i-1}') (l_i' - l_{i-1}')^3 - (b_n + b_n') \left(\frac{l_m}{2} \right)^3 \right\} \\ W_b &= \rho g b_n h l R_W, \quad I_b = \rho b_n h l^3 R_I \end{aligned} \right\} \dots\dots\dots (31)$$

in which $b_0 = b_0' = l_0 = l_0' = 0$

From Eq. (29) the frequency equation is expressed with the following determinant of the fourth order regardless of the value of steps n :

$$\begin{vmatrix} M_1 + \beta l \lambda_m M_2 & N_1 + \beta l \lambda_m N_2 \\ M_2 & N_2 \\ M_3 - R_I (I/I_b) (\beta l)^3 M_2 & N_3 - R_I (I/I_b) (\beta l)^3 N_2 \\ M_4 + R_W (W/W_b) (\beta l) M_1/2 & N_4 + R_W (W/W_b) (\beta l) N_1/2 \\ -M_1' & -N_1' \\ M_2' & N_2' \\ -M_3' & -N_3' \\ M_4' + R_W (W/W_b) (\beta l) M_1'/2 & N_4' + R_W (W/W_b) (\beta l) N_1'/2 \end{vmatrix} = 0 \dots\dots\dots (32)$$

Eq. (32) is a transcendental equation, as M_1, M_2, N_1, N_2 , etc. include the trigonometrical and hyperbolic functions of βl . Accordingly, since it is difficult in general to get the eigen value βl from the given value of W/W_b and I/I_b , in the numerical calculation we find the values of βl which make either the value of W/W_b or I/I_b positive when the other is given.

4. Case of $n=1$

4.1 Derivation of formulas

$n=1$ means that the cross section of the beam is uniform. In this case, substituting Eqs. (14) and (18) in Eq. (27), we find

$$\begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \\ M_3 & N_3 \\ M_4 & N_4 \end{pmatrix} = \begin{pmatrix} \sin \beta l \lambda_1 & \sinh \beta l \lambda_1 \\ \cos \beta l \lambda_1 & \cosh \beta l \lambda_1 \\ -\sin \beta l \lambda_1 & \sinh \beta l \lambda_1 \\ -\cos \beta l \lambda_1 & \cosh \beta l \lambda_1 \end{pmatrix} \dots\dots\dots (33)$$

Expanding the determinant of Eq. (32) after substitution of Eq. (33) and the equation of the right beam corresponding to Eq. (33) into Eq. (32), we obtain the frequency equation

$$\begin{aligned} & 2[2 \sin \beta l (\lambda_1 + \lambda_1') \sinh \beta l (\lambda_1 + \lambda_1') + \beta l \lambda_m \{ \sin \beta l (\lambda_1 + \lambda_1') \cosh \beta l \lambda_1 \cosh \beta l \lambda_1' \\ & + \cos \beta l \lambda_1 \cos \beta l \lambda_1' \sinh \beta l (\lambda_1 + \lambda_1') \}] - \frac{W}{W_b} R_W \beta l [2 \{ \sin \beta l \lambda_1 \sin \beta l \lambda_1' \sinh \beta l (\lambda_1 + \lambda_1') \\ & - \sin \beta l (\lambda_1 + \lambda_1') \sinh \beta l \lambda_1 \sinh \beta l \lambda_1' \} + \beta l \lambda_m \{ \sin \beta l \lambda_1 \sin \beta l \lambda_1' \cosh \beta l \lambda_1 \cosh \beta l \lambda_1' \\ & - \cos \beta l \lambda_1 \cos \beta l \lambda_1' \sinh \beta l \lambda_1 \sinh \beta l \lambda_1' \}] - 2 \frac{I}{I_b} R_I (\beta l)^3 \{ \sin \beta l (\lambda_1 + \lambda_1') \cosh \beta l \lambda_1 \cosh \beta l \lambda_1' \\ & - \cos \beta l \lambda_1 \cos \beta l \lambda_1' \sinh \beta l (\lambda_1 + \lambda_1') \} + \frac{W}{W_b} \frac{I}{I_b} R_W R_I (\beta l)^4 \zeta_4(\beta l \lambda_1) \zeta_4(\beta l \lambda_1') = 0 \dots\dots\dots (34) \end{aligned}$$

