539.384.2:624.073.6

Large Deflection Analysis of Circular Ring Plates under Uniform Transverse Force along the Inner Edge*

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Numerical calculations of large deflection of circular ring plates subjected to uniform transverse force along the inner edge, are carried out for three edge conditions and some geometric configurations. Iterative procedure developed by H. B. Keller and E. L. Reiss is applied here, and a finite difference method is used to solve the set of differential equations, which appears in each step of iterative procedures. The purposes of this paper are mainly to reveal the limitations of small deflection theory and the limitations of the approximate solutions of large deflection theory. Experimental results are given for the case of the circular ring plates which are clamped along inner and outer edges. These results are quite satisfactory, and the Keller-Reiss method is also proved to be valid for this problem.

1. Introduction

This paper deals with a large deflection problem of circular ring plates as shown in Fig. 1. Though they are important as fundamental elements of structures, there have been few investigations on this problem; these are Hamada's approximate solutions⁽¹⁾ for the case of circular ring plates with both edges clamped, and Wempner and Schmidts' solutions⁽²⁾ by a power series method for the case of simply supported edges. The former assumes constant radial membrane stresses, and therefore it is predicted that if the membrane stress distribution is unlike that of the assumption, the solution may become a poor approximation. As for the latter, the general characteristics of circular ring plates are not made clear because of few numerical examples, and the authors admit that there is no assurance that the solution converges for all values of the load parameter. In this connection, the problem of

clamped circular ring plates under uniform pressure was solved by O. Tamate and H. Abe⁽³⁾.

In this paper, the basic nonlinear equations of the problem are solved numerically using a high speed digital computer, since it seems to be very difficult to solve them by analytical method. The iterative procedures are applied here, which were originally developed by H. B. Keller and E. L. Reiss⁽⁴⁾⁽⁵⁾, and the solutions are given for various boundary conditions and for a very wide range of the load parameter in order to reveal the limits of applicabilities of Hamada's approximate solutions and the classical linear theory.

Experiments are carried out for the case of the circular ring plates with both edges clamped for two radius ratios, and the measured results of the maximum deflections and the stress distributions are found to be consistent sufficiently with the calculated results.

2. Basic equations

Equilibrium equations of the circular ring plates under the lateral force uniformly distributed along the inner edge $P/2\pi b$ (Fig. 1), are

$$\left. \begin{aligned} N_r - N_t + r \frac{dN_r}{dr} &= 0 \\ D \left(\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) &= N_r \frac{dw}{dr} + \frac{P}{2\pi r} \end{aligned} \right\} \dots\dots\dots (1)$$

and the compatibility condition is

$$r \frac{d}{dr} (N_r + N_t) + \frac{hE}{2} \left(\frac{dw}{dr} \right)^2 = 0 \dots\dots\dots (2)$$

where w is the vertical deflection, N_r and N_t are

Fig. 1 Plate geometry and coordinates

* Received 26th May, 1964.

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the radial and circumferential forces per unit length, respectively, and D is the flexural rigidity,

$$D = Eh^3/12(1-\nu^2)$$

Introducing new dimensionless variables z , p , q , a stress function ϕ , and a load parameter k in the following forms⁽⁶⁾;

$$\left. \begin{aligned} z &= \frac{r^2}{a^2}, & q &= -\left(\frac{2\pi b^2 E}{P} \frac{a^2}{b^2} \frac{h}{a}\right)^{1/3} \frac{a}{r} \frac{dw}{dr} \\ p &= -\left(\frac{2\pi b^2 E}{P} \frac{a^2}{b^2} \frac{h}{a}\right)^{2/3} \frac{1}{E} \frac{1}{r} \frac{d\phi}{dr} \\ k &= \frac{a}{h} \left(\frac{P}{2\pi b^2 E} \frac{b^2}{a^2} \frac{h}{a}\right)^{1/3} \{12(1-\nu^2)\}^{1/2} \\ N_r &= \frac{h}{r} \frac{d\phi}{dr}, & N_t &= h \frac{d^2\phi}{dr^2} \end{aligned} \right\} \dots\dots\dots (3)$$

the basic Eqs. (1) and (2) become

$$\frac{4}{k^2} \frac{d^2}{dz^2} (zq) = -\frac{1}{z} - pq \dots\dots\dots (4)$$

$$8 \frac{d^2}{dz^2} (zp) = q^2 \dots\dots\dots (5)$$

The variable q is proportional to the slope, p is proportional to the radial membrane stress and the load parameter k is related to the external force P .

