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Theory of Non-Symmetrical Bending State for Cylindrical Shell*

(Stresses in a Cylindrical Shell with a Local Radial Load)

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In this paper the fundamental equations for bending states of cylindrical shells are derived in the terms of stress function and radial displacement component. These equations have the same accuracy as Flügge's equations. We can easily give the solutions of these equations especially for a stress boundary value problem.

Stresses in a cylindrical vessel with a local radial load were calculated as a specific application. We obtained a good agreement with the results calculated by P. P. Bijlaard.

1. Introduction

The fundamental differential equations for bending states of cylindrical shells have already been introduced in the text books written by S. Timoshenko⁽¹⁾, W. Flügge⁽²⁾ and so forth. Since these equations are expressed by displacement components, they seem to be complicated especially to deal with stress boundary value problems. On the other hand, W. Zerna⁽³⁾ or Vlasov⁽⁴⁾ derived equations from applying the conception of a stress function. But all of the above seem to have less accuracy because of some unknown terms being neglected.

We have obtained more explicit and rigorous equations in the terms of a stress function F and a displacement component w . These equations have the same accuracy as the Flügge's and show a correlation with the membrane theory.

As an example, we calculated stresses in an infinite cylindrical shell with a local radial load using our basic equations. These results may be practically applied to a cylindrical pressure vessel which is locally loaded with thermal expansion of pipes or earth quake forces. A similar problem was treated by P. P. Bijlaard⁽⁵⁾. But he assumed such pipe as square attachment.

2. Basic equations for cylindrical shells of uniform thickness

2.1 Derivation of fundamental differential equations

The co-ordinates are shown in Fig. 1. We apply the Flügge's method for definition of membrane

forces and bending or twisting moments, and for introduction of strain components. Unknown quantities are ten stress resultant components and three displacement components. The relations among these quantities are represented as follows;

a) Equilibrium equations

$$\left. \begin{aligned} N_x' + N_{\theta x}' + aX &= 0 \\ N_{\theta}' + N_{x\theta}' - Q_{\theta} + aY &= 0 \end{aligned} \right\} \dots\dots\dots (1.a)$$

$$Q_x' + Q_{\theta}' + N_{\theta} + aZ = 0 \dots\dots\dots (1.b)$$

$$\left. \begin{aligned} M_{\theta}' + M_{x\theta}' - aQ_{\theta} &= 0 \\ M_x' + M_{\theta x}' - aQ_x &= 0 \end{aligned} \right\} \dots\dots\dots (1.c)$$

$$M_{\theta x} + a(N_{x\theta} - N_{\theta x}) = 0 \dots\dots\dots (1.d)$$

b) Relations between stresses and displacements

$$\left. \begin{aligned} N_{\theta} &= \frac{K}{a} [\dot{v} + \mu u' - w] - \frac{D}{a^3} [w + \ddot{w}] \\ N_x &= \frac{K}{a} [u' + \mu \dot{v} - \mu w] + \frac{D}{a^3} w'' \end{aligned} \right\} \dots\dots\dots (2.a)$$

$$N_{x\theta} = \frac{1-\mu}{2} \frac{K}{a} [\dot{u} + v'] + \frac{1-\mu}{2} \frac{D}{a^3} [v' + \dot{w}'] \dots\dots\dots (2.b)$$

$$\left. \begin{aligned} M_{\theta} &= -\frac{D}{a^2} [w + \ddot{w} + \mu w''] \\ M_x &= -\frac{D}{a^2} [w'' + \mu w + \mu \ddot{w}] - \frac{h^2}{12a} N_x \end{aligned} \right\} \dots\dots\dots (2.c)$$

$$\left. \begin{aligned} M_{\theta x} &= -\frac{1-\mu}{2} \frac{D}{a^2} [2w'' + v' - \dot{u}] \\ M_{x\theta} &= M_{\theta x} - \frac{h^2}{12a} N_{x\theta} \end{aligned} \right\} \dots\dots\dots (2.d)$$

where $x/a = \xi$, $(\cdot)' = \frac{\partial(\cdot)}{\partial \xi}$, $(\cdot)' = \frac{\partial(\cdot)}{\partial \theta}$

Fig. 1 Co-ordinates

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$$K = \frac{Eh}{(1-\mu^2)}, \quad D = \frac{Eh^3}{12(1-\mu^2)}$$

μ : Poisson's ratio, E : Young's modulus
 a : radius, h : thickness

u, v, w : displacement components

X, Y, Z : external surface load components

From the above 13 equations, we introduce the homogeneous solutions for the case of $X=Y=Z=0$ in the order of approximation of $1+(h^2/12a^2) \doteq 1$.