In the special case of $\lambda_1 = \lambda_1'$, i.e., in the case of the beam and the added mass of symmetry with respect to the center of the beam, Eq. (34) can be solved into factors as follows:

$$\begin{aligned} & \sin 2\beta l \lambda_1 \sinh 2\beta l \lambda_1 \left\{ 4 + \beta l \lambda_m (\coth \beta l \lambda_1 + \cot \beta l \lambda_1) \right\} \\ & - \frac{I}{I_b} R_I (\beta l)^3 (\coth \beta l \lambda_1 - \cot \beta l \lambda_1) \left[1 - \frac{1}{4} \frac{W}{W_b} R_W \beta l (\tan \beta l \lambda_1 - \tanh \beta l \lambda_1) \right] = 0 \dots\dots\dots (35) \end{aligned}$$

It will be seen that there is no coupling between vertical and angular vibration.

Ignoring the width l_m of the added mass in addition to the assumption of symmetry, Eq. (35) becomes

$$\sin \beta l \sinh \beta l \left\{ 4 - \frac{1}{12} \frac{I}{I_b} (\beta l)^3 \left(\coth \frac{\beta l}{2} - \cot \frac{\beta l}{2} \right) \right\} \left\{ 1 - \frac{1}{4} \frac{W}{W_b} \beta l \left(\tan \frac{\beta l}{2} - \tanh \frac{\beta l}{2} \right) \right\} = 0 \dots\dots\dots (36)$$

The fundamental natural frequency ω_1 of a simple supported beam with a concentrated mass at its center may be, as well known, obtained approximately from the term including W/W_b in Eq. (36) as

$$\omega_1^2 = \frac{48 E J g}{l^3 W \{ 1 + 0.486 (W_b/W) + 0.003 (W_b/W)^2 \}} \dots\dots\dots (37)$$

which means that the value of ω_1 is obtained by adding about a half of the beam weight to the weight of the concentrated mass in the known formula derived from ignoring the beam weight.

Let us now find the approximate formula of the fundamental natural frequency of a beam with a concentrated moment of inertia of body. From the term including I/I_b in Eq. (36)

$$\frac{I_b}{I} = \frac{1}{6} \left(\frac{\beta l}{2} \right)^3 \left(\coth \frac{\beta l}{2} - \cot \frac{\beta l}{2} \right) \dots \dots \dots (38)$$

Putting $\xi = \beta l/2$, $\xi < \pi$ for the fundamental frequency and expanding Eq. (38) after substitution of ξ in Eq. (38) leads to

$$9 \frac{I_b}{I} = \xi^4 + \frac{2}{315} \xi^8 + \frac{2}{31185} \xi^{12} + \dots \dots \dots (39)$$

Setting $9I_b/I = \eta$

and $\xi^4 = \eta + a\eta^2 + b\eta^3 + \dots$

and substituting these in Eq. (39), we get

$$\eta = \eta + \left(a + \frac{2}{315} \right) \eta^2 + \left(b + \frac{4}{315} a + \frac{2}{31185} \right) \eta^3 + \dots$$

Equating the coefficients of terms of the same order with respect to η , the values of a , b , etc. are obtained. Then

$$\xi^4 = \frac{\beta_1^4 l^4}{16} = 9 \frac{I_b}{I} \left\{ 1 - \frac{2}{35} \left(\frac{I_b}{I} \right) + \frac{118}{13475} \left(\frac{I_b}{I} \right)^2 - \dots \right\}$$

Meanwhile from definition it follows that

$$\beta_1^4 = \frac{\omega_1^2}{a^2} = \omega_1^2 \frac{\rho S}{EJ} = \omega_1^2 \frac{12I_b}{EJl^3}$$

