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Deformation of and Stress in a Short Pipe Grooved Axially under Torsion*

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When a thin cylindrical pipe of finite length with a groove cut out through the length axially is twisted with both ends fixed on grips as shown in Fig. 1, the influence of fixed ends on the angle of twist and the stress is discussed, using the fundamental differential equation for cylindrical shell introduced by one of the present authors.

The angle of twist per unit length of the short pipe with a groove is very small compared with that of the pipe of infinite length and this difference is so extremely great that the application of the equations for the infinitely long pipe to a short pipe may be inadequate. Besides this remarkable difference, there exist great axial thrusts at both fixed ends.

These theoretical results are ascertained by experiments.

1. Solution of problem

The fundamental differential equation of a cylindrical shell with surface loads acting normally and tangentially, has been introduced by one of the present authors as follows⁽¹⁾:

$$\begin{aligned} A \frac{\partial^3 w}{\partial x^3} + \frac{4B}{a^2} \frac{\partial^3 w}{\partial x^2 \partial \theta^2} + \frac{6C}{a^4} \frac{\partial^3 w}{\partial x^4 \partial \theta^4} + \frac{8-2\nu^2}{a^4} \frac{\partial^3 w}{\partial x^4 \partial \theta^2} + \frac{6L}{a^4} \frac{\partial^4 w}{\partial x^4} \\ + \frac{4}{a^6} \frac{\partial^2}{\partial x^2} \left[\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right] + \frac{1}{a^8} \frac{\partial^2}{\partial \theta^2} \left[\frac{\partial^6 w}{\partial \theta^6} + 2 \frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} \right] \\ = \frac{1}{D} \left[A \frac{\partial^4 p}{\partial x^4} + \frac{2U}{a^2} \frac{\partial^4 p}{\partial x^2 \partial \theta^2} + \frac{B}{a^4} \frac{\partial^4 p}{\partial \theta^4} \right] - \frac{1}{D} \left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \frac{\partial^5 q}{\partial x^4 \partial \theta} - \frac{(2+\nu)}{a^2} \frac{\partial^3 q}{\partial x^2 \partial \theta} \right. \\ \left. + \frac{4-3\nu+\nu^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \frac{\partial^5 q}{\partial x^2 \partial \theta^3} + \frac{1}{12a^4} \left(\frac{h}{a} \right)^2 \frac{\partial^3 q}{\partial \theta^3} - \frac{1}{a^4} \frac{\partial^3 q}{\partial \theta^3} \right] \dots \dots \dots (1) \end{aligned}$$

where

$$\left. \begin{aligned} A &= 1 + \frac{1}{3} \left(\frac{h}{a} \right)^2, & B &= 1 + \frac{1}{12} \left(\frac{h}{a} \right)^2, \\ C &= 1 + \frac{1-\nu^2}{72} \left(\frac{h}{a} \right)^2, & L &= 2(1-\nu^2) \left(\frac{a}{h} \right)^2 A, \\ U &= 1 + \frac{1+(1-\nu)^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 \end{aligned} \right\} \dots \dots \dots (2)$$

and a is the radius of the cylinder, h is the thickness, $D = Eh^3/12(1-\nu^2)$, w is the radial displacement, and p, q are the radial and tangential surface loads per unit area respectively.

We take the origin of x -coordinate at one of the fixed ends and denote by l the length of the pipe. Referring to Figs. 1 and 2, the boundary conditions which should be

Fig. 1

Fig. 2 Element of the cylindrical shell

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satisfied are as follows:

1) at $\theta = \pm \alpha$

$$T_\theta = 0, \quad S = 0, \quad M_\theta = 0, \quad N_\theta + \frac{\partial \mathcal{M}}{\partial x} = 0,$$

2) at $x = 0$ and $x = l$

$$u = 0, \quad w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad v = \pm v_0 \text{ (const.)}$$

Now, as shown in Fig. 3, we imagine a grooved cylindrical pipe of given section and infinite length acted upon at intervals of l by line loads of equal magnitude in opposite directions alternately. If we take one interval of such an infinite pipe, the first and third terms of the boundary conditions 2)

$$A \frac{d^8 W_{pj}}{dx^8} - \frac{4B}{a^2} j^2 \frac{d^6 W_{pj}}{dx^6} + \frac{6C}{a^4} j^4 \frac{d^4 W_{pj}}{dx^4} - \frac{8-2\nu^2}{a^4} j^2 \frac{d^2 W_{pj}}{dx^2} + \frac{6L}{a^4} \frac{d^4 W_{pj}}{dx^4} - \frac{4}{a^6} j^2 (j^2 - 1)^2 \frac{d^2 W_{pj}}{dx^2} + \frac{1}{a^8} j^4 (j^2 - 1)^2 W_{pj} = \frac{1}{D} \left[A \frac{d^4 P_j}{dx^4} - \frac{2U}{a^2} j^2 \frac{d^2 P_j}{dx^2} + \frac{B}{a^4} j^4 P_j \right] \quad (4)$$

Applying Fourier transform to both sides of Eq. (4), i.e.,

$$\bar{W}_{pj} = \int_{-l}^l W_{pj} \cos \frac{n\pi x}{l} dx \quad (5)$$

and noting the fact that even times derivatives of W_{pj} with respect to x produce even functions of x and odd times derivatives produce odd ones, and P_j in the right-hand side of Eq. (4) is considered to be δ function for x , this Fourier transform gives the following equation from the nature of δ function

$$\left[A \left(\frac{n\pi}{l} \right)^8 + \frac{4B}{a^2} \left(\frac{n\pi}{l} \right)^6 j^2 + \frac{6C}{a^4} \left(\frac{n\pi}{l} \right)^4 j^4 - \frac{8-2\nu^2}{a^4} \left(\frac{n\pi}{l} \right)^2 j^2 + \frac{6L}{a^4} \left(\frac{n\pi}{l} \right)^4 + \frac{4}{a^6} \left(\frac{n\pi}{l} \right)^2 j^2 (j^2 - 1)^2 + \frac{1}{a^8} j^4 (j^2 - 1)^2 \right] \bar{W}_{pj} = \frac{2P_j}{D} \left[A \left(\frac{n\pi}{l} \right)^4 + \frac{2U}{a^2} \left(\frac{n\pi}{l} \right)^2 j^2 + \frac{B}{a^4} j^4 \right] \quad (6)$$

According to the characters of δ function, the even terms of n vanish and only the odd terms remain in the right-hand side of Eq. (6). Taking the inverse transform of Eq. (6), we obtain for w_{pj} as follows:

$$w_{pj} = a\alpha_j \sin j\theta \sum_{n=1}^{\infty} \xi_{nj} \cos \frac{n\pi x}{l} \quad (7)$$

where

$$\left. \begin{aligned} \xi_{nj} &= \frac{A\mu^4 + 2Uj^2\mu^2 + Bj^4}{H(\mu, j)}, \quad \mu = \frac{n\pi a}{l}, \\ H(\mu, j) &= A\mu^8 + 4Bj^2\mu^6 + \{6Cj^4 - (8-2\nu^2)j^2 + 6L\} \mu^4 + 4j^2(j^2-1)^2\mu^2 + j^4(j^2-1)^2 \end{aligned} \right\} \quad (8)$$

$n = 1, 3, 5, 7, \dots$

and the arbitrary constant α_j is determined from the boundary conditions.