[Introduction of stress function]

Eliminating Q_x, Q_θ and $N_{\theta x}$ from Eqs. (1.a) and

$$\left\{ \begin{aligned} (Eh/a)\dot{v} &= (N_\theta - M_\theta/a) - \mu N_x + (Eh/a)w \\ (Eh/a)u' &= N_x - \mu(N_\theta - M_\theta/a) - (1-\mu^2)(D/a^3)w'' \end{aligned} \right\} \quad \dots\dots\dots(4)$$

Substituting Eq. (4) into Eqs. (2.b) and (2.d),

$$\left\{ \begin{aligned} 2(1+\mu)N_{x\theta}'' &= \{N_x - \mu(N_\theta - M_\theta/a)\}'' + \{(N_\theta - M_\theta/a) - \mu N_x\}'' + (Eh/a)w'' \\ 2(1+\mu)M_{\theta x}'' &= -(1-\mu^2)(D/a^2)(2w'''' + w'') \\ &\quad + (h^2/12a)[\{N_x - \mu(N_\theta - M_\theta/a)\}'' - \{(N_\theta - M_\theta/a) - \mu N_x\}'] \end{aligned} \right\} \quad \dots\dots\dots(5)$$

Elimination of $M_{\theta x}$ and $M_{x\theta}$ from Eq. (1.a)' using the second of Eq. (2.d) gives

$$N_x'' + (N_\theta - M_\theta/a)'' + 2N_{x\theta}'' = 0$$

Substituting the first of Eq. (5) into the above equation, we obtain

$$\nabla^2(N_x + N_\theta - M_\theta/a) + (Eh/a)w'' = 0$$

where

$$\nabla^2(\) = (\)'' + (\)''$$

The above equation can be expressed by the stress function in the form of:

$$\nabla^2(\nabla^2 + 1)F + (Eh/a)w'' + (Eh/a)\frac{1-\mu}{k^2}\nabla^2(w'' + w - w'') = 0 \quad \dots\dots\dots(6)$$

where

$$k^2 = 12(1-\mu^2)a^2/h^2$$

In the next place, we substitute Eq. (2.c) and the second of Eq. (2.d) into Eq. (1.b)'.

$$-\frac{D}{a^3}[\nabla^2(\nabla^2 + 1)w - (1-\mu)(2w'''' + w'')] + N_\theta + 2\frac{M_{\theta x}''}{a} - \frac{h^2}{12a^2}(N_x'' + N_{x\theta}'') = 0$$

Since $N_x'' + N_{x\theta}'' = -M_{\theta x}''/a$ from the first of Eq. (1.a)', the last term in the above equation can be neglected in comparison with the third term $2M_{\theta x}''/a$.

Substituting of the second of Eq. (5) and Eq. (3) into the above equation, we get

$$(Eh/a)\nabla^2(\nabla^2 + 1)w - k^2F'' - (1-\mu)\nabla^2(F'' + F - F'') = 0 \quad \dots\dots\dots(7)$$

Eqs. (6) and (7) are the fundamental equations with regard to the stress function F and the radial displacement component w , and are equivalent to the following single equation.

$$\nabla^2(\nabla^2 + 1)\Phi + i\left(\frac{1-\mu}{k}\right)\nabla^2(\Phi'' + \Phi - \Phi'') + ik\Phi'' = 0 \quad \dots\dots\dots(8)$$

where

$$\Phi = (Eh/a)w + ikF, \quad i = \sqrt{-1}$$

Once the complex stress function $\Phi(\xi, \theta)$ is determined, w and F are obtained as the real and imaginary part of Φ respectively.

[Stresses and displacements]

All other unknown quantities can be represented with w and F as follows.

$$\left\{ \begin{aligned} N_\theta &= F'', \quad N_x = F'' + F - (D/a^3)[w'' + \mu(w + \ddot{w})], \\ M_\theta &= -(D/a^2)[w + \ddot{w} + \mu w''], \quad M_x = -(D/a^2)[w'' + \mu(w + \ddot{w})] - (1-\mu^2)(a/k^2)(F'' + F) \end{aligned} \right\} \quad \dots\dots\dots(9.a)$$

$$\left\{ \begin{aligned} 2N_{x\theta}'' &= -(2F'''' + F'') + (D/a^3)[\nabla^2(w'' - w - \ddot{w}) + (1+\mu)w''] \\ 2M_{\theta x}'' &= -(1-\mu)(D/a^2)[2w'''' + w''] - (1-\mu)(a/k^2)[\nabla^2(F'' - F - F'') + (1+\mu)F''] \end{aligned} \right\} \quad \dots\dots\dots(9.b)$$

$$\left\{ \begin{aligned} (Eh/a)u' &= F'' + F - \mu F'' - 2(D/a^3)[w'' + \mu(w + \ddot{w})] \\ (Eh/a)\dot{v} &= F'' - \mu(F'' + F) + (Eh/a)w + (D/a^3)[2\mu w'' + (1+\mu^2)(w + \ddot{w})] \end{aligned} \right\} \quad \dots\dots\dots(9.c)$$

$$Q_x = -(F'' + F)' - N_{x\theta}, \quad Q_\theta = F''' + N_{x\theta}' \quad \dots\dots\dots(9.d)$$

$$N_{\theta x} = N_{x\theta} + M_{\theta x}/a, \quad M_{x\theta} = M_{\theta x} - (1-\mu^2)(a/k^2)N_{x\theta} \quad \dots\dots\dots(9.e)$$

(1.b) by Eqs. (1.c) and (1.d), we get

$$\left\{ \begin{aligned} N_x' + (N_{x\theta} + M_{\theta x}/a)' &= 0 \\ (N_\theta - M_\theta/a)' + (N_{x\theta} - M_{x\theta}/a)' &= 0 \end{aligned} \right\} \quad \dots\dots\dots(1.a)'$$

$$M_\theta'' + (M_{x\theta} + M_{\theta x})'' + M_x'' + aN_\theta = 0 \quad \dots\dots\dots(1.b)'$$

We can eliminate $N_{x\theta}, M_{\theta x}$ and $M_{x\theta}$ from the above equations by inserting the second of Eq. (2.d).

$$N_\theta'' + N_\theta = (N_x - M_x/a)''$$

This equation yields the following stress function $F(\xi, \theta)$.

$$N_\theta = F'', \quad N_x - M_x/a = F'' + F \quad \dots\dots\dots(3)$$

[Derivation of fundamental differential equations]

Eq. (2.a) gives the displacements u and v .