Finally we get the approximate formula for ω_1 as follows:

$$\omega_1^2 = \frac{12EJ}{II} \left\{ 1 - \frac{2}{35} \left(\frac{I_b}{I} \right) + \frac{118}{13475} \left(\frac{I_b}{I} \right)^2 - \dots \right\} \doteq \frac{12EJ}{II \{ 1 + 0.05714(I_b/I) - 0.00550(I_b/I)^2 \}} \dots \dots \dots (40)$$

In other words: the fundamental natural frequency can be obtained by adding about five per cent of moment of inertia I to moment of inertia of the beam with respect to the center in the formula $\omega_1^2 = \sqrt{12EJ/II}$ which is derived from disregarding moment of inertia of the beam.

Next let us find the normal modes of vibration of the beam. Substituting the eigen value $\beta_j l$ obtained from Eq. (34) into the first to third equations of Eq. (29), we have

$$\left. \begin{aligned} \frac{C_{1j}}{A_{1j}} &= - \frac{2 \sin \beta_j l (\lambda_1 + \lambda_1') \sinh \beta_j l \lambda_1' + \cosh \beta_j l \lambda_1 \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1') - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1') \}}{2 \sin \beta_j l \lambda_1' \sinh \beta_j l (\lambda_1 + \lambda_1') + \cosh \beta_j l \lambda_1 \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1') - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1') \}} \\ \frac{C_{1j}'}{A_{1j}'} &= - \frac{2 \sin \beta_j l (\lambda_1 + \lambda_1') \sinh \beta_j l \lambda_1 + \cosh \beta_j l \lambda_1' \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1) - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1) \}}{2 \sin \beta_j l \lambda_1 \sinh \beta_j l (\lambda_1 + \lambda_1') + \cosh \beta_j l \lambda_1' \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1) - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1) \}} \\ \frac{A_{1j}'}{A_{1j}} &= \frac{2 \sin \beta_j l \lambda_1 \sinh \beta_j l (\lambda_1 + \lambda_1') + \cosh \beta_j l \lambda_1' \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1) - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1) \}}{2 \sin \beta_j l \lambda_1' \sinh \beta_j l (\lambda_1 + \lambda_1') + \cosh \beta_j l \lambda_1 \{ \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1') - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1') \}} \end{aligned} \right\} \dots \dots \dots (41)$$

From these the j th natural modes of vibration are

$$\left. \begin{aligned} \frac{Y_{1j}(x)}{A_{1j}} &= \sin \beta_j x + \frac{C_{1j}}{A_{1j}} \sinh \beta_j x \\ \frac{Y_{1j}'(x)}{A_{1j}'} &= \frac{A_{1j}'}{A_{1j}} \left(\sin \beta_j x + \frac{C_{1j}'}{A_{1j}'} \sinh \beta_j x \right) \end{aligned} \right\} \dots \dots \dots (42)$$

For the case of symmetry, however, the natural modes are not obtained only by putting $\lambda_1 = \lambda_1'$ in Eq. (41), and it is necessary for us to use three proper equations of Eq. (29). For example, if we put $\lambda_1 = \lambda_1'$, C_{1j}/A_{1j} in Eq. (41), the first equation becomes

$$\frac{C_{1j}}{A_{1j}} = - \frac{\cos \beta_j l \lambda_1 \{ 4 \sin \beta_j l \lambda_1 \sinh \beta_j l \lambda_1 + \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1) - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1) \}}{\cosh \beta_j l \lambda_1 \{ 4 \sin \beta_j l \lambda_1 \sinh \beta_j l \lambda_1 + \beta_j l \lambda_m \zeta_3(\beta_j l \lambda_1) - (I/I_b) R_I(\beta_j l)^3 \zeta_4(\beta_j l \lambda_1) \}} \dots \dots \dots (43)$$

The contents in the brackets, { }, of numerator and denominator in Eq. (43) are one and the same equation, and they become zero if the second factor of Eq. (35) is applied to them. It means that Eq. (35) does not express the natural mode of angular vibration. If β_j makes, however, the third factor of Eq. (35) zero, i.e., for the vertical natural vibration, the common factors of numerator and denominator in Eq. (43) do not become zero. Then reducing the common factors yields

$$\frac{C_{1j}}{A_{1j}} = -\frac{\cos \beta_j l \lambda_1}{\cosh \beta_j l \lambda_1} \quad (44)$$