Applying the same transform and inverse transform to the tangential loads, we obtain the following equation for w_{qj}

$$w_{qj} = a\beta_j \sin j\theta \sum_{n=1}^{\infty} \eta_{nj} \cos \frac{n\pi x}{l} \quad (9)$$

where

$$\eta_{nj} = \frac{\left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^4 + (2+\nu)\mu^2 + \frac{4-3\nu+\nu^2}{12(1-\nu)} \left(\frac{h}{a} \right)^2 j^2 \mu^2 + \frac{1}{12} \left(\frac{h}{a} \right)^2 j^4 + j^2 \right] j}{H(\mu, j)} \quad (10)$$

and H , μ and n are the same with those of Eq. (8).

The other displacements u_{pj} , v_{pj} , u_{qj} and v_{qj} for Eqs. (7) and (9) are determined from the relations⁽²⁾ of $w \sim u$ and $w \sim v$ which contain no terms of p or q as follows,

$$\left. \begin{aligned} u_{pj} &= a\alpha_j \sin j\theta \sum_{n=1}^{\infty} \kappa_{nj} \sin \frac{\mu x}{a}, \\ v_{pj} &= a\alpha_j \cos j\theta \sum_{n=1}^{\infty} \eta_{nj} \cos \frac{\mu x}{a}, \\ u_{qj} &= a\beta_j \sin j\theta \sum_{n=1}^{\infty} \sigma_{nj} \sin \frac{\mu x}{a}, \\ v_{qj} &= a\beta_j \cos j\theta \sum_{n=1}^{\infty} \tau_{nj} \cos \frac{\mu x}{a} \end{aligned} \right\} \quad (11)$$

are satisfied automatically according to the nature of symmetry. Therefore, all of the boundary conditions 2) can be satisfied by taking the magnitudes of circumferential direction of these line loads properly.

Let us first consider the case where the force p acts only. We assume the following expression as the solution of Eq. (1) for $p_j = P_j \sin j\theta$

$$w_{pj} = W_{pj} \sin j\theta$$

where P_j and W_{pj} are functions of x only and

$$j = m\pi/\alpha, \quad m = 0, 1, 2, 3, \dots \quad (3)$$

Substituting these expressions into Eq. (1), we obtain

Fig. 3

where

$$\left. \begin{aligned} \kappa_{\mu j} &= \frac{\left[\frac{(1+\nu)(2-\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 j^2 \mu^2 - \nu A \mu^2 + \frac{(1+\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 j^4 + V j^2 \right] \mu}{H(\mu, j)}, \\ \sigma_{\mu j} &= \frac{\left[\frac{(1+\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^4 - \frac{\nu(2-\nu)}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^2 + 1 + \frac{(1+\nu)}{6(1-\nu)} \left(\frac{h}{a}\right)^2 j^2 \mu^2 + \frac{(1+\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 j^4 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a}\right)^2 j^2 \right] j \mu}{H(\mu, j)}, \\ \tau_{\mu j} &= \frac{\frac{1}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^6 + 2(1+\nu) \mu^2 + \frac{5-\nu}{12(1-\nu)} \left(\frac{h}{a}\right)^2 j^2 \mu^4 + \frac{2-\nu}{6(1-\nu)} \left(\frac{h}{a}\right)^2 j^4 \mu^2 + \frac{1}{12} \left(\frac{h}{a}\right)^2 j^6 + j^2}{H(\mu, j)}, \\ V &= 1 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \end{aligned} \right\} \dots\dots\dots (12)$$

Besides the particular solutions mentioned above, we take the following expressions as complementary solutions of Eq. (1)

$$w_{cp} = a\alpha_j \sum_{n=1}^{\infty} \Re C_{pj i}^{\mu} \xi_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a}, \quad w_{cq} = a\beta_j \sum_{n=1}^{\infty} \Re C_{qj i}^{\mu} \xi_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a} \dots\dots\dots (13)$$

where

$$\xi_i^{\mu} = A\mu^4 + 2U(m_i^{\mu})^2 \mu^2 + B(m_i^{\mu})^4 \dots\dots\dots (14)$$

and m_i^{μ} in these equations are the roots of an equation

$$m^8 + 2(2\mu^2 - 1)m^6 + \{6C\mu^4 - 8\mu^2 + 1\}m^4 + 2\mu^2\{2B\mu^4 - (4 - \nu^2)\mu^2 + 2\}m^2 + \mu^4(A\mu^4 + 6L) = 0 \dots\dots\dots (15)$$

which may be factorized with sufficient accuracy in the following way⁽⁸⁾.

$$\left. \begin{aligned} &\left[m^4 + \left\{ 2\mu^2 - 1 + \frac{i(1+\nu^2)}{K} \mu^2 \right\} m^2 + \mu^2(\mu^2 - 2 + iK) \right] \\ &\times \left[m^4 + \left\{ 2\mu^2 - 1 - \frac{i(1+\nu^2)}{K} \mu^2 \right\} m^2 + \mu^2(\mu^2 - 2 - iK) \right] = 0, \\ &K = \sqrt{6L + 4\mu^2 - 4}, \quad i = \sqrt{-1} \end{aligned} \right\} \dots\dots\dots (16)$$

