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Thermal Stress Due to Disturbances of Uniform Heat Flow by an Insulated Spherical Cavity in a Circular Cylinder*

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The linear thermoelastic problem is solved by using potential of thermoelastic displacement and stress function for the insulated spherical cavity in a circular cylinder with axis parallel to the heat flow. Stress distributions are given for a range of some various sizes of cavity.

1. Introduction

The investigations on the effects of temperature in the study of the elastic behavior of a circular cylinder have been done for some time. However, it may be said that they have been limited to obtaining the applicability of the solution about a solid, composite or hollow cylinder.

In this paper solutions are found to the problem of thermal stress induced by an axisymmetric flow past a spherical cavity inside a long circular cylinder. About similar investigations on the steady heat flow, solutions have been given by Florence and Goodier for the cases of circular hole⁽¹⁾, spherical cavity⁽¹⁾, ovaloid hole⁽²⁾ and spheroidal cavity⁽³⁾, but they have been studied for an unlimited elastic region.

2. Temperature field

The steady heat flow with uniform temperature gradient at infinity τ passes a spherical cavity inside a cylinder, situated as shown in Fig. 1. Since the undisturbed temperature field may be taken as τz , the temperature in the presence of the cavity is written as

$$T = \tau z + T^* \quad (1)$$

$$T^* = \frac{2}{\pi} \int_0^\infty \frac{K_0(p\rho) I_0'(p) - K_0'(p) I_0(p\rho)}{I_0'(p)} \sin pz \sum_{s=0}^\infty (-1)^s a_s \frac{p^{2s+1}}{(2s+1)!} dp \quad (6)$$

Where I_n and K_n are respectively modified Bessel functions of the first and second kind of order n , and I_n' and K_n' are the derivatives of the functions. Using the equations

$$\left. \begin{aligned} \int_0^\infty \sin pz K_0(p\rho) p^{2s+1} dp &= \frac{\pi}{2} (-1)^s \frac{(2s+1)!}{r^{2s+2}} P_{2s+1}(\cos \theta), \\ \sin pz I_0(p\rho) &= \sum_{s=0}^\infty (-1)^s p^{2s+1} \frac{r^{2s+1}}{(2s+1)!} P_{2s+1}(\cos \theta) \end{aligned} \right\} \quad (7)$$

we have

$$T^* = \sum_{s=0}^\infty \left\{ \frac{a_s}{r^{2s+2}} - \frac{2}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{s+j} a_j p^{2s+2j+2} K_0'(p)}{(2s+1)!(2j+1)! I_0'(p)} dp r^{2s+1} \right\} P_{2s+1}(\cos \theta) \quad (8)$$

Denote the cylindrical and spherical coordinates of a point by (ρ, ϕ, z) and (r, θ, ϕ) . Regarding, for convenience, r, z, ρ and λ as dimensionless quantities referring to a typical length a , they are related with each other by

$$\left. \begin{aligned} z &= r \cos \theta \\ \rho &= r \sin \theta \end{aligned} \right\} \quad (2)$$

T^* in Eq. (1) must be an odd function of z and satisfy the following conditions:

$$\nabla^2 T^* = 0 \quad (3)$$

$$\left. \begin{aligned} \partial T^* / \partial z &\rightarrow 0 \text{ as } z \rightarrow \infty \\ \partial T^* / \partial \rho &= 0 \text{ for } \rho = 1 \end{aligned} \right\} \quad (4)$$

Note that ∇^2 denotes the Laplacian operator and in cylindrical coordinates,

$$\nabla^2 = \frac{1}{a^2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\rho \partial \rho} + \frac{\partial^2}{\partial z^2} \right) \quad (5)$$

Hence the appropriate solution will be⁽⁴⁾

Fig. 1 Notation for cylinder

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Since the cavity is insulated, we must have

$$(\partial T / \partial r)_{r=\lambda} = 0 \quad (9)$$

The coefficients a_s in Eq. (6) are found so as to satisfy the condition (9) with the relations:

$$\left. \begin{aligned} a_0 &= \frac{\lambda^3}{2} \left\{ \tau - \frac{2}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j a_j p^{2j+2} K_0'(p)}{(2j+1)! I_0'(p)} dp \right\}, \\ a_s &= \frac{\lambda^{4s+3}}{(2s+2)} \left\{ -\frac{2}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{s+j} a_j p^{2s+2j+2} K_0'(p)}{(2s)!(2j+1)! I_0'(p)} dp \right\}, \quad s \geq 1 \end{aligned} \right\} \quad (10)$$

