The Longitudinal Impact of Coil Springs*

By Setuo KOBAYASHI** and Yoshiaki TAKENOUCHI**

The impact problems of an elastic body treated by Saint Venant, Bousinessq and Donnel are only about the transient state in the shocked surface of the body. Pipes transformed a shocked system into the equivalent electric circuit and solved it with the operative method.

In this paper the authors introduce Dirac's δ(t)-function into the boundary condition in the parallel system of a coil spring and a damper, and solved it with the method of Laplace transformation.

The results are summarized as follows:

(1) The condition that a falling body separates from a spring after impact is obtained.
(2) The maximum stress at the fixed end of a spring and the approximate equation for it are found.
(3) The effects of a damping coefficient of the restitution coefficient are made clear.
(4) The behavior that a damping coefficient affects the spring stress due to impact and the transmission force at the fixed end of the spring is known.

1. Introduction

The impact problems of an elastic body treated by Saint Venant(1), Bousinessq(2) and Donnel(3) are only about the transient state in the shocked surface of the body. Pipes(4) transformed a shocked system into the equivalent electric circuit and solved it with the operative method. In addition to these, there are the following recent papers on the elastic impact, that is, the study of the numerical method(5), the study of the case of a coil contacting with the next coil(6), the study of the impact with friction like ring springs(7), and the proposal of a new operator for the analysis(8).

In this paper the authors introduce Dirac's δ(t)-function into the boundary condition in the parallel system of a coil spring and a damper, and solved it with the method of Laplace transformation.

2. Nomenclature

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<th>Symbol</th>
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<td>d</td>
<td>wire diameter</td>
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<td>D</td>
<td>mean diameter of spring</td>
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<td>c</td>
<td>spring index</td>
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<td>n</td>
<td>number of active coils</td>
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<td>l</td>
<td>length of active coils</td>
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<td>G</td>
<td>torsional modulus of elasticity</td>
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<td>ρ</td>
<td>mass of the spring material per unit of volume</td>
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\[ k = Gd/(8nD^2) \]: spring rate
\[ k_2 = k l \]: spring rate per unit of length
\[ g \]: acceleration of gravity
\[ W \]: weight of a mass striking a spring
\[ W_s = (\pi/4)d^4lp_g \]: weight of active coils
C*: damping coefficient
\[ C_r = 2\sqrt{Wk/g} \]: critical damping coefficient
\[ \gamma = C^*/C \]
\[ a = \sqrt{\frac{kd^2}{W_s}} \cdot \left( \frac{1}{1 + (1/2c^2)} \right) = \frac{1}{c} \sqrt{\frac{G}{2\rho}} \cdot \left( \frac{1}{1 + (1/2c^2)} \right) \]: velocity of propagation of shock wave (surge velocity)
\[ T = 2l/a \]: time required to go and return once in a spring for shock wave (surge time)
\[ v_0 \]: impact velocity
\[ t \]: time
\[ t_1 \]: contact duration of a mass and a spring
\[ \gamma = \frac{a^2W}{kd^2} = \frac{1}{W_s} \cdot \left( \frac{1}{1 + (1/2c^2)} \right) \], \[ \lambda = \frac{kd^2}{aW} \]
\[ \xi = \frac{a^2W}{kd^2} \]: \[ \eta = \xi \cdot \frac{a}{v_0} \], \[ \xi_1 = \xi \cdot (1 + 2\sqrt{\gamma}) \], \[ \xi_2 = \xi \cdot (1 - 2\sqrt{\gamma}) \], \[ \mu = \xi_1/\xi_2 \]
\[ x \]: wire length along coils from the free end of a spring
\[ y(x, t) \]: displacement of point x at time t
\[ v(x, t) \]: velocity of point x at time t
\[ p(x, t) \]: force of point x at time t
\[ W_k = k_2v_0 = p(0, 0) \]: initial force of the struck end of spring

The theory for a coil spring in this paper is

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also applicable to the longitudinal impact problem of uniform bar by replacing $k$ and $a$ as follows:

$A$: cross-sectional area of a bar
$E$: Young's modulus
$l$: length of a bar
$W$: weight of a bar

\[ k_0 = EA, \quad k = k_0/l, \quad a = \sqrt{\frac{k_0 g}{W_s}} \]

3. Equations of motion

3.1 Assumptions

We take the following assumptions in this paper.