In Eq. (13), m_i^{μ} shows m_1^{μ} or m_2^{μ} , i. e., two kinds of the roots of Eq. (16) which are not conjugate to each other and the right-hand sides of Eq. (13) mean that two such expressions for $i=1$ and 2 should be added. $C_{pj i}^{\mu}$ and $C_{qj i}^{\mu}$ are complex arbitrary constants to be determined from the boundary conditions 1) and notation $\Re$ denotes to take real parts of the functions. Substituting Eq. (13) into the relations of $w \sim u$ and $w \sim v$, we obtain the corresponding equations of u and v as follows:

$$\left. \begin{aligned} u_{cp} &= a\alpha_j \sum_{n=1}^{\infty} \Re C_{pj i}^{\mu} \kappa_i^{\mu} \sin m_i^{\mu} \theta \sin \frac{\mu x}{a}, & v_{cp} &= a\alpha_j \sum_{n=1}^{\infty} \Re C_{pj i}^{\mu} \eta_i^{\mu} \cos m_i^{\mu} \theta \cos \frac{\mu x}{a}, \\ u_{cq} &= a\beta_j \sum_{n=1}^{\infty} \Re C_{qj i}^{\mu} \kappa_i^{\mu} \sin m_i^{\mu} \theta \sin \frac{\mu x}{a}, & v_{cq} &= a\beta_j \sum_{n=1}^{\infty} \Re C_{qj i}^{\mu} \eta_i^{\mu} \cos m_i^{\mu} \theta \cos \frac{\mu x}{a} \end{aligned} \right\} \dots\dots\dots (17)$$

where

$$\left. \begin{aligned} \kappa_i^{\mu} &= \left[\frac{(1+\nu)(2-\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 (m_i^{\mu})^2 \mu^2 - \nu A \mu^2 + \frac{(1+\nu)}{12(1-\nu)} \left(\frac{h}{a}\right)^2 (m_i^{\mu})^4 + V(m_i^{\mu})^2 \right] \mu, \\ \eta_i^{\mu} &= \left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a}\right)^2 \mu^4 + \frac{4-3\nu+\nu^2}{12(1-\nu)} \left(\frac{h}{a}\right)^2 (m_i^{\mu})^2 \mu^2 + (2+\nu) \mu^2 + \frac{1}{12} \left(\frac{h}{a}\right)^2 (m_i^{\mu})^4 + (m_i^{\mu})^2 \right] m_i^{\mu} \end{aligned} \right\} \dots\dots\dots (18)$$

m_i^{μ} and suffix i have the same meaning as in Eq. (13). Substituting these expressions of displacements into the relation of stresses and strains, we obtain the following equations as the stress resultants and couples. Regarding the particular solutions of p group, they are

$$\left. \begin{aligned} \frac{h^2 T_{x p j}}{12D} &= \alpha_j \sin j\theta \sum_{n=1}^{\infty} \zeta_{\mu j} \cos \frac{\mu x}{a}, & \frac{h^2 T_{\theta p j}}{12D} &= \alpha_j \sin j\theta \sum_{n=1}^{\infty} \nu_{\mu j} \cos \frac{\mu x}{a}, \\ \frac{h^2 S_{p j}}{6(1-\nu)D} &= -\alpha_j \cos j\theta \sum_{n=1}^{\infty} \omega_{\mu j} \sin \frac{\mu x}{a}, & \frac{M_{x p j}}{D} &= -\frac{\alpha_j}{a} \sin j\theta \sum_{n=1}^{\infty} z_{\mu j} \cos \frac{\mu x}{a}, \\ \frac{M_{\theta p j}}{D} &= -\frac{\alpha_j}{a} \sin j\theta \sum_{n=1}^{\infty} \chi_{\mu j} \cos \frac{\mu x}{a}, & \frac{a^2}{D} \left[N_{\theta p j} + \frac{\partial \mathfrak{M}_{p j}}{\partial x} \right] &= -\alpha_j \cos j\theta \sum_{n=1}^{\infty} \varepsilon_{\mu j} \cos \frac{\mu x}{a} \end{aligned} \right\} \dots\dots\dots (19)$$

where

$$\left. \begin{aligned}
\zeta_{\mu j} &= \frac{\left[\frac{(2-\nu)}{12} \left(\frac{h}{a} \right)^2 \mu^4 + (1-\nu^2) I \mu^2 + \frac{(1-\nu)^2}{12} \left(\frac{h}{a} \right)^2 j^2 \mu^2 - \frac{\nu}{12} \left(\frac{h}{a} \right)^2 j^2 (j^2-1) \right] j^2}{H(\mu, j)}, \\
\nu_{\mu j} &= \frac{(1-\nu^2) A \mu^4 - \frac{4-\nu^2}{12} \left(\frac{h}{a} \right)^2 j^2 \mu^4 - \frac{1}{3} \left(\frac{h}{a} \right)^2 j^2 (j^2-1) \mu^2 - \frac{1}{12} \left(\frac{h}{a} \right)^2 j^4 (j^2-1)}{H(\mu, j)}, \\
\omega_{\mu j} &= \frac{\left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^4 + 2(1+\nu) I \mu^2 + \frac{(1-\nu)}{6} \left(\frac{h}{a} \right)^2 j^2 \mu^2 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2 j^2 (j^2-1) \right] j \mu}{H(\mu, j)}, \\
z_{\mu j} &= \frac{A \mu^6 + (2U + \nu V) j^2 \mu^4 + \left\{ 1 + 2\nu + \frac{1-\nu^2}{12} \left(\frac{h}{a} \right)^2 \right\} j^4 \mu^2 - \nu(2+\nu) j^2 \mu^2 + \nu j^4 (j^2-1)}{H(\mu, j)}, \\
\chi_{\mu j} &= \frac{\nu A \mu^6 + (2\nu U + V) j^2 \mu^4 + (2+\nu) j^2 (j^2-1) \mu^2 + j^4 (j^2-1)}{H(\mu, j)}, \\
\varepsilon_{\mu j} &= \frac{[(2-\nu) \mu^6 + (5-2\nu) j^2 \mu^4 - 2(1-\nu)(2+\nu) \mu^4 + (4-\nu) j^2 (j^2-1) \mu^2 + j^4 (j^2-1)] j}{H(\mu, j)}, \\
I &= 1 + \frac{\nu}{6(1+\nu)} \left(\frac{h}{a} \right)^2
\end{aligned} \right\} \dots\dots\dots (20)$$

From the equilibrium condition of the axial direction, we obtain

$$\frac{(1-\nu)}{2} \omega_{\mu j} j = \zeta_{\mu j} \mu \dots\dots\dots (21)$$

The expressions for particular solutions of q group have the same form as Eqs. (19), with α_j , $\nu_{\mu j}$, $\omega_{\mu j}$, $z_{\mu j}$, $\chi_{\mu j}$ and $\varepsilon_{\mu j}$ replaced by β_j , $-\delta_{\mu j}$, $\theta_{\mu j}$, $\pi_{\mu j}$, $-0_{\mu j}$ and $-n_{\mu j}$ respectively,