It should be noted that there are following difficulties except those from nonlinearity in solving Eqs. (4) and (5); that is,

- (i) There is a singularity at $z=0$.
- (ii) The edge-effect appears as $k \rightarrow \infty$.

Boundary conditions at the inner edge $z=\rho=(b/a)^2$ which is assumed to be always clamped are $q(\rho)=0$, $2\rho p'(\rho)+(1-\nu)p(\rho)=0 \dots\dots\dots (6)$

At the outer edge, three types of support are conceivable:

Case A: Clamped edge

$$q(1)=0, \quad 2p'(1)+(1-\nu)p(1)=0 \dots\dots\dots (7)$$

(no slope and no radial displacement)

Case B: Pinned edge free to rotate

$$\left. \begin{aligned} 2q'(1)+(1+\nu)q(1) &= 0 \\ 2p'(1)+(1-\nu)p(1) &= 0 \end{aligned} \right\} \dots\dots\dots (8)$$

(no bending moment and no radial displacement)

Case C: Simply supported edge

$$2q'(1)+(1+\nu)q(1)=0, \quad p(1)=0 \dots\dots\dots (9)$$

(no bending moment and no radial membrane stress)

The radial and circumferential bending stresses σ_b and σ_{bt} , the radial and circumferential membrane stresses σ_r and σ_t , and the vertical deflection w can be calculated from p , q and their derivatives by the following relations:

$$\frac{\sigma_b a^2}{Eh^2} = 6k \{12(1-\nu^2)\}^{-3/2} \{2zq' + (1+\nu)q\} \dots\dots\dots (10)$$

$$\frac{\sigma_r a^2}{Eh^2} = -k^2 \{12(1-\nu^2)\}^{-1} p \dots\dots\dots (11)$$

$$\frac{\sigma_{bt} a^2}{Eh^2} = 6k \{12(1-\nu^2)\}^{-3/2} \{2\nu zq' + (1+\nu)q\} \dots\dots\dots (12)$$

$$\frac{\sigma_t a^2}{Eh^2} = -k^2 \{12(1-\nu^2)\}^{-1} (p + 2zp') \dots\dots\dots (13)$$

$$\frac{w}{h} = -\frac{k}{2} \{12(1-\nu^2)\}^{-1/2} \int q dz \dots\dots\dots (14)$$

3. Iterative procedures

The iterative procedures to solve Eqs. (4) and (5) can be formulated by the following steps of calculations:

(1) Set $m=1$.

(2) Solve

$$\frac{4}{k^2} \frac{d^2}{dz^2} (zq_m) = -\frac{1}{z} - p_{m-1} q_{m-1} \dots\dots\dots (15.a)$$

(3) Solve

$$8 \frac{d^2}{dz^2} (zp_m) = q_m^2 \dots\dots\dots (15.b)$$

(4) Set $m+1 \rightarrow m$ and jump to (2).

where p_0 and q_0 are taken to be zero. It should be noticed that the solution q_1 is nothing but one by the linear theory. For a sufficiently large number of m , the iterative solutions p_m and q_m are expected to converge to the solutions of Eqs. (4) and (5) together with the boundary conditions.

However, it was found by test calculations that the above iterative procedures are available only to small values of the parameter k ; that is, for the larger values than a certain value of k , the above iterative solutions diverge. Therefore, modifying the process of iteration in order to make the iterative solution converge for all values of the parameter k , the following procedures are introduced:

(1) Set $m=1$.