1. Effect of pitch angle of springs is negligible.
2. The damping of springs is negligible.
3. Strain of spring is proportional to the stress. Deflection is under the elastic limit.
4. Neighboring coils do not touch each other.
5. Damping force of a damper is proportional to its velocity, and its weight is disregarded. Cylinder and piston in it are looked upon as rigid.

3.2 Consideration of the impact force

Since a shock wave in a coil spring is transmitted from one end to the other, we can neglect the mass of the moving coil at the moment of impact. As the origin of time we take the moment when a rigid body impacts the end of a coil spring, and we assume the velocity change in the short time after the striking moment as shown in Fig.1, so the end velocity will be shown as follows.

\[ v = v_0 u(t) \]

where $u(t)$ is the unit step function.

The phenomenon at the impact moment mentioned above is equivalent to the phenomenon that a rigid body $W$ resting on the top of a coil spring is given the velocity change as shown in Fig.1.

The instantaneous force required for this change is

\[ \frac{d}{dt} \left( \frac{W}{g} v_0 u(t) \right) = \frac{W}{g} v_0 \delta(t) \]

where $\delta(t)$ is a Dirac's $\delta$-function, viz., $\delta(t) = 0$ at $t \neq 0$, $\delta(0) = \infty$.

\[ \int_{-\infty}^{\infty} \delta(t) dt = 1 \]

3.3 The equation of motion and the boundary condition

We take the $y$-axis vertically downward from the free end of a coil spring at rest and the $x$-axis along the coil from the free end to the rigid end. If the damping force acting on a spring is proportional to the deforming velocity, we can write the equation of motion of the coil element as follows:

\[ \frac{\partial^2 y}{\partial t^2} + \beta \frac{\partial y}{\partial t} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{3} \]

where $a$ is the so-called surge velocity

\[ a = \frac{1}{c} \sqrt{\frac{G}{2 \rho}} \frac{1}{1 + (1/2 \epsilon^2)} \tag{4} \]

and $\beta$ is the constant depending upon the inherent damping coefficient $\epsilon$ of a spring.

\[ \beta = \frac{4 \epsilon}{\pi d^2 \rho} \frac{1}{1 + (1/2 \epsilon^2)} \tag{5} \]

For simplicity, we neglect the inherent damping force and put $\beta = 0$ in Eq. (3), i.e.,

\[ \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \tag{6} \]

The restoring force $p(x, t)$ and the particle velocity $v(x, t)$ in the $y$-direction at any point $x$ may be written as

\[ p = -k_0 \frac{\partial y}{\partial x} \tag{7} \]

\[ v = \frac{\partial v}{\partial t} \tag{8} \]

If a coil spring in the parallel system of the spring and a damper as shown in Fig.2 is at rest before impact, the initial conditions are

\[ y(x, 0) = 0 \tag{9} \]

\[ v(x, 0) = 0 \tag{10} \]

The boundary conditions for the free and fixed end may be written from section 3.2 as follows, respectively:

\[ \frac{W}{g} \frac{d}{dt} v(0, t) = \frac{W}{g} v_0 \delta(t) + W - p(0, t) - \epsilon v(0, t) \]

\[ v(t, t) = 0 \]