A formal solution by successive approximation is as follows:

Write, for $s \geq 0$,

$$a_s = \sum_{r=0}^\infty a_s^{(r)} \quad (11)$$

where

$$\left. \begin{aligned} a_0^{(0)} &= \lambda^3 \tau / 2, \quad a_s^{(0)} = 0, \quad (s=1, 2, \dots) \\ a_0^{(r)} &= -\frac{\lambda^3}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^j a_j^{(r-1)} p^{2j+2} K_0'(p)}{(2j+1)! I_0'(p)} dp, \\ a_s^{(r)} &= -\frac{2}{\pi} \frac{\lambda^{4s+3}}{(2s+2)} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{s+j} a_j^{(r-1)} p^{2s+2j+2} K_0'(p)}{(2s)!(2j+1)! I_0'(p)} dp \end{aligned} \right\} \quad (12)$$

3. Potential of thermoelastic displacement

To find the state of stress, we shall use the potential of thermoelastic displacement Φ , which is determined by the Poisson equation

$$\nabla^2 \Phi = \frac{(1+\nu)}{(1-\nu)} \alpha T^* \quad (13)$$

where ν is Poisson's ratio and α is a coefficient of thermal expansion.

The solution is given by the function

$$\Phi = -a^2 \frac{(1+\nu)\alpha}{(1-\nu)\pi} \int_0^\infty \left\{ \rho K_1(p\rho) - \frac{K_1(p)}{I_1(p)} \rho I_1(p\rho) \right\} \sin pz \sum_{s=0}^\infty \frac{(-1)^s a_s p^{2s}}{(2s+1)!} dp \quad (14)$$

In expressing the function in terms of spherical coordinates, the following relations are useful.

$$\left. \begin{aligned} \int_0^\infty K_1(p\rho) p^{2s} \sin pz dp &= \frac{\pi}{2} (-1)^{s+1} \frac{(2s+1)!}{r^{2s+1}} P_{2s}^{-1}(\cos \theta), \\ I_1(p\rho) \sin pz &= \sum_{s=0}^\infty \frac{(-1)^s p^{2s+2}}{(2s+1)!} r^{2s+2} P_{2s+2}^{-1}(\cos \theta) \end{aligned} \right\} \quad (15)$$

Consequently, we have

$$\begin{aligned} \Phi &= -a^2 \frac{(1+\nu)}{(1-\nu)} \alpha \sum_{s=0}^\infty \left\{ \frac{a_s}{(8s+2)r^{2s}} [P_{2s+1}(\mu) - P_{2s-1}(\mu)] \right. \\ &\quad \left. + \frac{1}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{s+j} a_j p^{2s+2j+2} K_0'(p)}{(4s+5)(2s+1)!(2j+1)! I_0'(p)} dp r^{2s+3} [P_{2s+1}(\mu) - P_{2s+3}(\mu)] \right\} \end{aligned} \quad (16)$$

where

$$\mu = \cos \theta \quad (17)$$

Having the function Φ , we can calculate the stresses. The normal and tangential stresses are given by

$$\left. \begin{aligned} \sigma_{\rho, \rho} &= \frac{E}{a^2(1+\nu)} \frac{\partial^2 \Phi}{\partial \rho^2} - \frac{E\alpha}{1-\nu} T^* = -\frac{E\alpha}{1-\nu} \frac{1}{\pi} \int_0^\infty \left\{ [K_0(p\rho) + p\rho K_1(p\rho)] \right. \\ &\quad \left. + \frac{K_1(p)}{I_1(p)} [I_0(p\rho) - p\rho I_1(p\rho)] \right\} \sin pz \sum_{s=0}^\infty \frac{(-1)^s a_s p^{2s+1}}{(2s+1)!} dp, \\ \tau_{\rho, \theta} &= \frac{E}{a^2(1+\nu)} \frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Phi}{\partial \theta} \right) = \frac{E\alpha}{1-\nu} \frac{1}{\pi} \int_0^\infty \left\{ \rho K_0(p\rho) \right. \\ &\quad \left. + \frac{K_1(p)}{I_1(p)} \rho I_0(p\rho) \right\} \cos pz \sum_{s=0}^\infty \frac{(-1)^s a_s p^{2s+2}}{(2s+1)!} dp \end{aligned} \right\} \quad (18)$$