In the same way, we get the following equations from the second and third equations of Eq. (41) respectively:

$$\frac{C_{1j}'}{A_{1j}'} = -\frac{\cos \beta_j l \lambda_1}{\cosh \beta_j l \lambda_1}, \quad \frac{A_{1j}'}{A_{1j}} = 1 \quad (45)$$

Hence $Y_{1j}(x) = Y_{1j}'(x)$. From which we see that, in the case of a symmetrical beam, the natural modes of vertical vibration are symmetrical with respect to the perpendicular at the mid-point of the line connecting both ends of the beam.

The natural modes of angular vibration are obtained by putting $\lambda_1 = \lambda_1'$ in the first, second, and fourth equations of a set of four linear equations which are obtained by substituting Eq. (33) in Eq. (29). But in fact the modes can be obtained easily without such troublesome means, as follows. In the case of a symmetrical beam, the natural modes of angular vibration may be symmetrical with respect to the focus of the line connecting both ends of the beam, viz., $Y_{1j}(x) = -Y_{1j}'(x)$. Hence putting $A_{1j} = -A_{1j}'$ and $C_{1j} = -C_{1j}'$ in the first equation of a set of four linear equations mentioned above, we get

$$\left. \begin{aligned} \frac{C_{1j}}{A_{1j}} &= \frac{C_{1j}'}{A_{1j}'} \\ &= -\frac{2 \sin \beta_j l \lambda_1 + \beta_j l \lambda_m \cos \beta_j l \lambda_1}{2 \sinh \beta_j l \lambda_1 + \beta_j l \lambda_m \cosh \beta_j l \lambda_1}, \\ \frac{A_{1j}'}{A_{1j}} &= -1 \end{aligned} \right\} \quad (46)$$

4.2 Numerical examples

The eigen values of vertical and angular vibration for symmetrical beam are given up to the

Fig. 3

fourth order in Fig. 3. The full lines in this show the case of $\lambda_m = l_m/l = 0.05$, and the dotted lines show the case of ignoring the width of the added mass. The higher the order of vibration is, the more both lines separate from each other. For instance, for the fourth order of vertical vibration and when $W/W_b = 25$, $\beta_4 l = 21.50$ if the width of the added mass is considered, while 20.43 if it is ignored. Since the natural frequency is in proportion to β^2 from Eq. (5), the difference of both frequencies reaches about 11.1 per cent. There is little difference between vertical and angular vibration for higher frequencies than the second order.

Fig. 4 gives the natural mode shapes of vertical vibration of the symmetrical beam for $W/W_b = 25$. The added mass scarcely moves in the vibration with a higher natural frequency. If we disregard the width of the added mass, one of nodal points never fails to arise at the mid-point of the beam, but there is no such contradiction so far as the width of the added mass is given consideration. The amplitude takes the largest value at the upper or lower peak always next the added mass.

The natural mode shapes of the angular vibration of the symmetrical beam for $I/I_b = 1$ are shown in Fig. 5. Also in this case, one of nodal points arises at the mid-point of the beam, if the width of the added mass is not considered.

As it is seen in the above examples, we should not disregard the width of the added mass especially for the vibration with a higher frequency, unless it is very small compared with the total length of the beam.

As an example of asymmetry, the eigen values for $\lambda_a = 0.3$ and 0.4 and $\lambda_m = 0.05$ are given in Fig.

Fig. 4

6. The natural mode shapes for this case are shown in Fig. 7. In the higher vibration than the third order, one hand of the beam divided in two by the added mass hardly moves when the other hand vibrates severely. The mode shapes of the same order for the case of $\lambda_a=0.3$ and 0.4 are not always similar to each other. For instance, the fourth vibration of $\lambda_a=0.3$ correspond to the fifth of $\lambda_a=0.4$.