(2) Solve

$$8 \frac{d^2}{dz^2} (zp_m) = q_{m-1}^2 \dots\dots\dots (16.a)$$

(3) Solve

$$\frac{4}{k^2} \frac{d^2}{dz^2} (zq_m^*) = -\frac{1}{z} - p_m q_{m-1} \dots\dots\dots (16.b)$$

(4) Calculate q_m by

$$q_m = \lambda q_m^* + (1-\lambda)q_{m-1}, \quad 0 < \lambda \leq 1 \dots\dots\dots (16.c)$$

(5) Set $m+1 \rightarrow m$ and jump to (2).

where λ is a so-called relaxation parameter⁽⁵⁾, and the solutions at a smaller value of the parameter k are used as initial estimates p_0 and q_0 .

In practice of calculations, the value of the parameter λ should be presumed. For the case of circular plates, H. B. Keller and E. L. Reiss obtained the optimum values of the relaxation parameter from both test calculations and analytical procedures, and they discussed the relation between a convergence of iterative procedures and a value

of the parameter. And the optimum values of the relaxation parameter obtained theoretically are found to have remarkable difference from those by test calculations so that the use of the theoretical results seems to be rather dangerous because of increasing computing time. Consequently, in this paper, the values of the parameter obtained from test calculations which are considered to be nearly optimum are adopted. From a practical point of view, it is not difficult to seek for such values (see Table 1).

$$u_n' = (2\bar{h})^{-1}(-u_{n-1} + u_{n+1}) + O(\bar{h}^2) \quad (17\cdot a)$$

$$u_n'' = (\bar{h})^{-2}(u_{n-1} - 2u_n + u_{n+1}) + O(\bar{h}^2) \quad (17\cdot b)$$

Eqs. (15·a) to (15·b) and (16·a) to (16·c) become

$$z_{n+1}q_m(z_{n+1}) - 2z_nq_m(z_n) + z_{n-1}q_m(z_{n-1}) = -\frac{1}{4}k^2\bar{h}^2\left[\frac{1}{z_n} + p_{m-1}(z_n)q_{m-1}(z_n)\right] \quad (15\cdot c)$$

$$z_{n+1}p_m(z_{n+1}) - 2z_np_m(z_n) + z_{n-1}p_m(z_{n-1}) = \frac{1}{8}\bar{h}^2[q_m(z_n)]^2 \quad (15\cdot d)$$

$$z_{n+1}p_m(z_{n+1}) - 2z_np_m(z_n) + z_{n-1}p_m(z_{n-1}) = \frac{1}{8}\bar{h}^2[q_{m-1}(z_n)]^2 \quad (16\cdot d)$$

$$z_{n+1}q^*(z_{n+1}) - 2z_nq^*(z_n) + z_{n-1}q_m^*(z_{n-1}) = -\frac{1}{4}k^2\bar{h}^2\left[\frac{1}{z_n} + p_m(z_n)q_{m-1}(z_n)\right] \quad (16\cdot e)$$

$$q_m(z_n) = \lambda q_m^*(z_n) + (1-\lambda)q_{m-1}(z_n) \quad (16\cdot f)$$

respectively, where

$$\bar{h} = (1-\rho)/N, \quad z_n = \rho + n\bar{h}, \quad n=0, 1, 2, \dots, N \quad (18)$$

Using the approximation

$$u_n' = \bar{h}^{-1}(u_{n+1} - u_n) + O(\bar{h}) \quad (19)$$

Eq. (6), the boundary conditions at the inner edge, becomes

$$q_m(z_0) = 0, \quad \left[1 - \frac{\bar{h}(1-\nu)}{2\rho}\right]p_m(z_0) - p_m(z_1) = 0$$

and Eqs. (7) to (9), the conditions at $z=1$, become

$$\text{Case A: } q_m(z_N) = 0, \quad \left[1 + \frac{\bar{h}(1-\nu)}{2}\right]p_m(z_N) - p_m(z_{N-1}) = 0$$

$$\text{Case B: } \left[1 + \frac{\bar{h}(1+\nu)}{2}\right]q_m(z_N) - q_m(z_{N-1}) = 0, \quad \left[1 + \frac{\bar{h}(1-\nu)}{2}\right]p_m(z_N) - p_m(z_{N-1}) = 0$$

$$\text{Case C: } \left[1 + \frac{\bar{h}(1+\nu)}{2}\right]q_m(z_N) - q_m(z_{N-1}) = 0, \quad p_m(z_N) = 0$$