Under the above initial and boundary conditions, we will solve Eq. (6) by applying the following Laplace transformation;
\[ Y(x,s) = \int_0^\infty y(x,t)e^{-st}dt \]
\[ V(x,s) = \int_0^\infty v(x,t)e^{-st}dt \]
\[ P(x,s) = \int_0^\infty p(x,t)e^{-st}dt \]  
Equation (6) is transformed under the initial conditions (9) and (10) to
\[ \frac{d^2}{dx^2} Y(x,s) - \frac{s^2}{a^2} Y(x,s) = 0 \]  
The general solution of this differential equation is given explicitly by
\[ Y(x,s) = A(s)e^{-s(x/a)^2} + B(s)e^{s(x/a)^2} \]  
where \( A(s) \) and \( B(s) \) are integral constants. By substitution of Eq. (15) into the driving transforms of Eqs. (7) and (8), the following are found:
\[ P(x,s) = -\frac{k_b}{a} s \left[ A(s) - B(s) \right] - e^{ks} \left( A(s) + B(s) \right) \]  
From Eq. (15) and the driving transform of Eq. (12), the following is found:
\[ A(s)e^{-s(x/a)^2} + B(s)e^{s(x/a)^2} = 0 \]
\[ P(x,s) = \sum_{i=1}^\infty \left[ \frac{s - \xi_i s}{s + \xi_i s} \right]^{-1} \left[ e^{-\left[2i(1-\iota)\pi/\lambda\right]s/a} + e^{-\left[2i(1+\iota)\pi/\lambda\right]s/a} \right] \]  
In the above three equations, the terms including \( e^{-((2i-1)\pi/\lambda)s/a} \) denote the incident waves and the terms including \( e^{-((2i+1)\pi/\lambda)s/a} \) denote the reflected waves.

4. Derivation of calculating formula

4.1 Case of spring without damper (\( \gamma = 0 \))

Let us obtain the force \( p(0,t) \) at the free end of a coil spring. Putting \( \gamma = 0 \) (in this case, \( \xi_1 = \xi_2 = \xi \)) and \( \omega = 0 \) in Eq. (25), we get
\[ \frac{P(0,s)}{W\lambda} = \frac{1}{s + \xi} + \frac{\xi}{\lambda} \frac{1}{s(s + \xi)} + \sum_{i=1}^\infty \left[ \left( \frac{s - \xi_i s}{s + \xi_i s} \right)^{i-1} \left( \frac{s - \xi s}{s + \xi s} \right)^{i-1} \right] \]  
For convenience sake, the next three equations are derived by expanding the left hand side of each equation in the partial fractions and performing inverse transformation on every term.
\[ \begin{cases} (s - \xi s)^{-1}\ e^{-\alpha s} & 0 < t < \alpha \\ (s+\xi s)^i \ e^{t(s-a)} \sum_{j=1}^\infty (-\xi)^{j-1}A_{ij} \left[ (t-\alpha)^{j-1} - 1 \right] & \alpha \leq t \end{cases} \]  
\[ \begin{cases} (s - \xi s)^{-1}\ e^{-\alpha s} & 0 < t < \alpha \\ (s+\xi s)^i \ e^{t(s-a)} \sum_{j=1}^\infty (-1)^{j-1}B_{ij} \left[ (t-\alpha)^{j-1} - 1 \right] & \alpha \leq t \end{cases} \]  

The integral constants \( A(s) \) and \( B(s) \) may be obtained from Eqs. (18) and (19) as follows:
\[ A(s) = \frac{v_0 + \frac{\xi}{s}}{s(s + \xi_1)(1 - Re^{-s(2i-1)\pi/\lambda\xi})} \]  
\[ B(s) = \frac{v_0 + \frac{\xi}{s}}{s(s + \xi_1)(1 - Re^{-s(2i+1)\pi/\lambda\xi})} \]  
where
\[ R_0 = \frac{s - \xi s}{s + \xi_1}, \quad \xi_1 = \xi(1 + 2\gamma\sqrt{\eta}), \quad \xi = \frac{k_b}{aW} \]