5. Case of $n \geq 2$

5.1 Derivation of formulas

Expanding the determinant of left side in Eq. (32), we have

$$\begin{aligned}
 & (M_1 N_2 - M_2 N_1)(M_3' N_4' - M_4' N_3') + (M_1 N_3 - M_3 N_1)(M_2' N_4' - M_4' N_2') \\
 & + (M_1 N_4 - M_4 N_1)(M_2' N_3' - M_3' N_2') + (M_2 N_3 - M_3 N_2)(M_1' N_4' - M_4' N_1') \\
 & + (M_2 N_4 - M_4 N_2)(M_1' N_3' - M_3' N_1') + (M_3 N_4 - M_4 N_3)(M_1' N_2' - M_2' N_1') \\
 & + \beta l \lambda_m \{ (M_2 N_3 - M_3 N_2)(M_2' N_4' - M_4' N_2') + (M_2 N_4 - M_4 N_2)(M_2' N_3' - M_3' N_2') \} \\
 & - \frac{W}{W_b} R_w \beta l \left[\{ (M_1 N_2 - M_2 N_1)(M_1' N_3' - M_3' N_1') + (M_1 N_3 - M_3 N_1)(M_1' N_2' - M_2' N_1') \} \right. \\
 & \quad \left. + \frac{1}{2} \beta l \lambda_m \{ (M_1 N_2 - M_2 N_1)(M_2' N_3' - M_3' N_2') \right. \\
 & \quad \left. + (M_2 N_3 - M_3 N_2)(M_1' N_2' - M_2' N_1') \} \right] \\
 & - \frac{I}{I_b} R_f (\beta l)^3 \{ (M_1 N_2 - M_2 N_1)(M_2' N_4' - M_4' N_2') \\
 & \quad + (M_2 N_4 - M_4 N_2)(M_1' N_2' - M_2' N_1') \} \\
 & + \frac{W}{W_b} \frac{I}{I_b} R_w R_f (\beta l)^4 (M_1 N_2 - M_2 N_1) \\
 & \quad \times (M_1' N_2' - M_2' N_1') = 0 \dots\dots\dots (47)
 \end{aligned}$$

In the special case of a symmetrical beam, since $M_i = M_i'$, $N_i = N_i'$ ($i=1, 2, 3, 4$), Eq. (47) can be solved into factors in the same manner as the case of $n=1$. But in reality the vertical

Fig. 5

Fig. 6

Fig. 7

and angular natural frequency are found simply from the following equation derived by putting $I/I_b=0$ and $W/W_b=0$ in Eq. (47), respectively:

For vertical vibration

$$(M_1N_2-M_2N_1)(M_3N_4-M_4N_3)+(M_1N_3-M_3N_1)(M_2N_4-M_4N_2) \\ + (M_1N_4-M_4N_1)(M_2N_3-M_3N_2)+\beta l\lambda_m(M_2N_3-M_3N_2)(M_2N_4-M_4N_2) \\ - \frac{W}{W_b}R_W\beta l\left\{(M_1N_2-M_2N_1)(M_1N_3-M_3N_1)+\frac{1}{2}\beta l\lambda_m(M_1N_2-M_2N_1)(M_2N_3-M_3N_2)\right\}=0 \dots\dots\dots(48)$$

and for angular vibration

$$(M_1N_2-M_2N_1)(M_3N_4-M_4N_3)+(M_1N_3-M_3N_1)(M_2N_4-M_4N_2) \\ + (M_1N_4-M_4N_1)(M_2N_3-M_3N_2)+\beta l\lambda_m(M_2N_3-M_3N_2)(M_2N_4-M_4N_2) \\ - \frac{I}{I_b}R_I(\beta l)^3(M_1N_2-M_2N_1)(M_2N_4-M_4N_2)=0 \dots\dots\dots(49)$$