The approximation of Eq. (19) is generally adopted to such type of a problem as dealt with in this paper because of its simplicity, but the order of its error estimates is only $O(\bar{h})$ while at the inner mesh points it is $O(\bar{h}^2)$ as shown in Eqs. (17·a) and (17·b). In the stage of numerical calculations, the

Table 1 Values of λ and the number of iterations N' [values of k of the solutions used as the initial estimates are shown in brackets. (0) means that $p_0 = q_0 = 0$]

b/a	Case A			Case B			Case C		
	k	λ	N'	k	λ	N'	k	λ	N'
0.5	2 (0)	1	3	2 (0)	1	6	4 (0)	1	12
	4.5 (0)	1	9	4 (2)	0.5	28	6 (4)	0.2	23
	6 (4.5)	0.5	12	5 (4)	0.2	31	6.5 (6)	0.2	30
	6.5 (6.0)	0.4	14	6 (5)	0.15	39			
0.25	2 (0)	1	4	2 (0)	1	34	2 (0)	1	7
	4 (2)	0.6	11	4 (2)	0.2	37	4 (2)	0.2	38
	6 (4)	0.3	24	6 (4)	0.1	36	6 (4)	0.15	61
0	2 (0)	1	7	1 (0)	1	5	2 (0)	1	14
	4 (2)	0.5	39	2 (1)	0.5	18	4 (2)	0.2	45
	5 (4)	0.3	34	4 (2)	0.12	65	5 (4)	0.2	75

solutions by Eqs. (17) and (19) will undoubtedly converge to the exact solutions as $\bar{h} \rightarrow 0$, but practically speaking, it is impossible to increase the mesh points unlimitedly under restraints of the capacity and computing time of a computer. Hence, it is desirable to use the following approximation, whose order of error estimates is equal to that at the inner mesh points, in order to decrease the mesh points:

$$u_n' = (2\bar{h})^{-1}[-3u_n + 4u_{n+1} - u_{n+2}] + O(\bar{h}^2) \dots \dots (20)$$

Using Eq. (20), for instance, the boundary conditions for Case A become

$$\begin{aligned} q_m(z_0) &= 0, & \left[3 - \frac{\bar{h}(1-\nu)}{\rho}\right] p_m(z_0) - 4p_m(z_1) \\ & & + p_m(z_2) = 0 \\ q_m(z_N) &= 0, & p_m(z_{N-2}) - 4p_m(z_{N-1}) \\ & & + [3 + \bar{h}(1-\nu)] p_m(z_N) = 0 \end{aligned}$$

Comparisons of accuracy between the results by Eq. (19) and the results by Eq. (20) will be presented later.

At each step of iteration, the problem is reduced to solving the linear ordinary differential equations (15·a) to (15·b) or (16·a) to (16·c), and by means of the finite difference method, these equations together with the boundary conditions become systems of linear algebraic equations or tridiagonal equations, which can be easily solved by the elimination method in which only two sweeps are required. The elimination method seems to be valid even in the domain where the edge-effect occurs⁽⁷⁾. Practical procedures are as follows:

(i) At first, a sufficiently small value of the parameter k is selected. Then the iteration by Eqs. (15·c) to (15·d) where $p_0 = q_0 = 0$ is applied.

(ii) When the value of the parameter k for which the solutions by Eqs. (15·c) to (15·d)

diverge is reached, the iteration by Eqs. (16·d) to (16·f) should be applied. As previously discussed, the optimum value of the parameter λ which is influenced by the parameter k and the initial estimates p_0 and q_0 , should be presumed by test calculations. The last solutions by the procedure (i) should be used as the initial estimates.

(iii) To obtain the solutions in which the value of the parameter k is larger, the initial estimates are taken to be equal to the solutions obtained for a slightly smaller value of the parameter k than under consideration. The parameter λ , generally, is to be made smaller than what is applied to the solutions for the smaller value of the parameter k .

As a numerical criterion for convergence of the iterative procedures, we put

$$\begin{aligned} |p_m(z_n) - p_{m-1}(z_n)| &< \eta \\ |q_m(z_n) - q_{m-1}(z_n)| &< \eta \end{aligned} \quad n=0, 1, 2, \dots, N$$

The accuracy of the solutions can be determined arbitrarily by selecting the values of η . Here we put

$$\eta = 0.00005$$

The numerical calculations were performed on the Osaka University NEAC-2203 and NEAC-2206 digital computers.