2.2 Procedure to solve the fundamental differential equation

In this paper we restrict our interest to uni-valued stresses and displacements with respect to θ . Though stresses are uni-valued, the stress function F need not be a uni-valued function. Multi-valued function in F will be discussed later on. In this place we introduce uni-valued solutions of Eq. (8) which can be expanded to Fourier series, i. e., $\Phi = \sum \phi(\xi) \cos n\theta$.

Eq. (8) is rewritten with $\phi(\xi)$ as

$$\left[(L^2 - n^2)(L^2 - n^2 + 1) + i \left(\frac{1 - \mu}{k} \right) (L^2 - n^2)(1 - n^2 - L^2) + ikL^2 \right] \phi = 0$$

where $L(\xi) = \frac{d(\xi)}{d\xi}$,

then

$$\left(1 - i \frac{1 - \mu}{k} \right) L^4 + (1 - 2n^2 + ik) L^2 + \left(1 + i \frac{1 - \mu}{k} \right) n^2(n^2 - 1) = 0$$

This characteristic equation about L^2 can easily be solved as

$$L^2 = \frac{1}{2} \left[-(1 - 2n^2 + ik) \pm \sqrt{(1 - 2n^2 + ik)^2 - 4n^2(n^2 - 1)} \right] \left(1 + i \frac{1 - \mu}{k} \right)$$

We distinguish between the above plus and minus signs with the suffix "m" and "b" respectively.

$$L_m^2 = \frac{-2n^2(n^2 - 1) \left(1 + i \frac{1 - \mu}{k} \right)}{[1 - 2n^2 + ik \{1 + \sqrt{1 + 2i(2n^2 - 1)/k}\}]} \quad \dots\dots\dots (10 \cdot a)$$

when $n=0$ and 1, $L_m^2 = 0$

$$L_b^2 = -\frac{1}{2} \left(1 + i \frac{1 - \mu}{k} \right) [1 - 2n^2 + ik \{1 + \sqrt{1 + 2i(2n^2 - 1)/k}\}] \quad \dots\dots\dots (10 \cdot b)$$

when $n=0$ and 1, $L_b^2 = 2n^2 - \mu - ik$

The solution by L_m^2 corresponds to membrane theory as shown in the next paragraph. On the other hand the solution by L_b^2 is proper to bending theory. In what follows we designate the former as "membrane solution" and the latter as "bending solution". The sum of membrane and bending solutions gives the general solution of Eq. (8).

[Oscillating solution; in the case of $L^2 \neq 0$]

We consider the bending solution for all values of n and the membrane solution for $n \geq 2$.

Let $L^2 = A + ikB$, i. e., $L^2(\phi) - (A + ikB)\phi = 0 \quad \dots\dots\dots (11 \cdot a)$

A and B are obtained from Eq. (10) as follows.

$$\left. \begin{aligned} A_m &= S_0 \{-2 + \mu + 2n^2 - (1 - \mu)S_1 + kS_2\}, & B_m &= S_0 \left\{ 1 + S_1 + \frac{1 - \mu}{k^2} (2n^2 - 1 + kS_2) \right\} \\ A_b &= \frac{1}{2} \left\{ -\mu + 2n^2 + (1 - \mu)S_1 + kS_2 \right\}, & B_b &= -\frac{1}{2} \left\{ 1 + S_1 - \frac{1 - \mu}{k^2} (2n^2 - 1 + kS_2) \right\} \end{aligned} \right\} \quad \dots\dots\dots (11 \cdot b)$$

where

$$S_0 = \frac{2n^2(n^2 - 1)}{k^2(1 + S_1)^2 + (2n^2 - 1 + kS_2)^2}, \quad S_1 = \sqrt{\{1 + \sqrt{1 + 4(2n^2 - 1)^2/k^2}\}/2},$$

$$S_2 = \sqrt{\{-1 + \sqrt{1 + 4(2n^2 - 1)^2/k^2}\}/2}$$

Then

$$L = \pm \sqrt{A + ikB} = \pm (\alpha + i\beta) \quad \text{i. e.,} \quad \{L + (\alpha + i\beta)\} \{L - (\alpha + i\beta)\} \phi = 0 \quad \dots\dots\dots (12 \cdot a)$$

where

$$\alpha = \sqrt{\frac{A + \sqrt{A^2 + k^2 B^2}}{2}} \quad (>0), \quad \beta = \frac{kB}{2\alpha} \quad \dots\dots\dots (12 \cdot b)$$

From Eq. (12 \cdot a),

$$\phi = (C_1 + iD_1)e^{(\alpha + i\beta)\xi} + (C_2 + iD_2)e^{-(\alpha + i\beta)\xi}$$

where C and D are real constants of integration.