Substituting Eqs. (20) and (21) into Eqs. (16), (17) and (15), we get the following three equations.
\[ \begin{align*} P(x,s) &= \frac{k_b}{a} \left[ \frac{v_0 + \frac{\xi}{s}}{s(s + \xi_1)(1 - Re^{-s(2i-1)\pi/\lambda\xi})} \\ V(x,s) &= \frac{v_0 + \frac{\xi}{s}}{s(s + \xi_1)(1 - Re^{-s(2i-1)\pi/\lambda\xi})} \\ Y(x,s) &= \frac{v_0 + \frac{\xi}{s}}{s(s + \xi_1)(1 - Re^{-s(2i+1)\pi/\lambda\xi})} \end{align*} \]  
As it is difficult to perform inverse transformation on Eqs. (22), (23) and (24), let these equations be expanded in power series as follows:
\[ \left( \frac{s - \xi s}{s + \xi s} \right)^{-1} \sum_{i=1}^\infty \left[ \frac{s - \xi s}{s + \xi s} \right]^{i-1} \left[ e^{-\left[2(1-i)\pi/\lambda\right]s/a} + e^{-\left[2(1+i)\pi/\lambda\right]s/a} \right] \]  

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\[
\frac{(s-\xi)^{i-1}}{s^2(s+\xi)^t} e^{-\alpha} \sum_{k=0}^{t-1} \frac{(-1)^k}{\xi^2} \left( (2i-1) - \xi(t) - \alpha - e^{-t(i-\alpha)} \sum_{j=1}^{i-1} C_{j,i} (t-\alpha)^{i-1} \right) ; \quad t \geq \alpha
\]

where

\[
A_{i,j} = \frac{2i-1}{(j-1)!} \left( i - 1 \right) \quad (j-1)!
\]

\[
B_{i,j} = \frac{2i-1}{(j-1)!} \left( \frac{1}{\xi^2} \right) \left( \frac{(-1)^k}{k} \right) \left( i - 1 \right) \quad (j-1)!
\]

\[
C_{i,j} = \frac{2i-1}{(j-1)!} \sum_{k=0}^{j} \frac{(-1)^k}{k} \left( i - 1 \right) \quad (j-1)!
\]

The values of \(A_{i,j}\), \(B_{i,j}\), and \(C_{i,j}\) are shown in Tables 1, 2, and 3, respectively. Applying Eqs. (29) and (30) to Eq. (28) and symbolizing the inverse transform of terms including \(e^{-t(2i-1)/\alpha}\) by \(p_0(0,t)\), the following is obtained:

\[
p_0(0,t) = \begin{cases} 
0; & \frac{t}{T} < i-1 \\
\frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right) e^{-t/\alpha} & \frac{t}{T} \geq i-1
\end{cases}
\]

Eq. (35) is applied to the case of \(i \geq 2\), and for the special case of \(i = 1\) the equation corresponding to Eq. (35) is

\[
p_0(0,t) = \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right) e^{-t/\alpha}
\]

Then, the spring force of the free end \(p(0,t)\) in the interval \(n-1 \leq t/T \leq n\) is given by

\[
p(0,t) = \sum_{i=1}^{n} p_0(0,t)
\]

Let us find the particle velocity \(v(0,t)\) and the deformation \(y(0,t)\) of the free end of a spring. Putting \(\gamma = 0\) and \(x = 0\) in Eqs. (26) and (27), we get

\[
V(0,s) = \frac{1}{s+\xi} + \frac{\xi}{s(s+\xi)} + \sum_{i=1}^{\infty} \left[ \frac{(s-\xi)^{i-1}}{s(s+\xi)} \right] \frac{1}{\lambda} e^{-t(1-i)/\alpha}
\]

\[
Y(0,s) = \frac{\xi \gamma}{2s(s+\xi)} + \frac{\xi \gamma}{2s^2(s+\xi)^2} + \sum_{i=1}^{\infty} \left[ \frac{(s-\xi)^{i-1}}{s(s+\xi)} \right] \frac{1}{\lambda} e^{-t(1-i)/\alpha}
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Putting \( r = 0 \) and \( x = 1 \) in Eq. (25) to find the spring force \( p(l, t) \) of the fixed end, the following is obtained:

\[
\frac{P(l, t)}{W \lambda} = \sum_{i=1}^{\infty} \left( \frac{(s - \xi)}{(s + \xi)} \right)^{l-1} \frac{e^{-[(\tau - \xi)/(\lambda)] l}}{s(s + \xi)} + \frac{\xi}{\lambda} \frac{e^{-[(\tau - \xi)/(\lambda)] l}}{s(s + \xi)} \] \quad \text{-----------------------------------------------(41)}
\]

Applying Eqs. (29), (30) and (31) to Eqs. (29) and (40) and symbolizing the inverse transform of terms including \( e^{-[(\tau - \xi)/(\lambda)] l} \) by \( v_i(0, t) \) and \( y_i(0, t) \), respectively, the following are obtained:

\[
v_i(0, t) = \left\{ \begin{array}{l}
0: \frac{T}{l} < i - 1 \\
\frac{\lambda}{2} \left[ 2 - \frac{e^{-(\tau - \xi)}(B_{t_{i-1}}(\tau_{i-1}))^{l-1} + \sum_{n=1}^{l-1} (B_{t_{i-1}} + B_{t_{i-1}})(\tau_{i-1})^{l-1}]}{2 \lambda} \right] \quad \text{if } \frac{T}{l} \geq i - 1 \\
\end{array} \right.
\] \quad \text{-----------------------------------------------(42)}

\[
y_i(0, t) = \left\{ \begin{array}{l}
0: \frac{T}{l} < i - 1 \\
\frac{\lambda}{2} \left[ -2\tau_{i-1} - \frac{e^{-(\tau - \xi)}(C_{t_{i-1}}(\tau_{i-1}))^{l-1} + \sum_{n=1}^{l} (C_{t_{i-1}} + C_{t_{i-1}})(\tau_{i-1})^{l-1}]}}{2 \lambda} \right] \quad \text{if } \frac{T}{l} \geq i - 1 \\
\end{array} \right.
\] \quad \text{-----------------------------------------------(43)}

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Table 1: Values of \( B_{t_f} \)

Table 2: Values of \( C_{t_f} \)
Eqs. (42) and (43) are applied to the case of \( i \geq 2 \), and for the special case of \( i = 1 \) the equations corresponding to Eqs. (42) and (43) are respectively

\[
\begin{align*}
y_i(0, t) &= \frac{1}{\lambda} + \left(1 - \frac{1}{\lambda}\right) e^{-\tau_0} \\
y_i(0, t) / v_0 T &= \frac{7}{2\lambda} \left(\lambda - 1 + \tau_0\right) + (1 - \lambda) e^{-\tau_0} \quad (45)
\end{align*}
\]

The particle velocity \( v_i(0, t) \) and the deformation \( y_i(0, t) \) of the free end of a spring in the interval \( n - 1 \leq t / T \leq n \) may be expressed by the above obtained \( v_i(0, t) \) and \( y_i(0, t) \) as follows:

\[
\begin{align*}
v_i(0, t) &= \sum_{i=1}^{n} v_i(0, t) \\
y_i(0, t) &= \sum_{i=1}^{n} y_i(0, t) \quad (46) \quad (47)
\end{align*}
\]

Applying Eqs. (29) and (30) to Eq. (41) and symbolizing the inverse transform of the terms including \( e^{-[(i(i-1)/2)]/\sigma^2} \) by \( p_i(l, t) \), the following is obtained:

\[
p_i(l, t) / W \lambda = \begin{cases} 0; & t \leq 1/2 \\
\left[2 e^{-\tau_0} \sum_{j=1}^{n} A_{ij} \tau_0 (\tau_0 - 1) / \lambda \right] \left[1 - e^{-\tau_0} \sum_{j=1}^{n} B_{ij} \tau_0 (\tau_0 - 1) / \lambda \right] \end{cases} \quad (48)
\]

where \( \tau_0 = (2/\pi)(4T - t - 1/2) \).