Let us next get the natural modes of vibration. From the first, second, and third equations of Eq. (29), three equations corresponding to Eq. (41) are obtained for the eigen values $\beta_j l$ settled by Eq. (47) as follows:

$$\left. \begin{aligned} \frac{C_{1j}}{A_{1j}} &= - \frac{(M_{1j}+\beta_j l\lambda_m M_{2j})(M_{2j}'N_{3j}'-M_{3j}'N_{2j}')-M_{2j}(M_{3j}'N_{1j}'-M_{1j}'N_{3j}')}{(N_{1j}+\beta_j l\lambda_m N_{2j})(M_{2j}'N_{3j}'-M_{3j}'N_{2j}')-N_{2j}(M_{3j}'N_{1j}'-M_{1j}'N_{3j}') \\ &\quad + \{M_{3j}'-(I/I_b)R_I(\beta_j l)^3 M_{2j}'\}(M_{1j}'N_{2j}'-M_{2j}'N_{1j}')} \\ \frac{C_{1j}'}{A_{1j}'} &= - \frac{(M_{1j}'+\beta_j l\lambda_m M_{2j}') (M_{2j}N_{3j}-M_{3j}N_{2j})-M_{2j}'(M_{3j}N_{1j}-M_{1j}N_{3j})}{(N_{1j}'+\beta_j l\lambda_m N_{2j}') (M_{2j}N_{3j}-M_{3j}N_{2j})-N_{2j}'(M_{3j}N_{1j}-M_{1j}N_{3j}) \\ &\quad + \{M_{3j}'-(I/I_b)R_I(\beta_j l)^3 M_{2j}'\}(M_{1j}N_{2j}-M_{2j}N_{1j})} \\ \frac{A_{1j}'}{A_{1j}} &= \frac{(N_{1j}'+\beta_j l\lambda_m N_{2j}') (M_{2j}N_{3j}-M_{3j}N_{2j})-N_{2j}'(M_{3j}N_{1j}-M_{1j}N_{3j})}{(N_{1j}+\beta_j l\lambda_m N_{2j})(M_{2j}'N_{3j}'-M_{3j}'N_{2j}')-N_{2j}(M_{3j}'N_{1j}'-M_{1j}'N_{3j}') \\ &\quad + \{N_{3j}'-(I/I_b)R_I(\beta_j l)^3 N_{2j}'\}(M_{1j}N_{2j}-M_{2j}N_{1j})} \end{aligned} \right\} \dots\dots\dots(50)$$

From Eq. (20) we have

$$\begin{pmatrix} A_{ij}/A_{1j} \\ B_{ij}/A_{1j} \\ C_{ij}/A_{1j} \\ D_{ij}/A_{1j} \end{pmatrix} = \frac{1}{2^{i-1}} \Gamma_{ij} \begin{pmatrix} 1 \\ C_{1j}/A_{1j} \end{pmatrix} \dots\dots\dots(51)$$

Then, the mode shapes of the j th order are obtained from the following equations:

$$\left. \begin{aligned} 0 \leq x \leq l_1 : \frac{Y_{1j}(x)}{A_{1j}} &= \sin \beta_j x + \frac{C_{1j}}{A_{1j}} \sinh \beta_j x \\ l_{i-1} \leq x \leq l_i : \frac{Y_{ij}(x)}{A_{1j}} &= \frac{A_{ij}}{A_{1j}} \sin \beta_j x + \frac{B_{ij}}{A_{1j}} \cos \beta_j x + \frac{C_{ij}}{A_{1j}} \sinh \beta_j x + \frac{D_{ij}}{A_{1j}} \cosh \beta_j x \\ 0 \leq x \leq l_1' : \frac{Y_{1j}'(x)}{A_{1j}'} &= \frac{A_{1j}'}{A_{1j}} \left(\sin \beta_j x + \frac{C_{1j}'}{A_{1j}'} \sinh \beta_j x \right) \\ l_{i-1}' \leq x \leq l_i' : \frac{Y_{ij}'(x)}{A_{1j}'} &= \frac{A_{1j}'}{A_{1j}} \left(\frac{A_{ij}'}{A_{1j}'} \sin \beta_j x + \frac{B_{ij}'}{A_{1j}'} \cos \beta_j x + \frac{C_{ij}'}{A_{1j}'} \sinh \beta_j x + \frac{D_{ij}'}{A_{1j}'} \cosh \beta_j x \right) \end{aligned} \right\} \dots\dots\dots(52)$$