Let $\phi = g + ikf$

where $F = f(\xi) \cos n\theta$, $(Eh/a)w = g(\xi) \cos n\theta \quad \dots\dots\dots (13)$

ϕ is separated in real and imaginary parts.

$$\left. \begin{aligned} g &= e^{\alpha\xi} (C_1 \cos \beta\xi - D_1 \sin \beta\xi) + e^{-\alpha\xi} (C_2 \cos \beta\xi + D_2 \sin \beta\xi) \\ kf &= e^{\alpha\xi} (C_1 \sin \beta\xi + D_1 \cos \beta\xi) - e^{-\alpha\xi} (C_2 \sin \beta\xi - D_2 \cos \beta\xi) \end{aligned} \right\} \quad \dots\dots\dots (14 \cdot a)$$

The derivatives of the above are

$$\begin{cases} L(g) = e^{\alpha\xi}(\bar{C}_1 \cos \beta\xi - \bar{D}_1 \sin \beta\xi) + e^{-\alpha\xi}(\bar{C}_2 \cos \beta\xi + \bar{D}_2 \sin \beta\xi) \\ kL(f) = e^{\alpha\xi}(\bar{C}_1 \sin \beta\xi + \bar{D}_1 \cos \beta\xi) - e^{-\alpha\xi}(\bar{C}_2 \sin \beta\xi - \bar{D}_2 \cos \beta\xi) \end{cases} \quad (14 \cdot b)$$

where

$$\bar{C}_1 = \alpha C_1 - \beta D_1, \quad \bar{D}_1 = \alpha D_1 + \beta C_1, \quad \bar{C}_2 = -\alpha C_2 + \beta D_2, \quad \bar{D}_2 = -\alpha D_2 - \beta C_2$$

The second derivatives of g and f can be reduced to g and f themselves with Eq. (11.a).

That is,

$$L^2(g) = Ag - k^2 Bf, \quad L^2(f) = Af + Bg \quad (14 \cdot c)$$

Finally all the other quantities of Eq. (9) are represented as follows.

$$\begin{aligned} N_\theta &= [Af + Bg] \cos n\theta \\ N_x &= \left[(1+B-n^2)f - \frac{A+\mu(1-n^2)}{k^2} g \right] \cos n\theta \\ M_\theta/a &= \left[\mu Bf - \frac{A+\mu(1-n^2)}{k^2} g \right] \cos n\theta \\ M_x/a &= \left[\left\{ B - \frac{(1-\mu^2)(1-n^2)}{k^2} \right\} f - \frac{A+\mu(1-n^2)}{k^2} g \right] \cos n\theta \\ N_{x\theta} &= \left[\left\{ 2n^2 - 1 - B + \frac{n^2(1-n^2)}{A^2+k^2B^2} B \right\} L(f) + \left\{ A + \mu + \frac{n^2(1-n^2)}{A^2+k^2B^2} A \right\} \frac{L(g)}{k^2} \right] \frac{\sin n\theta}{2n} \\ M_{\theta x}/a &= \left[- \left\{ A + \mu + \frac{n^2(1-n^2)}{A^2+k^2B^2} A \right\} L(f) + \left\{ 2n^2 - 1 - B + \frac{n^2(1-n^2)}{A^2+k^2B^2} B \right\} L(g) \right] \frac{(1-\mu) \sin n\theta}{2nk^2} \\ (Eh/a)u &= \left[- \left\{ \mu + \frac{1-n^2}{A^2+k^2B^2} (2\mu B - A) \right\} L(f) - \left\{ \frac{2}{k^2} + \frac{1-n^2}{A^2+k^2B^2} \left(B + \frac{2\mu}{k^2} A \right) \right\} L(g) \right] \cos n\theta \\ (Eh/a)v &= \left[\left\{ A - 2\mu B - \mu(1-n^2) \right\} f + \left\{ 1 + B + \frac{2\mu A + (1+\mu^2)(1-n^2)}{k^2} \right\} g \right] \frac{\sin n\theta}{n} \end{aligned} \quad (15)$$

when $n=0$; $v=M_{\theta x}=N_{x\theta}=0$

[Non-oscillating solution; in the case of $L^2=0$]

The membrane solutions for $n=0$ and 1 are obtained from the equation $L^2(\phi)=0$.

$$\phi = (C_1 + iD_1)\xi + C_2 + iD_2,$$

i. e.,

$$g = C_1\xi + C_2, \quad kf = D_1\xi + D_2 \quad (16)$$

These solutions have special meanings as follows.