where

$$\left. \begin{aligned}
\delta_{\mu j} &= \frac{\left[\frac{(2+\nu)}{12} \mu^6 + \frac{5+2\nu}{12} j^2 \mu^4 - \frac{(2-\nu)(1+\nu)}{6} \mu^4 + \frac{4+\nu}{12} j^2 (j^2-1) \mu^2 + \frac{1}{12} j^4 (j^2-1) \right] j \left(\frac{h}{a} \right)^2}{H(\mu, j)}, \\
\theta_{\mu j} &= \frac{\left[\frac{1}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^6 + \frac{2-\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2 j^2 \mu^4 + 2(1+\nu) \mu^2 + \frac{\nu(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 j^2 \mu^2 \right. \\
&\quad \left. + \frac{1-2\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2 j^4 \mu^2 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2 j^4 (j^2-1) \right] \mu}{H(\mu, j)}, \\
\pi_{\mu j} &= \frac{\left[\frac{1}{3} \left(\frac{h}{a} \right)^2 \mu^6 + \frac{1}{3} \left(\frac{h}{a} \right)^2 j^2 \mu^4 + (2+\nu) \mu^4 + \frac{1-\nu^2}{12} \left(\frac{h}{a} \right)^2 j^4 \mu^2 + (1+\nu)^2 j^2 \mu^2 - 2\nu(1+\nu) \mu^2 + \nu j^2 (j^2-1) \right] j}{H(\mu, j)}, \\
0_{\mu j} &= \frac{\left[\frac{(1-\nu)}{6} \left(\frac{h}{a} \right)^2 \mu^6 + \frac{(1-\nu)^2}{12} \left(\frac{h}{a} \right)^2 j^2 \mu^4 - \nu(2+\nu) \mu^4 - 2(1+\nu) (j^2-1) \mu^2 - j^2 (j^2-1) \right] j}{H(\mu, j)}, \\
n_{\mu j} &= \frac{\frac{1}{3} \left(\frac{h}{a} \right)^2 \mu^6 + \frac{1}{3} \left(\frac{h}{a} \right)^2 j^2 \mu^4 + \frac{1-\nu^2}{12} \left(\frac{h}{a} \right)^2 j^4 \mu^2 - (4-\nu^2) j^2 \mu^4 + 4(1-\nu^2) \mu^4 - 4j^2 (j^2-1) \mu^2 - j^4 (j^2-1)}{H(\mu, j)}
\end{aligned} \right\} \dots\dots\dots (22)$$

Next, the expressions of stress resultants and couples of p group for complementary solutions (13) become as follows:

$$\left. \begin{aligned}
\frac{h^2 T_{xcp}}{12D} &= \alpha_j \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} \zeta_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a}, & \frac{h^2 T_{\theta cp}}{12D} &= \alpha_j \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} \nu_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a}, \\
\frac{h^2 S_{cp}}{6(1-\nu)D} &= -\alpha_j \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} \omega_i^{\mu} \cos m_i^{\mu} \theta \sin \frac{\mu x}{a}, & \frac{M_{xcp}}{D} &= -\frac{\alpha_j}{a} \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} z_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a}, \\
\frac{M_{\theta cp}}{D} &= -\frac{\alpha_j}{a} \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} \chi_i^{\mu} \sin m_i^{\mu} \theta \cos \frac{\mu x}{a}, & \frac{a^2}{D} \left[N_{\theta cp} + \frac{\partial \mathfrak{M}_{cp}}{\partial x} \right] &= -\alpha_j \sum_{n=1}^{\infty} \Re C_{pji}^{\mu} \varepsilon_i^{\mu} \cos m_i^{\mu} \theta \sin \frac{\mu x}{a}
\end{aligned} \right\} \dots\dots\dots (23)$$

where

$$\left. \begin{aligned}
\zeta_i^{\mu} &= \frac{(2-\nu)}{12} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^2 \mu^4 + (1-\nu^2) I (m_i^{\mu})^2 \mu^2 + \frac{(1-\nu)^2}{12} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^4 \mu^2 - \frac{\nu}{12} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^4 \{ (m_i^{\mu})^2 - 1 \}, \\
\nu_i^{\mu} &= (1-\nu^2) A \mu^4 - \frac{4-\nu^2}{12} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^2 \mu^4 - \frac{1}{3} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^2 \{ (m_i^{\mu})^2 - 1 \} \mu^2 - \frac{1}{12} \left(\frac{h}{a} \right)^2 (m_i^{\mu})^4 \{ (m_i^{\mu})^2 - 1 \},
\end{aligned} \right\}$$

$$\left. \begin{aligned} \omega_i^\mu &= m_i^\mu \mu \left[\frac{(2-\nu)}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^4 + 2(1+\nu) I \mu^2 + \frac{(1-\nu)}{6} \left(\frac{h}{a} \right)^2 (m_i^\mu)^2 \mu^2 - \frac{\nu}{6(1-\nu)} \left(\frac{h}{a} \right)^2 (m_i^\mu)^2 \{ (m_i^\mu)^2 - 1 \} \right], \\ z_i^\mu &= A \mu^6 + (2U + \nu V) (m_i^\mu)^2 \mu^4 + \left\{ 1 + 2\nu + \frac{1-\nu^2}{12} \left(\frac{h}{a} \right)^2 \right\} (m_i^\mu)^4 \mu^2 - \nu(2+\nu) (m_i^\mu)^2 \mu^2 + \nu (m_i^\mu)^4 \{ (m_i^\mu)^2 - 1 \}, \\ \chi_i^\mu &= \nu A \mu^6 + (2\nu U + V) (m_i^\mu)^2 \mu^4 + (2+\nu) (m_i^\mu)^2 \{ (m_i^\mu)^2 - 1 \} \mu^2 + (m_i^\mu)^4 \{ (m_i^\mu)^2 - 1 \}, \\ \varepsilon_i^\mu &= m_i^\mu \left[(2-\nu) \mu^6 + (5-2\nu) (m_i^\mu)^2 \mu^4 - 2(1-\nu)(2+\nu) \mu^4 \right. \\ &\quad \left. + (4-\nu) (m_i^\mu)^2 \{ (m_i^\mu)^2 - 1 \} \mu^2 + (m_i^\mu)^4 \{ (m_i^\mu)^2 - 1 \} \right] \end{aligned} \right\} \dots (24)$$

And from the same reason of Eq. (21), we obtain

$$\frac{(1-\nu)}{2} \omega_i^\mu m_i^\mu = \zeta_i^\mu \mu \dots (25)$$

In Eqs. (23), suffix i also indicates addition of the values for $i=1$ and 2 as in Eq. (13). The expressions for q group are the same as those for p group except that α_j and C_{pji}^μ should be replaced by β_j and C_{qji}^μ respectively. Among the coefficients $\xi_{\mu j}$, $\eta_{\mu j}$, $\kappa_{\mu j}$ etc., only $n_{\mu j}$ contains a term of μ^3 in the numerator, so we make a transformation as follows:

$$n_{\mu j} = \frac{1}{3A} \left(\frac{h}{a} \right)^2 - \frac{n_{\mu j}'}{A} \dots (26)$$

where

$$n_{\mu j}' = \frac{\left(\frac{h}{a} \right)^2 j^2 \mu^6 + \left(\frac{h}{a} \right)^2 \left(2 - \frac{1-\nu^2}{12} \right) j^4 \mu^4 + (4-\nu^2) \left\{ 1 - \frac{1}{3} \left(\frac{h}{a} \right)^2 \right\} j^2 \mu^2 + j^2 (j^2 - 1) (4 + j^2) \left\{ 1 + \frac{1}{3} \left(\frac{h}{a} \right)^2 j^2 \right\}}{H(\mu, j)} \dots (27)$$