Eq. (48) may be used for the case of \( i \geq 1 \). The spring force of the fixed end \( p(l, t) \) in the interval \( n - 1/2 \leq t / T \leq n + 1/2 \) is given by

\[
p(l, t) = \sum_{i=1}^{n} p_i(l, t) \quad (49)
\]

4-2 Case of spring with damper

Putting \( z = 0 \) in Eqs. (25), (26) and (27) to find \( p(0, t) \), \( v(0, t) \) and \( y(0, t) \), the following are obtained, respectively:

\[
\begin{align*}
P(0, s) / W \lambda &= \frac{1}{s + \xi_1} + \frac{\xi_1}{\lambda s(s + \xi_1)} + \sum_{i=2}^{n} \left[\frac{(s - \xi_2)^{i-1}}{s(s + \xi_1)^i} + \frac{(s - \xi_2)^{i-2}}{(s + \xi_1)^{i-1}}\right] \quad (50) \\
V(0, s) / v_0 &= \frac{1}{s + \xi_1} + \frac{\xi_1}{\lambda s(s + \xi_1)} + \sum_{i=2}^{n} \left[\frac{(s - \xi_2)^{i-1}}{s(s + \xi_1)^i} - \frac{(s - \xi_2)^{i-2}}{(s + \xi_1)^{i-1}}\right] s e^{-[(i(i-1))/\sigma^2]} \quad (51) \\
Y(0, s) / v_0 T &= \frac{\xi_1}{2 s(s + \xi_1)} + \frac{\xi_1}{2 \lambda s(s + \xi_1)} + \sum_{i=2}^{n} \left[\frac{\xi_1}{2 s(s + \xi_1)} - \frac{(s - \xi_2)^{i-2}}{(s + \xi_1)^{i-1}}\right] s e^{-[(i(i-1))/\sigma^2]} \quad (52)
\end{align*}
\]

On the other hand, substituting \( z = 1 \) into Eq. (23), we get

\[
P(l, s) / W \lambda = \sum_{i=1}^{\mu} \left[\frac{s - \xi_2}{(s + \xi_1)^{i-1}} + \frac{\xi_1}{\lambda s(s + \xi_1)^i}\right] e^{-[(i(i-1))/\sigma^2]} \quad (53)
\]

where \( \mu = 1/(1+27/\sqrt{\eta}) \).

The inverse transforms of \( (s - \xi_2)^{i-1} / (s + \xi_1)^i \), \( (s - \xi_2)^{i-2} / (s + \xi_1)^{i-1} \), etc. in Eqs. (50), (51), (52) and (53) are the complicated functions of \( A_{ih}, B_{ih}, C_{ih} \) and \( \mu \), so that it is difficult to express them with the general equations. We performed, therefore, the inverse transformations on them for the individual value of \( i \).

The force \( F(t) \) transmitted to the foundation through the spring-damper system is given by

\[
F(t) / W \lambda = \frac{c p(0, t) + p(l, t)}{W \lambda} = 27 \sqrt{\eta} \left[\frac{p(0, t)}{v_0} + \frac{p(l, t)}{p(0, t)}\right] \quad (55)
\]

5. Consideration of the calculated results

5-1 Case of spring without damper

Fig. 3 shows three examples of the spring force \( p(0, t) \) at the free end calculated by using Eq. (38).

The curve of \( \lambda = 1.5, \eta = 3 \) crosses the transverse axis in the interval \( 3 \leq t / T \leq 4 \). It means that an impact body loses contact with a coil spring at the time \( t \), and thus the phenomenon of impact will be
finished. It is usual that the curve of \( p(0,t) \) in the impact will increase discontinuously when \( t/T \) is equal to the integer, and it will decrease constantly and cross the transverse axis finally. On the contrary, the curve of \( \lambda=1, \eta=0.51 \) in Fig. 3 is tangent to the transverse axis and the curve of \( \lambda=0.4, \eta=1 \) does not cross it. The further explanation with regard to this point will be added in consideration of Fig. 5.