(a) For $n=0$;

From Eq. (16),

$$(Eh/a)w = C_1\xi + C_2, \quad kF = D_1\xi + D_2$$

Other quantities are

$$N_\theta = 0, \quad N_x = D_2/k, \quad M_\theta/a = -\mu D_2/k^3,$$

$$M_x/a = -D_2/k^3, \quad (Eh/a)u = (D_2/k)\xi + E_1,$$

$$(Eh/a)v = 2(1+\mu)E_2\xi + E_3, \quad N_{x\theta} = E_2,$$

$$M_{\theta x}/a = -(1-\mu^2)E_2/k^2$$

where $C_1 = D_1 = 0, \quad C_2 = \mu D_2/k$

- P : axial tensile force
- V : shearing force
- T : torsional moment
- M : bending moment (at $x=0$)
- D_P : axial displacement
- D_V : lateral displacement
- θ_T : circumferential rotation
- θ_M : rotation about the direction of $\theta = \pi/2$ (at $x=0$)

Fig. 2 Resultant loads and rigid body translations

E_1, E_2, E_3 : arbitrary constants

The integration constants in the above can be determined by the resultant loads P, T , and the translations of a rigid body D_P, θ_T , which are shown in Fig. 2.

P and T are defined as

$$P = \int_0^{2\pi} N_x a d\theta, \quad T = \int_0^{2\pi} (aN_{x\theta}) a d\theta$$

Generally, all quantities are composed of the membrane and bending solutions. But we easily see in Eq. (15) that the bending parts of N_x and $N_{x\theta}$ for $n=0$ are zero. ($A_0 = -\mu, B_0 = -1$)

Then P and T can be determined only by the membrane part.

$$\begin{cases} P = \int_0^{2\pi} \frac{D_2}{k} a d\theta & \therefore D_2 = \frac{P}{2\pi a} k \\ T = \int_0^{2\pi} a^2 E_2 d\theta & \therefore E_2 = \frac{T}{2\pi a^2} \end{cases}$$

In the second place D_P and θ_T are given as

$$(Eh/a)D_P = E_1, \quad Eh\theta_T = E_3$$

Finally the membrane solutions for $n=0$ are represented as follows.

$$\begin{cases} N_\theta = 0, \quad N_x = \frac{P}{2\pi a}, \quad M_\theta = -\frac{\mu P}{2\pi k^2}, \\ M_x = -\frac{P}{2\pi k^2}, \quad N_{x\theta} = \frac{T}{2\pi a^2}, \quad M_{\theta x} = -\frac{(1-\mu^2)T}{2\pi a k^2}, \\ Q_x = Q_\theta = 0, \quad u = \frac{P}{2\pi Eh} \xi + D_P, \\ v = \frac{(1+\mu)T}{\pi Eh a} \xi + a\theta_T, \quad w = \frac{\mu P}{2\pi Eh} \end{cases} \quad (17)$$

(b) For $n=1$;

From Eq. (16),

$$\begin{cases} (Eh/a)w = (C_1\xi + C_2)\cos\theta \\ kF = (D_1\xi + D_2)\cos\theta \end{cases}$$

Substitution of the above F into Eq. (3) yields insignificant solutions, i. e., $N_\theta=0$ and $N_x-M_x/a=0$.

In order to avoid this absurdness, a multi-valued solution of F is needed.

Under the condition that stresses must be uni-valued, Eq. (3) is rewritten as

$$F'' = n_\theta(\xi)\cos n\theta, \quad F'' + F = n_x(\xi)\cos n\theta$$

Integration of the above equations shows that F can include multi-valued functions, $\theta \sin \theta$ and

$$\begin{cases} (Eh/a)w = \left[C_1\xi + C_2 + \frac{1}{k}(D_1\xi^3/6 + D_2\xi^2/2) \right] \cos\theta, & kF = (D_1\xi + D_2) \frac{\theta \sin\theta}{2} - \frac{\mu}{k^2}(D_1\xi^3/6 + D_2\xi^2/2) \cos\theta, \\ N_x = \frac{D_1\xi + D_2}{k} \cos\theta, & N_\theta = M_\theta/a = -\frac{\mu}{k^2}N_x, & M_x/a = -\frac{(2-\mu^2)}{k^2}N_x, & N_{x\theta} = -\frac{D_1}{k} \sin\theta, \\ M_{x\theta}/a = \frac{1-\mu}{k^2}N_{x\theta}, & Q_\theta = \frac{\mu}{k^3}(D_1\xi + D_2) \sin\theta, & Q_x = -\frac{3-\mu-\mu^2}{k^3}D_1 \cos\theta, \\ (Eh/a)u = \left[\frac{D_1}{k} \left(\frac{\xi^2}{2} + 2 + \mu \right) + \frac{D_2}{k} \xi + C_1 \right] \cos\theta, \\ (Eh/a)v = \left[\frac{D_1}{k} \left(\frac{\xi^3}{6} - \mu\xi \right) + \frac{D_2}{k} \left(\frac{\xi^2}{2} - \mu \right) + C_1\xi + C_2 \right] \sin\theta \end{cases}$$

Integration constants in the above are determined by the resultant loads V , M , and the translations of a rigid body, D_V , Θ_M which are shown in Fig. 2.