Using Eq. (26), we obtain

$$\frac{a^2}{D} \left[N_{\theta q j} + \frac{\partial \mathfrak{N}_{q j}}{\partial x} \right] = \beta_j \cos j \theta \sum_{n=1}^{\infty} n_{\mu j} \cos \frac{\mu x}{a} = \frac{\beta_j}{3A} \left(\frac{h}{a} \right)^2 \cos j \theta \sum_{n=1}^{\infty} \cos \frac{\mu x}{a} - \frac{\beta_j}{A} \cos j \theta \sum_{n=1}^{\infty} n_{\mu j}' \cos \frac{\mu x}{a} \dots (28)$$

Since the first term in Eq. (28) indicates the Fourier expansion of δ function, it shows concentrated transverse shearing forces acting on the loading portions of line loads and the shearing forces through the whole length of the pipe are shown by the second terms of Eq. (28).

Substituting the above stress resultants and couples in the conditions 1), we obtain the following equations

$$\left. \begin{aligned} \Re C_{pji}^\mu \nu_i^\mu \sin m_i^\mu \alpha &= 0, & \Re C_{pji}^\mu \omega_i^\mu \cos m_i^\mu \alpha &= -(-1)^m \omega_{\mu j}, \\ \Re C_{pji}^\mu \chi_i^\mu \sin m_i^\mu \alpha &= 0, & \Re C_{pji}^\mu \varepsilon_i^\mu \cos m_i^\mu \alpha &= -(-1)^m \varepsilon_{\mu j}, \\ \Re C_{qji}^\mu \nu_i^\mu \sin m_i^\mu \alpha &= 0, & \Re C_{qji}^\mu \omega_i^\mu \cos m_i^\mu \alpha &= -(-1)^m \theta_{\mu j}, \\ \Re C_{qji}^\mu \chi_i^\mu \sin m_i^\mu \alpha &= 0, & \Re C_{qji}^\mu \varepsilon_i^\mu \cos m_i^\mu \alpha &= -(-1)^m \frac{n_{\mu j}'}{A} \end{aligned} \right\} \dots (29)$$

The left-hand sides of Eqs. (29) denote addition of the values for $i=1$ and 2 . C_{pji}^μ and C_{qji}^μ are determined from these expressions and for C_{pji}^μ we obtain as follows:

$$\left. \begin{aligned} C_{pji}^\mu &= \frac{\Delta_{pji}^\mu}{A^\mu} \\ \Delta^\mu &= -4\Im \bar{\nu}_i^\mu \sin \bar{m}_i^\mu \alpha \omega_i^\mu \cos m_i^\mu \alpha \Im \bar{\chi}_k^\mu \sin \bar{m}_k^\mu \alpha \varepsilon_k^\mu \cos m_k^\mu \alpha \\ &\quad + 4\Im \bar{\nu}_i^\mu \sin \bar{m}_i^\mu \alpha \chi_i^\mu \sin m_i^\mu \alpha \Im \bar{\omega}_k^\mu \cos \bar{m}_k^\mu \alpha \varepsilon_k^\mu \cos m_k^\mu \alpha \\ &\quad - 4\Im \bar{\nu}_i^\mu \sin \bar{m}_i^\mu \alpha \varepsilon_i^\mu \cos m_i^\mu \alpha \Im \bar{\omega}_k^\mu \cos \bar{m}_k^\mu \alpha \chi_k^\mu \sin m_k^\mu \alpha, \\ \Delta_{pji}^\mu &= -4i(-1)^m [\omega_{\mu j} \bar{\nu}_i^\mu \sin \bar{m}_j^\mu \alpha \Im \bar{\chi}_k^\mu \sin \bar{m}_k^\mu \alpha \varepsilon_k^\mu \cos m_k^\mu \alpha \\ &\quad + \varepsilon_{\mu j} \bar{\nu}_i^\mu \sin \bar{m}_i^\mu \alpha \Im \bar{\omega}_k^\mu \cos \bar{m}_k^\mu \alpha \chi_k^\mu \sin m_k^\mu \alpha \\ &\quad - \omega_{\mu j} \bar{\chi}_i^\mu \sin \bar{m}_i^\mu \alpha \Im \bar{\nu}_k^\mu \sin \bar{m}_k^\mu \alpha \varepsilon_k^\mu \cos m_k^\mu \alpha \\ &\quad + \omega_{\mu j} \bar{\varepsilon}_i^\mu \cos \bar{m}_i^\mu \alpha \Im \bar{\nu}_k^\mu \sin \bar{m}_k^\mu \alpha \chi_k^\mu \sin m_k^\mu \alpha \\ &\quad - \varepsilon_{\mu j} \bar{\omega}_i^\mu \cos \bar{m}_i^\mu \alpha \Im \bar{\nu}_k^\mu \sin \bar{m}_k^\mu \alpha \chi_k^\mu \sin m_k^\mu \alpha \\ &\quad + \varepsilon_{\mu j} \bar{\chi}_i^\mu \sin \bar{m}_i^\mu \alpha \Im \bar{\nu}_k^\mu \sin \bar{m}_k^\mu \alpha \omega_k^\mu \cos m_k^\mu \alpha] \end{aligned} \right\} \dots (30)^{(4)}$$

In these expressions, the suffixes i and k indicate 1 or 2, and in the expression of Δ^μ they mean addition of the values for $i=1, k=2$ and $i=2, k=1$. For the expression of Δ_{pji}^μ the suffixes of both sides correspond to each other, that is, the expression of right-hand side of Δ_{pji}^μ in which i and k are interchanged corresponds to Δ_{pjk}^μ . The symbol $\Im$ indicates that the imaginary part of the complex numbers is taken.

For the expression of C_{qji}^μ , we may only replace $\omega_{\mu j}$ and $\varepsilon_{\mu j}$ with $\theta_{\mu j}$ and $n_{\mu j}'/A$ respectively.