Fig. 4 shows the relation of \( t_i/T \) and \( \eta \) with a parameter \( \lambda \). Every curve increases discontinuously when \( t_i/T \) is equal to integer. For example, in the case of \( \lambda=1 \), the value of \( t_i/T \) jumps from 4 to 4.31 at \( \eta=3.82 \) and \( t_i/T \) is equal to 5 at \( \eta=7.13 \). This means that, in the case of \( \lambda=1 \), the impact will finish in the interval \( 4.31 \leq t_i/T \leq 5 \) and separation can not occur in the interval \( 4 \leq t_i/T \leq 4.31 \) for \( \eta \) between 3.82 and 7.12. There is no curve of \( \lambda=1 \) in the range of \( \eta<0.5 \). This means that a striking body and a coil spring will not be separated and will maintain the vibration in the case of \( \lambda=1 \) and \( \eta<0.5 \). This is also shown in Fig. 5. As \( \lambda \) is proportional to the impact force \( p(0,0) \) or the impact velocity \( v_0 \) and inversely proportional to the weight \( W \) of a striking body, the curve of \( \lambda=\infty \) in Fig. 4 expresses not only the impact with a very large impact velocity of a striking body, but also the impact with \( p(0,0) \) much larger than \( W \), and so it expresses the horizontal impact as well as the falling impact.

The domains of the values of \( \eta \) and \( \lambda \) in which the impact will finish within the interval of the time are shown in Fig. 5. For example, the impact will finish in the interval \( 4 < t_i/T \leq 5 \) when the point \((\eta, \lambda)\) is in the domain of \([4,5]\). As shown in Fig. 5, the larger \( \eta \) is (namely, the larger \( W \) is), the more time will be necessary before the impact will finish. Every boundary line of the domains has an asymptote for \( \lambda=\infty \). This asymptote will approach to \( \eta \) at the discontinuous point of the curve of \( \lambda=\infty \) in Fig. 4. When the point \((\eta, \lambda)\) is in the domain drawn with oblique lines, that is, a light body falls on the spring at a low velocity, the striking body does not separate from the spring, but it will vibrate with the spring. In other words, the motion caused by \( \eta \) and \( \lambda \) in the domain of the oblique lines cannot be called an impact.

Fig. 6 shows the maximum spring forces \( p(0,t)_{\text{max}} \) and \( p(I,t)_{\text{max}} \) as functions of \( \eta \) with
parameters $\lambda$. As shown in Fig. 6, the larger the value of $\gamma$ is, i.e., the heavier weight $W$ of a striking body is, the larger the restoring force of a spring will be. When the values of $\lambda$ are equal, the maximum value of the spring force acting at the fixed end is always larger than one acting at the free end. The curve for the small value of $\lambda$ (this means that the impact velocity is slow) exists above one in high velocity. This is due to the division of $p$ by $W\lambda$ for making the dimensionless values. Therefore it should not be taken that the upper lines have the larger $p(0, t)$ or $p(l, t)$.

The dotted lines in Fig. 6 are approximate solutions to obtain the maximum force. They are derived as follows. When we neglect the weight of the spring, the equation of motion of a body $W$ may be written as

$$\frac{W}{g}\frac{d^2y}{dt^2} \cdot y(0, t) = W - e^{\alpha t}y(0, t) - ky(0, t) \quad \cdots (56)$$

By solving Eq. (56) under the initial condition

$$y(0, 0) = 0, \quad v(0, 0) = v_0 \quad \cdots (57)$$

the maximum reactive force at the free end of a coil spring is obtained as follows:

$$\frac{p(0, t)}{W\lambda} = \frac{1}{\lambda} \left[ 1 + \sqrt{1 + \alpha^2} \sin \left( \sqrt{\frac{2}{\eta}} \frac{t}{T} - \alpha \right) \right]$$

$$\alpha = \tan^{-1} \left( \frac{1}{\lambda \sqrt{\eta}} \right) \quad \cdots (58)$$