V and M are defined as

$$V = \int_0^{2\pi} (N_{x\theta} \sin\theta + Q_x \cos\theta) a d\theta, \quad M - xV = \int_0^{2\pi} (N_x a \cos\theta - M_x \cos\theta) a d\theta$$

Substituting Eq. (9) into the above,

$$\begin{aligned} V &= \int_0^{2\pi} (N_{x\theta} \sin\theta - N_{x\theta}' \cos\theta) a d\theta - \int_0^{2\pi} (F'' + F)' (\cos\theta) a d\theta \\ &= - \int_0^{2\pi} (N_{x\theta} \cos\theta)' a d\theta - \int_0^{2\pi} (F' \cos\theta + F \sin\theta)' a d\theta = - \left[a N_{x\theta} \cos\theta \right]_0^{2\pi} - a \left[F' \cos\theta + F \sin\theta \right]_0^{2\pi} \\ \frac{M}{a} - \xi V &= \int_0^{2\pi} (F'' + F) a \cos\theta d\theta = a \left[F' \cos\theta + F \sin\theta \right]_0^{2\pi} \end{aligned}$$

These do not vanish only when the multi-valued solution of F is introduced. Therefore they can be determined only by the membrane part in F for $n=1$, and so

$$V = -\frac{D_1}{k} \pi a \quad \therefore \quad \frac{D_1}{k} = -\frac{V}{\pi a}, \quad \frac{M}{a} - \xi V = \frac{D_1\xi + D_2}{k} \pi a \quad \therefore \quad \frac{D_2}{k} = \frac{M}{\pi a^2}$$

In the next place, the rigid body translation D_V , Θ_M can be obtained from the integration constants in displacement components.

$$(Eh/a)D_V = C_2, \quad Eh\Theta_M = C_1$$

Finally the membrane solutions for $n=1$ are rewritten as follows.

$$\left. \begin{aligned} N_x &= \left(-\frac{V}{\pi a} \xi + \frac{M}{\pi a^2} \right) \cos\theta, & N_{x\theta} &= \frac{V}{\pi a} \sin\theta, & N_\theta &= \frac{M_\theta}{a} = -\frac{\mu}{k^2} N_x, \\ \frac{M_x}{a} &= -\frac{2-\mu^2}{k^2} N_x, & \frac{M_{x\theta}}{a} &= \frac{1-\mu^2}{k^2} N_{x\theta}, & Q_x &= \frac{3-\mu-\mu^2}{k^2} \frac{V}{\pi a} \cos\theta, \\ Q_\theta &= \frac{\mu}{k^2} \left(-\frac{V}{\pi a} \xi + \frac{M}{\pi a^2} \right) \sin\theta, & u &= \left[-\frac{V}{\pi Eh} \left(\frac{\xi^2}{2} + 2 + \mu \right) + \frac{M}{\pi Eha} \xi + a\Theta_M \right] \cos\theta, \\ v &= \left[-\frac{V}{\pi Eh} \left(\frac{\xi^3}{6} - \mu\xi \right) + \frac{M}{\pi Eha} \left(\frac{\xi^2}{2} - \mu \right) + a\Theta_M \xi + D_V \right] \sin\theta, \\ w &= \left[-\frac{V}{\pi Eh} \frac{\xi^3}{6} + \frac{M}{\pi Eha} \frac{\xi^2}{2} + a\Theta_M \xi + D_V \right] \cos\theta \end{aligned} \right\} \dots\dots\dots (18)$$

2.3 Correlation with the membrane theory

The solution by L_m^2 has a correlation with the membrane theory, as mentioned in the previous

$\xi\theta \sin\theta$, only when $n=1$. Then the solution of Eq. (8), i. e., Φ which contains the multi-valued functions takes the form of

$$\Phi = g \cos\theta + i \left\{ (D_1\xi + D_2) \frac{\theta \sin\theta}{2} + f \cos\theta \right\}$$

By insertion of Φ into Eq. (8), f and g are determined.

$$\begin{cases} f = -\frac{\mu}{k^2} (D_1\xi^3/6 + D_2\xi^2/2) + \frac{1}{k^2} (D_1\xi + D_2) \\ g = \frac{1}{k} (D_1\xi^3/6 + D_2\xi^2/2) + \frac{2}{k^3} (D_1\xi + D_2) \end{cases}$$

Finally the membrane solutions for $n=1$ are obtained as follows.

In the boundary condition of $x=\infty$, the oscillating solution of Eq. (15) must be zero, i.e., the integration constants C_1 and D_1 in Eq. (14) are made zero. The boundary conditions at $x=0$ are considered as follows.

a) Continuity conditions of displacements

In order that the two half infinite cylinders for $x \geq 0$ and $x \leq 0$ may be continuous at $x=0$ after deformed, the following conditions must be satisfied.

$$w'|_{x=0}=0 \quad \dots\dots\dots(21 \cdot a)$$

$$u|_{x=0}=0 \quad \dots\dots\dots(21 \cdot b)$$

b) Equilibrium conditions of forces

In the first step, we consider a distributed load $T(\theta)$, shown in Fig. 4 (b).

$$\begin{cases} \left(N_{x\theta} - \frac{M_{x\theta}}{a}\right)|_{x=0} = 0 \\ \left(Q_x + \frac{M_{x\theta}}{a}\right)|_{x=0} = -\frac{W}{2\pi a} \left(\frac{1}{2} + \cos \theta + \cos 2\theta + \dots\right) \end{cases}$$

Inspection of Eqs. (2·b), (2·d), (9·d) and (21) gives the following formulas instead of the above.

$$v'|_{x=0}=0 \quad \dots\dots\dots(21 \cdot c)$$

$$(F'' + F')|_{x=0} = \frac{W}{2\pi a} \left(\frac{1}{2} + \cos \theta + \cos 2\theta + \dots\right) \quad \dots\dots\dots(21 \cdot d)$$

Now the boundary conditions are all present and so the integration constants can be determined as follows.