To satisfy the second and fourth of the conditions 2), we expand $\sin m_i^\mu \theta$ and $\cos m_i^\mu \theta$ into Fourier series in the range $-\alpha < \theta < \alpha$, while the first and the third have been satisfied automatically as above

mentioned,

$$\left. \begin{aligned} \sin m_i \theta &= \sum_{\lambda=1}^{\infty} a_{i\lambda} \sin \frac{\lambda \pi \theta}{\alpha}, & \cos m_i \theta &= \frac{b_{i0}}{2} + \sum_{\lambda=1}^{\infty} b_{i\lambda} \cos \frac{\lambda \pi \theta}{\alpha}, \\ a_{i\lambda} &= -2 \frac{(-1)^{\lambda} \pi \lambda \sin m_i \alpha}{\left\{ \frac{\lambda^2 \pi^2}{\alpha^2} - (m_i)^2 \right\} \alpha^2}, & b_{i\lambda} &= -2 \frac{(-1)^{\lambda} m_i \sin m_i \alpha}{\left\{ \frac{\lambda^2 \pi^2}{\alpha^2} - (m_i)^2 \right\} \alpha} \end{aligned} \right\} \dots \dots \dots (31)$$

$\lambda = 0, 1, 2, 3, \dots$

Using Eqs. (31), we obtain

$$\begin{aligned} w_j &= w_{pj} + w_{qj} + w_{pc} + w_{qc} \\ &= a\alpha_j \sin j\theta \sum_{n=1}^{\infty} \xi_{nj} \cos \frac{n\pi x}{l} + a\beta_j \sin j\theta \sum_{n=1}^{\infty} \eta_{nj} \cos \frac{n\pi x}{l} \\ &\quad + a\alpha_j \sum_{n=1}^{\infty} \Re C_{pj} \xi_{nj} \left[\sum_{\lambda=1}^{\infty} a_{i\lambda} \sin \frac{\lambda \pi \theta}{\alpha} \right] \cos \frac{\mu x}{a} \\ &\quad + a\beta_j \sum_{n=1}^{\infty} \Re C_{qj} \eta_{nj} \left[\sum_{\lambda=1}^{\infty} a_{i\lambda} \sin \frac{\lambda \pi \theta}{\alpha} \right] \cos \frac{\mu x}{a} \end{aligned}$$

Now, if we put

$$\sum_{n=1}^{\infty} \xi_{nj} = \phi_j, \quad \sum_{n=1}^{\infty} \eta_{nj} = \psi_j, \quad \sum_{n=1}^{\infty} \Re C_{pj} \xi_{nj} a_{i\lambda} = P_{\lambda j}^w, \quad \sum_{n=1}^{\infty} \Re C_{qj} \eta_{nj} a_{i\lambda} = Q_{\lambda j}^w \dots \dots \dots (32)$$

we have,

$$w_j = a\alpha_j \phi_j \sin j\theta + a\beta_j \psi_j \sin j\theta + a\alpha_j \sum_{\lambda=1}^{\infty} P_{\lambda j}^w \sin \frac{\lambda \pi \theta}{\alpha} + a\beta_j \sum_{\lambda=1}^{\infty} Q_{\lambda j}^w \sin \frac{\lambda \pi \theta}{\alpha}$$

Therefore to satisfy $\sum_{j=0}^{\infty} w_j = 0$ independently of θ , the following equation must be satisfied

$$\alpha_\lambda \phi_\lambda + \beta_\lambda \psi_\lambda + \sum_{j=0}^{\infty} (\alpha_j P_{\lambda j}^w + \beta_j Q_{\lambda j}^w) = 0, \quad \lambda \neq 0 \dots \dots \dots (33)$$

where α_λ means $\alpha_j = \lambda \pi / \alpha$ and β_λ , ϕ_λ and ψ_λ also have the same meaning as α_λ . Eq. (33) should be satisfied for all values of λ , and if so, it is certain that the value of w at $x=l$, i.e., $\sum_{j=0}^{\infty} w_j$ becomes also equal to zero. Hence the second of the conditions 2) is satisfied.

Next, we take

$$\begin{aligned} v_j &= v_{pj} + v_{qj} + v_{pc} + v_{qc} \\ &= a\alpha_j \cos j\theta \sum_{n=1}^{\infty} \tau_{nj} \cos \frac{\mu x}{a} + a\beta_j \cos j\theta \sum_{n=1}^{\infty} \tau_{nj} \cos \frac{\mu x}{a} \\ &\quad + a\alpha_j \sum_{n=1}^{\infty} \Re C_{pj} \tau_{nj} \left[\frac{b_{i0}}{2} + \sum_{\lambda=1}^{\infty} b_{i\lambda} \cos \frac{\lambda \pi \theta}{\alpha} \right] \cos \frac{\mu x}{a} \\ &\quad + a\beta_j \sum_{n=1}^{\infty} \Re C_{qj} \tau_{nj} \left[\frac{b_{i0}}{2} + \sum_{\lambda=1}^{\infty} b_{i\lambda} \cos \frac{\lambda \pi \theta}{\alpha} \right] \cos \frac{\mu x}{a} \end{aligned}$$

Using the following notations

$$\left. \begin{aligned} \sum_{n=1}^{\infty} \tau_{nj} &= \Gamma_j, & \frac{1}{2} \sum_{n=1}^{\infty} \Re C_{pj} \tau_{nj} b_{i0} &= P_{0j}^v, & \sum_{n=1}^{\infty} \Re C_{pj} \tau_{nj} b_{i\lambda} &= P_{\lambda j}^v, \\ \frac{1}{2} \sum_{n=1}^{\infty} \Re C_{qj} \tau_{nj} b_{i0} &= Q_{0j}^v, & \sum_{n=1}^{\infty} \Re C_{qj} \tau_{nj} b_{i\lambda} &= Q_{\lambda j}^v \end{aligned} \right\} \dots \dots \dots (34)$$

we obtain

$$v_j = a\alpha_j \Gamma_j \cos j\theta + a\beta_j \Gamma_j \cos j\theta + a\alpha_j P_{0j}^v + a\alpha_j \sum_{\lambda=1}^{\infty} P_{\lambda j}^v \cos \frac{\lambda \pi \theta}{\alpha} + a\beta_j Q_{0j}^v + a\beta_j \sum_{\lambda=1}^{\infty} Q_{\lambda j}^v \cos \frac{\lambda \pi \theta}{\alpha}$$

Therefore, to satisfy $\sum_{j=0}^{\infty} v_j = v_0$ independently of θ , we have

$$\alpha_0 \phi_0 + \beta_0 \Gamma_0 + \sum_{j=0}^{\infty} (\alpha_j P_{0j}^v + \beta_j Q_{0j}^v) = \frac{v_0}{a}, \quad \alpha_\lambda \phi_\lambda + \beta_\lambda \Gamma_\lambda + \sum_{j=0}^{\infty} (\alpha_j P_{\lambda j}^v + \beta_j Q_{\lambda j}^v) = 0, \quad \lambda \neq 0 \dots \dots \dots (35)$$

where α_λ , β_λ , Γ_λ , etc. have the same meaning as Eq. (33). Under the requirements (35), it is obvious that the condition at $x=l$ will be also satisfied. For the left-hand sides of Eqs. (32) and (34), the suffix i means addition of the two expressions for $i=1$ and 2, and we may take $\alpha_0=0$ referring to Eq. (7).