If the system is symmetrical with respect to the mid-point of the beam, the natural modes of vertical vibration may be symmetrical in regard to the perpendicular of the line connecting both ends of the beam, as discussed in the case of $n=1$. Hence putting $M_{1j}=M_{1j}'$, $N_{1j}=N_{1j}'$, $A_{1j}=A_{1j}'$, and $C_{1j}=C_{1j}'$ in the first equation of Eq. (29), we find

$$\frac{C_{1j}}{A_{1j}} = \frac{C_{1j}'}{A_{1j}'} = -\frac{M_{2j}}{N_{2j}}, \quad \frac{A_{1j}'}{A_{1j}} = 1 \dots\dots\dots(53)$$

Meanwhile, since the natural modes of angular vibration may be symmetrical with respect to the mid-point of the line connecting both ends of the beam, putting $M_{1j}=M_{1j}'$, $N_{1j}=N_{1j}'$, $A_{1j}=-A_{1j}'$, and $C_{1j}=-C_{1j}'$ in the first equation of Eq. (29), we find

$$\left. \begin{aligned} \frac{C_{1j}}{A_{1j}} = \frac{C_{1j}'}{A_{1j}'} &= -\frac{2M_{1j}+\beta_j l\lambda_m M_{2j}}{2N_{1j}+\beta_j l\lambda_m N_{2j}}, \\ \frac{A_{1j}'}{A_{1j}} &= -1 \end{aligned} \right\} \dots\dots\dots(54)$$

The j th mode shape for the case of the symmetrical beam can be obtained by Eq. (53) or (54) and Eqs. (51) and (52).

5.2 Numerical examples

The calculations are executed for the following beam:

$$\begin{aligned} n=3, \quad \lambda_m=0.05, \quad b_1=b_1', \quad b_3=b_3'=3b_1, \\ b_2=b_2'=2b_1, \quad l_3=3l_1, \quad l_2=2l_1, \quad l_3'=3l_1', \\ l_2'=2l_1' \end{aligned}$$

First the natural frequencies of the symmetrical beam are given in Fig. 8. As compared with Fig. 3, it will be seen that there is little difference in vertical and angular frequencies between the cases of $n=1$ and $n=3$.

Fig. 9 gives the natural frequencies for the case of $\lambda_a=0.4$. Also in this case, if Fig. 9 is compared with the Fig. 6, both frequencies are nearly equal except a little difference in the second and third frequencies between them.

The mode shapes for the case of $\lambda_a=0.4$, $W/W_b=25$, and $I/I_b=1$ are shown in Fig. 10. As compared with Fig. 7, both shapes closely resemble each other, but when $n=1$ the largest amplitude arises at the upper or lower peak next the added mass, while when $n=3$ it arises at the upper or lower peak next the supported end.

6. Conclusions

(1) The formulas of the natural frequencies and modes were found for the beam equivalent to a leaf spring with an added mass and an added moment of inertia.

(2) There is no coupling between vertical and angular vibration in the case of the beam of symmetry with respect to the center of the added

mass.

(3) When the width of the added moment of inertia I is not considered, the fundamental frequency of the angular vibration of a symmetrical beam with a uniform cross section is obtained approximately by adding about 0.05 per cent of I_b to I in the conventional formula $\omega_1^2=12EJ/II$ which is derived with disregard of I_b .

Numerical examples are calculated by the digital computer FACOM 138A.

Acknowledgement

The author wishes to express his appreciation

Fig. 9

Fig. 10

Fig. 8

to Mr. A. Watari, Professor of Engineering Mechanics, University of Tokyo, for suggesting the subject and for valuable help offered by him in its

development. The author is also grateful to Mr. Y. Takenouchi and Mr. I. Saito for their assistance in the computation.