① $n=0$

membrane solution: $P=T=0$ in Eq. (17), bending solution: $A_b=-\mu$, $B_b=-1$,

$\alpha_b=\sqrt{(k-\mu)/2}$, $\beta_b=-\sqrt{(k+\mu)/2}$ in Eqs. (14), (15).

From Eq. (21·a), $\bar{C}_2^b=0$, From Eq. (21·b), $(Eh/a)D_p - \mu\bar{D}_2^b/k=0$, From Eq. (21·d), $\bar{D}_2^b/k = \frac{W}{4\pi a}$

Eq. (21·c) is automatically satisfied and θ_T is left undetermined, and may be put as zero.

② $n=1$

membrane solution: $M=0$, $V=-W/2$ in Eq. (18), bending solution: $A_b=2-\mu$, $B_b=-1$

$\alpha_b=\sqrt{(k+2-\mu)/2}$, $\beta_b=-\sqrt{(k-2+\mu)/2}$ in Eqs. (14), (15)

From Eq. (21·a), $Eh\theta_M + \bar{C}_2^b=0$

From Eq. (21·b), $\frac{W}{2\pi a}(2+\mu) + Eh\theta_M - \frac{\mu}{k}\bar{D}_2^b - \frac{2}{k^2}\bar{C}_2^b=0$

From Eq. (21·c), $-\frac{\mu W}{2\pi a} + Eh\theta_M + \frac{2+\mu}{k}\bar{D}_2^b + \frac{2\mu(2-\mu)}{k^2}\bar{C}_2^b=0$

Eq. (21·d) is automatically satisfied and D_V is left undetermined.

③ $n \geq 2$

Both membrane and bending solutions are given in Eqs. (14) and (15).

From Eq. (21·a), $\bar{C}_2^m + \bar{C}_2^b=0$

From Eq. (21·b), $\left\{\mu + \frac{(1-n^2)(2\mu B_m - A_m)}{A_m^2 + k^2 B_m^2}\right\} \frac{\bar{D}_2^m}{k} + \left\{\mu + \frac{(1-n^2)(2\mu B_b - A_b)}{A_b^2 + k^2 B_b^2}\right\} \frac{\bar{D}_2^b}{k} + \frac{(1-n^2)(B_m + 2\mu A_m/k^2)}{A_m^2 + k^2 B_m^2} \bar{C}_2^m + \frac{(1-n^2)(B_b + 2\mu A_b/k^2)}{A_b^2 + k^2 B_b^2} \bar{C}_2^b = 0$

From Eq. (21·c), $\{A_m - 2\mu B_m - \mu(1-n^2)\} \frac{\bar{D}_2^m}{k} + \{A_b - 2\mu B_b - \mu(1-n^2)\} \frac{\bar{D}_2^b}{k} + \{B_m + 2\mu A_m/k^2\} \bar{C}_2^m + \{B_b + 2\mu A_b/k^2\} \bar{C}_2^b = 0$

From Eq. (21·d), $(1-n^2)\left(\frac{\bar{D}_2^m}{k} + \frac{\bar{D}_2^b}{k}\right) = \frac{W}{2\pi a}$

Now the integration constants are all determined and the sum of each solution for $n=0, 1, 2, \dots$ gives a complete solution.

$$T(\theta) = \begin{cases} \frac{W}{2} / (2a\theta_0) \dots\dots |\theta| \leq \theta_0 \\ 0 \dots\dots\dots |\theta| > \theta_0 \end{cases}$$

$$= \frac{1}{2} a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \dots\dots$$

where

$$a_n = \frac{W}{2\pi a} \frac{\sin(n\theta_0)}{(n\theta_0)}$$

Equilibrium conditions of forces are

$$\begin{cases} \left(N_{x\theta} - \frac{M_{x\theta}}{a}\right)|_{x=0} = -T(\theta) \sin \theta \\ \left(Q_x + \frac{M_{x\theta}}{a}\right)|_{x=0} = -T(\theta) \cos \theta \end{cases}$$

Applying $\theta_0 \rightarrow 0$ to $T(\theta)$ in order to obtain conditions for the concentrated load $W/2$,

3.2 Solutions for a distributed load

Superposing the solutions for the concentrated loads, we can obtain the solution for the distributed load of Eq. (20). The stress distribution in an infinite cylindrical shell which is subjected to a normal load $Z(\xi, \bar{\theta}) \times a^2 d\xi d\bar{\theta}$ at the point $(\xi, \theta) = (\bar{\xi}, \bar{\theta})$ can be obtained by the following replacements made in the previous solution for a concentrated load W .

$$\begin{cases} |\xi - \bar{\xi}| & \text{instead of } \xi \\ \theta - \bar{\theta} & \text{instead of } \theta \\ a^2 Z d\xi d\bar{\theta} & \text{instead of } W \end{cases}$$

Solutions for the distributed load Z is obtained by integration of the solutions for $a^2 Z d\xi d\bar{\theta}$ with regard to $\bar{\theta}$ and $\bar{\xi}$ in the domain where Z is distributed, i. e., $\bar{\xi}^2 + \sin^2 \bar{\theta} \leq \epsilon^2$.