Since α_λ and β_λ are determined by Eqs. (33) and (35), the boundary conditions 1) and 2) are completely satisfied, hence the problem is solved.

2. Angle of twist

When the short pipe grooved axially like this example is twisted, the states of deformation are different at respective points, consequently the angles of twist per unit length are also different at respective sections. The angle of twist per unit length at a point of a section is the relative rotation per unit length along the axial direction, which is shown as follows⁽⁵⁾:

$$\tau = \frac{1}{a} \left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} \right) \dots\dots\dots (36)$$

The angle of twist per unit length at a section may be defined as the mean value of Eq. (36) through this section⁽⁶⁾. Denoting the mean value of the angle of twist expressed by Eq. (36) through the pipe length by $\bar{\theta}$, we obtain

$$\bar{\theta} = \frac{1}{al} \int_0^l \left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x} \right) dx = \frac{1}{al} \left(\frac{\partial w}{\partial \theta} - v \right)_0^l = \frac{2v_0}{al} \dots\dots\dots (37)$$

Eq. (37) represents the mean angle of twist per unit length through the whole length of pipe.

Next, let us consider the relationship between $\bar{\theta}$ and the torsional moment M .

$$S' - \frac{\mathfrak{M}}{a} = S - \frac{2\mathfrak{M}}{a}, \quad M = \int_{-\alpha}^{\alpha} \left(S - \frac{2\mathfrak{M}}{a} \right) a^2 d\theta \dots\dots\dots (38)$$

Considering the case of $x \rightarrow +0$ in Eq. (38), it is sure that the complementary functions in the expressions of S and $\mathfrak{M}$ converge to zero according to their natures as $x \rightarrow +0$ except $\theta = \pm \alpha$. Every integral of the particular solutions in Eq. (38) which all have terms of $\cos j\theta$ becomes zero except the case $j=0$ because of $j=m\pi/\alpha$. Therefore, observing $\alpha_0=0$, Eq. (38) reduces to

$$M = \int_{-\alpha}^{\alpha} \left(S_{q_0} - \frac{2\mathfrak{M}_{q_0}}{a} \right) a^2 d\theta \dots\dots\dots (39)$$

From Eqs. (19) and (22), we obtain

$$\frac{h^2 S_{q_0}}{6(1-\nu)D} = -\beta_0 \sum_{n=1}^{\infty} \theta_{\mu_0} \sin \frac{\mu x}{a}, \quad \theta_{\mu_0} = \frac{\left[\frac{1}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^6 + 2(1+\nu) \mu^2 \right] \mu}{A \mu^8 + 12(1-\nu^2) \left(\frac{a}{h} \right)^2 A \mu^4} = \frac{1}{6(1-\nu)A} \left(\frac{h}{a} \right)^2 \frac{1}{\mu} \dots\dots (40)$$

and $\mathfrak{M}_{qj}$ is obtained as follows from the relations of stresses and strains,

$$\frac{\mathfrak{M}_{qj}}{D} = \frac{\beta_j}{a} \cos j\theta \sum_{n=1}^{\infty} (1-\nu) \gamma_{\mu j} \sin \frac{\mu x}{a} \dots\dots\dots (41)$$

where

$$\gamma_{\mu j} = \frac{\left[\frac{1}{6(1-\nu)} \left(\frac{h}{a} \right)^2 \mu^6 + \frac{(1+\nu)}{12(1-\nu)} \left(\frac{h}{a} \right)^2 j^2 \mu^4 + \frac{\nu}{12} \left(\frac{h}{a} \right)^2 j^4 \mu^2 - (2+\nu) j^2 \mu^2 + 2(1+\nu) \mu^2 - j^2(j^2-1) \right] \mu}{H(\mu, j)} \dots\dots\dots (42)$$

Hence,

$$\begin{aligned} \frac{1}{D} \left[S_{q_0} - \frac{2\mathfrak{M}_{q_0}}{a} \right] &= -\frac{\beta_0}{Aa^2} \sum_{n=1}^{\infty} \frac{1}{\mu} \sin \frac{\mu x}{a} - \frac{\beta_0}{Aa^2} \sum_{n=1}^{\infty} \frac{1}{3} \left(\frac{h}{a} \right)^2 \frac{1}{\mu} \sin \frac{\mu x}{a} \\ &= -\frac{\beta_0}{Aa^2} \sum_{n=1}^{\infty} \left[1 + \frac{1}{3} \left(\frac{h}{a} \right)^2 \right] \frac{1}{\mu} \sin \frac{\mu x}{a} = -\frac{\beta_0}{a^2} \sum_{n=1}^{\infty} \frac{1}{\mu} \sin \frac{\mu x}{a} = -\frac{\beta_0 l}{4a^3} \\ \therefore M &= -\frac{\beta_0 l \alpha}{2a} D \dots\dots\dots (43) \end{aligned}$$

Now, if we put

$$\alpha_j = \frac{v_0}{a} \alpha_j', \quad \beta_j = \frac{v_0}{a} \beta_j' \dots\dots\dots (44)$$

then Eq. (43) becomes

$$M = -\frac{l v_0 \alpha}{2a^2} D \beta_0' \dots\dots\dots (43)'$$

Substituting Eq. (43)' into Eq. (37), we obtain

$$\bar{\theta} = -\frac{4M}{\alpha\beta_0 l D} \left(\frac{a}{l}\right) \dots\dots\dots(45)$$

The solutions for a pipe of infinite length with the same groove under torsion are very simple having expressions

$$w=0, \quad u=-ak\theta, \quad v=kx \dots\dots\dots(46)$$

The stress resultants and couples due to these displacements are shown as follows:

$$\left. \begin{aligned} T_x = T_\theta = S = M_x = M_\theta = 0, \\ \mathfrak{M} = -\frac{(1-\nu)D}{a} k \end{aligned} \right\} \dots\dots\dots(47)$$

and the angle of twist per unit length $\bar{\theta}_\infty$ is expressed by

$$\bar{\theta}_\infty = -\frac{1}{a} \frac{\partial v}{\partial x} = -\frac{M}{4(1-\nu)Da\alpha} \dots\dots\dots(48)$$

Considering the ratio of Eq. (48) to Eq. (45), we obtain

$$\frac{\bar{\theta}_\infty}{\bar{\theta}} = \frac{\beta_0'}{16(1-\nu)} \left(\frac{l}{a}\right)^2 \dots\dots\dots(49)$$

The calculation of Eq. (39) is performed for $x \rightarrow +0$ to simplify the evaluation, but it may be proved at any section. Eq. (48) coincides with that of Saint-Venant for the twist of a pipe with thin-walled open section.