Discussion

M. MIZUNO: (1) The laminated leaf springs for vehicles are usually considered as cantilever springs of trapezoidal profile for the convenience of analysis*, instead of the profile shown in Fig. 1 of this paper. In view of this profile, I think it might be better to treat these problems by the method used by T. B. Sato**.

T. KAWAI: (2) Please indicate the effect of idealization of the boundary condition (tension shackle) as well as that of the initial curvature of composite plate springs on the derived solution.

Extension of the solution to coupled vibration of the springs and a wheel-driving shaft will be expected.

Author's closure

(1) Replacing the variables according to the indicated thesis in the case of a leaf spring with the trapezoidal profile shown in Append-Fig. 1, we get the following equation as a general solution to the problem of the vibration of beam:

$$\begin{aligned} \eta = & A_0 F_3\left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}; \xi\right) \\ & + B \xi^{1/2} F_3\left(\frac{3}{2}, \frac{5}{4}, \frac{5}{4}; \xi\right) \\ & + C(1 + \log \xi) \xi^{1/4} F_3\left(\frac{5}{4}, 1, \frac{3}{4}; \xi\right) \\ & + D \xi^{1/4} \sum_{m=1}^{\infty} \left\{ \frac{1}{\left(\frac{5}{4}\right)_m (1)_m \left(\frac{3}{4}\right)_m} \right. \\ & \times \left. \frac{\xi_m}{m!} \sum_{r=1}^m \frac{32r^2 - 1}{\left(\frac{5}{4} + r - 1\right) r \left(\frac{3}{4} + r - 1\right)} \right\} \end{aligned} \quad \dots\dots\dots (a)$$

where A, B, C and D are arbitrary constants, and

$$\eta = lY, \quad \xi = \frac{1}{4^4} \frac{\rho S_n \omega^2}{E J_n c^4} (1 + cx)^4$$

$$c = \frac{(b_1/b_n) - 1}{l_n}, \quad J_n = \frac{b_n h^3}{12}, \quad S_n = b_n h$$

h : thickness

$$\begin{aligned} (\alpha)_m &= \alpha(\alpha+1)(\alpha+2)\dots(\alpha+m-1) \\ &= \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \quad (\alpha)_0 = 1 \end{aligned}$$

* A.M. Wahl: *Mechanical Springs*, (1963), p.179, McGraw-Hill.

** T.B. Sato: *Centenary Memorial Publication*, (Faculty of Engng. Keio University 1958), p.83, Appendix II.

Append-Fig. 1

$$\begin{aligned} & {}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; \xi) \\ &= \sum_{m=0}^{\infty} \frac{(\alpha_1)_m (\alpha_2)_m \dots (\alpha_p)_m \xi^m}{(\beta_1)_m (\beta_2)_m \dots (\beta_q)_m m!} \end{aligned}$$

As Eq. (a) may be a solution for the beams of either side in Append-Fig. 1, we take A, B, C , and D as the constants of the left beam, and A', B', C' , and D' as the constants of the right beam. Then, substituting both solutions in Eqs. (8) and (28), we have a set of eight linear equations with respect to the constants $A \sim D$ and $A' \sim D'$, so eliminating these constants, the frequency equation will be obtained. However, we should notice that the generalized hypergeometric series ${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; \xi)$ converges in the region $|\xi| < 1$. The variable ξ increases in proportion to the square of the natural circular frequency ω . Hence the solution for higher order may not be obtained from the above equations. The author intends to study still more about this point.

(2) Since the swing angle of a shackle increases with resonance of the vibrating system, the resonance diagram may not necessarily coincide with that of the case of simple support. However, in the case of free vibration as in this paper, if the initial deflection is not large, the swing angle of a shackle is small, too. Therefore the natural frequency obtained by an assumed simple support in this paper may be considered to be enough as the first approximation.

Leaf springs for vehicle suspension under no load are usually given the curvature which becomes nearly zero under normal load. Hence the analysis for curved beam may be unnecessary.

With regard to extension of the solution to the coupled vibration of the springs and a wheel-driving shaft, the author agrees with the discussor and intends to take it up as the future subject of study.