In other words, when we let $\phi_c = \frac{W}{\pi a} \phi_c(\xi)$ $\cos n\theta$ be a solution for the concentrated load, the corresponding solution for the distributed load can

be expressed in the form of:

Table 1 Numerical results

where

$$\gamma = a/h, \quad \epsilon = e/a$$

$$\eta = \epsilon \sqrt{k} = \epsilon \sqrt{\gamma} \sqrt{12(1-\mu^2)}$$

r	ϵ	η	$-N_\theta/W/h$	$-N_x/W/h$	M_θ/W	M_x/W
15	0.03	0.21	0.182	0.146	0.307	0.266
	0.125	0.88	0.167	0.132	0.159	0.121
	0.25	1.76	0.129	0.110	0.094 4	0.061 0
50	0.03	0.38	0.193	0.170	0.254	0.206
	0.125	1.61	0.138	0.143	0.112	0.070 0
	0.25	3.21	0.074 3	0.106	0.057 9	0.026 4
100	0.03	0.54	0.189	0.177	0.222	0.172
	0.125	2.27	0.108	0.138	0.086 1	0.045 8
	0.25	4.54	0.044 0	0.091 8	0.041 6	0.015 4
300	0.03	0.94	0.172	0.179	0.171	0.119
	0.125	3.94	0.055 6	0.113	0.052 3	0.020 5
	0.25	7.87	0.015 5	0.064 0	0.024 1	0.007 67

Fig. 5

Moment

(a) Bijlaard's problem

Moment

(b) Our problem

Fig. 6 Moment diagram

$$\Phi_D = \int_{-\epsilon}^{\epsilon} \int_{-\sin^{-1}\sqrt{\epsilon^2-\xi^2}}^{\sin^{-1}\sqrt{\epsilon^2-\xi^2}} \left[\frac{1}{\pi a} \frac{W}{\pi \epsilon^2} \phi_c(|\xi-\bar{\xi}|) \cos(\theta-\bar{\theta}) \right] d\bar{\theta} d\bar{\xi} = \frac{2}{\pi^2 \epsilon^2} \frac{W}{a} \cos n\theta \int_{-\epsilon}^{\epsilon} \phi_c(|\xi-\bar{\xi}|) h(\bar{\xi}) d\bar{\xi} \dots (22)$$

where

$$h(\bar{\xi}) = \begin{cases} \sin^{-1} \sqrt{\epsilon^2 - \bar{\xi}^2} & \dots n=0 \\ \frac{1}{n} \sin(n \sin^{-1} \sqrt{\epsilon^2 - \bar{\xi}^2}) & \dots n \geq 1 \end{cases}$$

The numerical results were obtained for the following parameters, using an IBM 7090 machine.

$$\begin{cases} \gamma (=a/h) = 15, 50, 100, 300 \\ \epsilon (=e/a) = 0.03, 0.125, 0.25 \\ \mu = 0.3 \end{cases}$$

Eq. (22) was numerically integrated utilizing the Gauss's integration method. The computations of the Fourier's series were done for values $n=100$. We saw that the numerical values of membrane forces (N_x, N_θ) and of bending moments (M_x, M_θ) were converged at $n=20 \sim 40$ and at $n=50 \sim 100$ respectively in the order of 1% accuracy. The maximum stress occurs at the origin point $(\xi, \theta) = (0, 0)$. These numerical results are shown in Table 1 and Fig. 5. The Bijlaard's results are also plotted on Fig. 5, where his results are modified in the following two points.

(i) The square loading surface is replaced with a circle of the same area.

(ii) He dealt with a definite cylinder with the ratio of $l/a=8$ which is simply supported at both edges. When we assume this cylinder as a beam, his problem is equivalent to treating the beam shown in Fig. 6(a). On the other hand the cylinder dealt with in this paper corresponds to the beam shown in Fig. 6(b).

Bijlaard's beam has the bending moment $Wl/4$ at $x=0$, and so Bijlaard's cylinder has the membrane force $N_x = -\frac{Wl}{4\pi a^2} \cos \theta$ at $x=0$. Therefore his values of N_x plotted in Fig. 5 are subtracted by

$$N_x/(W/h) = \frac{1}{4\pi} \frac{h}{a} \frac{l}{a} = + \frac{2}{\pi \gamma}$$

Fig. 5 shows good agreements between the two results, but a little disagreement is seen in the results of N_x and N_θ (especially N_x for $\gamma=50$ and

$$N_\theta \text{ for } \gamma=100, \epsilon = \frac{2}{\sqrt{\pi}} \times \frac{1}{32}.$$

4. Conclusions

(1) The characteristic equation of the fundamental differential equation can be easily solved, and further introduction of the stress function makes calculation of stress components easy.

(2) In this paper we solved a problem for a load in the direction of z . By the same way we shall be able to get solutions for loads in the directions of x and y or for moments around x and y axis.

(3) As one of the causes of a little disagreement between the Bijlaard's and our results, it is considered that the Timoshenko's basic equations used by Bijlaard have less accuracy comparing with Flugge's or ours.

(4) When we use practically the results in Fig. 5, we must consider supporting conditions at edges and modify the values of N_x .

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