For a numerical example, we take $a/h=25$, $l/a=6$, $\alpha=\pi$, and $\nu=0.3$ which yields $\beta_0'=389.35$. Substituting this value into Eq. (49), we obtain

Fig. 4 Shape of the test piece

Fig. 5 Angle of twist per unit length versus twist moment

$$\frac{\bar{\theta}_\infty}{\bar{\theta}} = \frac{389.35 \times 36}{16 \times 0.7} \cong 1250 \dots\dots\dots(50)$$

that is, the angle of twist per unit length of this example with both ends fixed is only 1/1250 compared with that of the pipe of infinite length. This difference is so great that we made an experiment to check whether the result is true or not. The test piece has the dimensions of $a=25$ mm, $h=1$ mm, and $l=150$ mm through which a slit is cut out, the whole length of the pipe being $l+2b$ as shown in Fig. 4. In Fig. 5, we show the experimental results by "○" and the theoretical values of $\bar{\theta}$ and $\bar{\theta}_\infty$ by lines I and II respectively. Strictly speaking, the line II should be drawn nearer to the abscissa than as shown in the figure. It is seen that the experimental values of $\bar{\theta}$ appear somewhat greater than the theoretical ones but they are extremely smaller than the values shown by line II. That the experimental results appear slightly greater than the theoretical values is due to some incompleteness of fixing at the ends. As pointed out in the previous papers⁽⁷⁾, the influence of the end condition upon the short pipe is so serious that even a slight decrease of the end thrusts due to the incompleteness of the fixing condition will exert a great influence upon the value of $\bar{\theta}$ according to the significant values of the thrust which will be discussed later. To make sure of this fact, another experiment has been made on a test piece with slightly weak grips and shorter grip length $b=20$ mm instead of 40 mm which is the grip length of the test piece used for the experiment mentioned above. The results yield somewhat greater values of them, in spite of the same size of the test piece. They are shown by "×" in Fig. 5. These results show that the slight disagreement between the theoretical and the experimental results is due to the incompleteness of the fixing. So long as we use the same materials for the test piece and the grips, this incompleteness of fixing cannot be removed. The line III in the figure shows the angle of twist per unit length of the pipe with no groove and of same dimensions. The figure indicates that the angle of twist of a short grooved pipe may approach rather that of the pipe without a groove. The angle of twist of a short pipe grooved axially under torsion is very small compared with that of infinite length and the difference is so great that the formula generally used for a pipe of infinite length can hardly be used for a short pipe.

The angles of twist per unit length at any point is given by Eq. (36). In Fig. 6, they are illustrated by curves I and II which are the values at the fixed ends and the mid section respectively.

In this figure, the abscissa indicates the angles at the circumference and the ordinate does the ratios of the values of Eq. (36) to the angles of twist of the pipe without groove $\bar{\theta}_0$. It is seen from these diagrams that the values of mid section along the circumference are nearly constant (completely constant in the case of infinite length); at the section of fixed end, however, they have pretty great variations which are caused by great axial thrusts as mentioned later. The straight lines parallel to the abscissa indicate the mean values of the angles of twist per unit length at fixed ends and mid section along the circumference. It is seen that at the fixed end this mean value coincides exactly with that of the pipe without groove and at the mid section this value is affected slightly by the nature of the pipe of infinite length with groove.

It is to be noted that the deformation expressed by Eq. (46) is inextensional, consequently this problem cannot be solved by membrane theory, moreover Eqs. (46) and (47) cannot be derived from Donnell's Equations.

3. Stress

The curves I and II in Fig. 7 indicate the values of shearing forces along the circumference at the fixed ends and mid section respectively, taking the angle of the circumference on the abscissa and the ratio S to S_0 (the shearing forces of the pipe without groove) on the ordinate. The curve II is nearly smooth, but the curve I has quite a peculiar look which is caused also by great thrusts at the ends. The curve I shows the limiting values for $x \rightarrow +0$; on the section extremely closed ends, however, it comes very near the curve

Fig. 6 Angle of twist per unit length along circumference

I but decreases abruptly to zero at the edge of slit. Such distribution may be found only at the section extremely near the ends and on the section at a little distance from the end, say, $0.3a$, the distribution almost coincides with the curve II at the mid section. In each case the values of S are significant but we find $S=0$ in Eqs. (47), which shows that the moment M is balanced by $\mathfrak{M}$ alone for the grooved pipe of infinite length. For the pipe of finite length, however, it is balanced by both S and $\mathfrak{M}$, and the contribution of $\mathfrak{M}$ to M increases as the length of the pipe increases. Fig. 8 shows the distribution of axial thrusts T_x at the ends, taking the circular angles on the abscissa and T_x/S_0 on the ordinate. It is seen that the

Fig. 7 Curves of shearing forces along circumference

Fig. 8 Distribution of axial thrust at the ends

Fig. 9 Bending moment of axial direction along the slit

axial thrust becomes extremely great at the edge of the groove. This fact may be confirmed by an experiment, i.e., applying a fairly great twisting moment, we can observe some yielding of the material at the position where the greatest axial thrust may occur. The diagrams mentioned above are all illustrated for one side with respect to the diameter through the slit because of their natures of symmetry or antisymmetry. Fig. 9 shows M_x of axial direction along the slit taking on the ordinate the ratio of M_x to the twist moment per unit length along the arc $M_0 = M/2\pi a$. It is seen that some bending moment occurs at the fixed ends, but at slightly inner portion a moment in opposite direction is found. The reason why such moments may occur will be explained by investigating the behavior of w along the slit which is illustrated in Fig. 10. In this figure, we take the ratio of w to the circumferential displacement v_0 on the ordinate.

4. Conclusion

We may conclude from the results mentioned above as follows:

1) In the case of a grooved pipe of finite length with fixed ends, the angle of twist per unit length is very small compared with that of infinite length. In such a case as $a/h=25$, $l/a=6$, $\alpha=\pi$, it is only $1/1250$ compared with that of infinite length, rather close to the value of the pipe without groove. As this difference is extremely great the formula

Fig. 10 Radial displacement along the slit

generally used for the pipe of infinite length can hardly be used for a short pipe.

2) At both fixed ends great axial thrusts occur, especially at the edge of the slit. Owing to these great thrusts, it is difficult to realize complete fixing conditions at the ends so long as we use the same material for the grips with that of test piece. Moreover it is to be noted that this axial thrust is more significant than the shearing force.

3) When the pipe of infinite length with groove is twisted, there exists no shearing force S , i.e., the moment M is entirely balanced by $\mathcal{M}$ only, but in the case of finite length the shearing force S occurs and moment M is balanced by both S and $\mathcal{M}$, and the contribution of S to M decreases as the length of pipe increases.

4) At both fixed ends some bending moment occurs.

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