5. Numerical results

Numerical results were presented for three types of boundary conditions and for three values of the radius ratios

$$b/a = 0.5[\rho = (1/2)^2], \quad 0.25[(1/4)^2], \quad 0(0)$$

Poisson's ratio was assumed as $\nu = 0.3$. It should be noticed that the circular ring plates where $b/a = 0$ are those under the concentrated force P at the center. For Case A, numerical calculations were carried out for $N=50, 100, 200$ and 400 .

Table 2 shows the results of the maximum

Table 2 Maximum deflections and bending stresses for $N=50, 100, 200$ and 400 , (Case A)

(i) $b/a=1/2$								
$k \backslash N$	$w_{\max}/h$				$(\sigma_b a^2/Eh^2)_{r=b}$			
	50	100	200	400	50	100	200	400
4.5	0.346 3	0.347 2	0.347 6	0.347 7	5.993	6.003	6.010	6.013
6	0.671 2	0.675 4	0.677 4	0.678 3	12.06	12.13	12.16	12.18
6.5	0.783 6	0.789 3	0.791 9	0.793 2	14.34	14.43	14.48	14.50
(ii) $b/a=1/4$								
2	0.127 5	0.127 3	0.127 3	0.127 3	1.284	1.289	1.290	1.291
4	0.745 1	0.752 3	0.755 6	0.757 2	8.008	8.136	8.182	8.202
6	1.448	1.471	1.481	1.486	18.00	18.48	18.65	18.73
(iii) $b/a=0$								
2	0.276 8	0.283 2	0.286 9	0.289 0	0.645 7	0.645 3	0.645 0	0.644 9
4	1.300	1.327	1.344	1.354	3.399	3.389	3.384	3.381
5	1.826	1.865	1.891	1.906	5.223	5.216	5.214	5.212

deflections and bending stresses for $N=50, 100, 200$ and 400 , and the results for $N=400$ are found to be sufficiently good approximations. Consequently, for Case B, numerical results for $N=400$ only are obtained. In the above calculations, the approximation of Eq. (19) was used, but it was found that the numerical results by the approximation

of Eq. (19) for $N=400$ are as accurate as those by Eq. (20) for $N=100$, as shown in Table 3. Therefore, numerical calculations for Case C were performed for $N=100$ using Eq. (20) in order to diminish computing time.

Fig. 2 shows the distributions of the solutions p and q for several values of the parameter k and

Fig. 2 Distributions of the solutions p and q

three cases of the inner and outer radius ratios. Fig. 3 shows the maximum deflections with relation to the dimensionless load

$$Pa^2/Eh^4 = 0.17412k^3 \quad (\nu=0.3)$$

The graphs of Fig. 4 are those for the radial bending stresses and the radial membrane stresses at both edges with relation to the maximum deflections where fine lines are the results by the clas-

Table 3 Comparisons of the results by Eq.(19) and Eq.(20), ($k=4.5$, Case A)
[Upper values are the results by Eq. (19) and lower values are by Eq.(20)]

N	Inner edge ($r=b$)			Outer edge ($r=a$)		
	$\sigma_r a^2/Eh^2$	$\sigma_b a^2/Eh^2$	w/h	$\sigma_r a^2/Eh^2$	$\sigma_b a^2/Eh^2$	w/h
50	0.441 1	5.993	0.346 3	0.271 5	-3.798	0
	0.410 9	6.012	0.348 2	0.256 5	-3.718	
100	0.423 9	6.003	0.347 2	0.262 3	-3.807	0
	0.409 2	6.015	0.348 0	0.254 4	-3.814	
200	0.415 9	6.010	0.347 6	0.258 0	-3.811	0
	0.408 7	6.016	0.348 0	0.254 2	-3.814	
400	0.412 2	6.013	0.347 7	0.256 0	-3.812	0
	0.408 5	6.016	0.348 0	0.254 0	-3.814

(a) Case A

(b) Case B and case C

Fig. 3 Maximum deflections

(a) Case A

(b) Case B and case C

Fig. 5 Graphs of σ_r/σ_b

sical theory.