From Eq. (58), the maximum reactive force $p_{\text{max}}/(W\lambda) = (1 + \sqrt{1 + \alpha^2})/\lambda$. Plotting this value into Fig. 6, it is smaller by about one than the mean value of $p(0, t)_{\text{max}}$ and $p(l, t)_{\text{max}}$. Hence the following equation can be used as the approximate equation of the maximum reactive force.

$$p_{\text{max}}/(W\lambda) \approx (\lambda + 1 + \sqrt{1 + \alpha^2})/\lambda \quad \cdots (59)$$

### 5.2 Case of spring with damper

The numerical results for the cases of $\gamma = 0$ and 0.4 from Eqs. (50) and (53) are shown in Figs. 7 and 8. In both figures, the spring forces $p(0, t)$ and $p(l, t)$ at the free end and the fixed end have the forms like the teeth of a hand-saw. The chain lines in the figures are the approximate solutions of Eq. (56) under the conditions of Eq. (57), and $p/(W\lambda) = ky(0, t)/(W\lambda)$ is plotted.

There are plotted in Fig. 9 the values of the particle velocity at the free end with parameter $\gamma$ from Eq. (51). The phenomenon of impact will be finished at the point of $O$-sign and every value of $t/T$-axis is nearly equal, hence the time of
impact finishing will be almost independent of the magnitude of a damping force.

The restitution coefficient $e$ for the case of a coil spring struck by a body $A$ with the velocity $v_0$ is defined as the ratio of the separating velocity $v(0, t_s)$ to $v_0$, i.e.,

$$e = \frac{v(0, t_s)}{v_0} \tag{60}$$

The value of $e$ is shown in Fig. 10.

Fig. 11 shows the deformation $y(0, t)$ of the free end of a coil spring with parameter $\gamma$ from Eq. (52). It denotes that the smaller the damping force is, the larger the maximum deformation of a spring will be. The approximate solutions in Figs. 9 and 11 are obtained from Eq. (56) under the conditions of Eq. (57).

Fig. 12 shows the relation of the spring force and the damping ratio $\gamma$. As shown in this figure, $y(t)$ decreases uniformly according to the increment of $\gamma$, but the maximum transmitted force $F(t)$ at the fixed end has the minimum value at $\gamma = 0.3$. Thus the larger the value of $\gamma$ is, the smaller the spring stress will be, but there is an optimum value of $\gamma$ for the minimum transmitted force.

6. Conclusion

The results in this paper are summarized as follows:

(1) When a coil spring is struck by a body of which weight and velocity are larger than the fixed values, both will be separated after the time $t_s$ depending upon the values of $\gamma$ and $\lambda$; that is called the phenomenon of impact, on the contrary, when the weight and velocity are smaller than the fixed values, both will not be separated, but a vibration will occur. The domain of the time $t_s$ required for the separation of the striking body from the spring, and the limit of two phenomena of impact and vibration are shown in Fig. 5.

(2) There is hardly any difference between the approximate solutions disregarding the weight of a spring and the solution in this paper so far as the deformation and the velocity at the struck end of the spring are concerned. There is, however, considerable difference in regard to the impact force as shown in Figs. 7 and 8. Hence another approximate solution for it is derived.

(3) In the parallel system of a coil spring and a damper, the spring force is inversely proportional to the value of the damping ratio $\gamma$, but the transmitted force at the fixed end has an optimum value of $\gamma$ to make itself minimum. An example is shown in Fig. 12.

(4) The restitution coefficient $e$ defined by the ratio of striking and separating velocities is indicated in Fig. 10, and it decreases as an exponential function according to the increment of the damping ratio $\gamma$.

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References