It is well-known that the membrane stresses play an important part in the large deflection problem of plates, and consequently the ratio of the membrane stress to the bending stress can be considered as one of criterions for the limit of the classical theory. Fig. 5 shows these ratios with relation to the parameter k , and the values of the parameter k where the slopes increase abruptly might be considered as the limits of the classical theory.

Discussing Hamada's approximate solutions, the maximum deflections are found to be good approximations except for $b/a=0$, and the same thing is true for the stresses. These facts mean that the assumption of uniform radial membrane stress is not reasonable for small values of b/a .

The approximate method⁽⁸⁾⁽⁹⁾ by calculus of variation is often used for a nonlinear problem of plates. Usually, the deflection is represented in the simple form

$$w = cw_0$$

where the fundamental function w_0 is the solution of the linear theory and c is an unknown constant. The bending stress by this approximate method is given by the linear equation

$$\sigma_b = \beta w_{\max}, \quad \text{where } \beta \text{ is a constant,}$$

and it is found to be not good approximation from Fig. 4. This fact suggests that the approximate method is valid for only small values of the deflection.

(a) Stresses at the inner edge; $b/a=1/2$

(b) Stresses at the outer edge; $b/a=1/2$

(d) Stresses at the outer edge; $b/a=1/4$

(c) Stresses at the inner edge; $b/a=1/4$

(e) Stresses at the outer edge; $b/a=0$

Fig. 4 Radial bending stresses and radial membrane stresses

6. Experiments

The experiments were carried out on a series of circular ring plates having an outer diameter of 300 mm, a thickness of 1.6 mm and the inner and outer radius ratios of 0.5 and 0.25. As shown

Fig. 6 Specimen and testing jig

Fig. 8 Experimental results of maximum deflections

Fig. 9 Experimental results of stress distributions; $b/a=0.5$

in Fig. 6 and Fig. 7, the specimens were held by two rings which were bolted and bonded with an epoxide resin type adhesive to the specimens. The load was applied to the inner boss which was also bolted and bonded. Young's modulus and Poisson's ratio were measured as $E=21\,100\text{ kg/mm}^2$, $\nu=0.294$.

Fig. 8 shows the maximum deflections with relation to the load, and Fig. 9 and Fig. 10 show the stress distributions of the upper and lower

Fig. 7 View of testing fixture

Fig. 10 Experimental results of stress distributions; $b/a=0.25$

surfaces for the cases of $b/a=0.5$ and 0.25 , respectively. Solid lines give the theoretical results which are observed to coincide well with the experimental results.

Wire strain gauges were used in experiments. When a plate thickness of specimens is so thin as in these experiments, the gauge thickness has an effect on the experimental results. In our case, the corrections due to this effect were made under an appropriate assumption of the gauge thickness.

Finally, the clamped edge condition of the specimens seems to be good on the basis of the experimental results, although the hardness and the creep property of the adhesive are unknown. And this clamping method is valid for this kind of experiments since a specimen by this method is more easily produced than a welded specimen or a lathed specimen of one body.

7. Summary

A numerical analysis of a finite deflection problem on circular ring plates has been made for three kinds of boundary conditions, and experiments have been carried out for a case of both edges clamped to show the propriety of the numerical results. The limits of the approximate solutions and the classical linear theory were clarified. It can be presumed that the Keller-Reiss method is an excellent one for numerical analyses of axisymmetric nonlinear problems of plates and shells, whose remarkable merit is that solutions can be obtained for almost all values of load parameters.

For example, the solution of this problem could be obtained for the large value of the load parameter such as

$$k=34(Pa^2/Eh^4=6\,844, \nu=0.3)$$

where the edge-effect was observed.

Finally, the influences of mesh numbers and selection of the approximations for the boundary conditions on the accuracy of the solutions were also examined.

8. Acknowledgment

The authors are indebted to Prof. K. Jō and Asst. Prof. S. Makinouchi for the usage of a computer, and they wish to express their appreciation to Prof. I. Nakahara for supplying the information about Ref. 9.

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