and

$$\begin{aligned}
\sigma_{r,\theta} &= \frac{E}{a^2(1+\nu)} \frac{\partial^2 \Phi}{\partial r^2} - \frac{E\alpha}{1-\nu} T^* = -\frac{E\alpha}{1-\nu} \sum_{s=0}^{\infty} \left\{ \frac{a_s}{(4s+1)r^{2s+2}} [(2s^2+5s+1)P_{2s+1}(\mu) \right. \\
&\quad \left. - (2s+1)sP_{2s-1}(\mu)] + \frac{1}{\pi} \int_0^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{s+j} a_j p^{2s+2j+2} K_0'(p)}{(2s+1)!(2j+1)!(4s+5)I_0'(p)} \right. \\
&\quad \left. \times dp r^{2s+1} [2(2s^2+s-2)P_{2s+1}(\mu) - (2s+2)(2s+3)P_{2s+3}(\mu)] \right\}, \\
\tau_{r\theta,\theta} &= \frac{E}{a^2(1+\nu)} \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right) = -\frac{E\alpha}{1-\nu} (1-\mu^2)^{1/2} \sum_{s=0}^{\infty} \left\{ \frac{a_s(2s+1)}{(8s+2)r^{2s+2}} [P_{2s+1}'(\mu) - P_{2s-1}'(\mu)] \right. \\
&\quad \left. - \frac{1}{\pi} \int_0^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{s+j} a_j (2s+2)p^{2s+2j+2}}{(2s+1)!(2j+1)!(4s+5)} \frac{K_0'(p)}{I_0'(p)} dp r^{2s+1} [P_{2s+1}'(\mu) - P_{2s+3}'(\mu)] \right\}
\end{aligned} \quad \dots\dots\dots (19)$$

where E is Young's modulus.

4. Stress functions

Since the function Φ does not satisfy the boundary conditions on the curved surface ($\rho=1$) of the cylinder and the inside spherical surface ($r=\lambda$) where tractions should vanish, we first complete the states $\sigma_{\rho,\theta}$ and $\tau_{\rho z,\theta}$ by the stresses σ_{ρ,z_1} and $\tau_{\rho z,z_1}$ respectively. The stress function is chosen in such a way that on the curved surface of the cylinder

$$(\sigma_{\rho,\theta} + \sigma_{\rho,z_1})_{\rho=1} = 0, \quad (\tau_{\rho z,\theta} + \tau_{\rho z,z_1})_{\rho=1} = 0 \quad \dots\dots\dots (20)$$

We may take the stress function as follows:

$$\chi_1 = a^3 \int_0^{\infty} \{ \phi_1(p) I_0(p\rho) + \phi_2(p) p \rho I_1(p\rho) \} \cos p z dp \quad \dots\dots\dots (21)$$

This function gives the following normal and tangential stresses

$$\begin{aligned}
\sigma_{\rho,z_1} &= \frac{1}{a^3} \frac{\partial}{\partial z} \left(\nu \nabla^2 \chi_1 - \frac{\partial^2 \chi_1}{\partial \rho^2} \right) = \int_0^{\infty} p^2 \left[\phi_1(p) \left\{ p I_0(p\rho) - \frac{I_1(p\rho)}{\rho} \right\} \right. \\
&\quad \left. + \phi_2(p) [(1-2\nu)p I_0(p\rho) + p^2 \rho I_1(p\rho)] \right] \sin p z dp, \\
\tau_{\rho z,z_1} &= \frac{1}{a^3} \frac{\partial}{\partial \rho} \left\{ (1-\nu) \nabla^2 \chi_1 - \frac{\partial^2 \chi_1}{\partial z^2} \right\} \\
&= \int_0^{\infty} p^3 \{ \phi_1(p) I_1(p\rho) + \phi_2(p) [2(1-\nu) I_1(p\rho) + p \rho I_0(p\rho)] \} \cos p z dp \quad \dots\dots\dots (22)
\end{aligned}$$

The condition (20) will be satisfied and χ_1 will be determined, provided that

$$\begin{aligned}
\phi_1(p) &= \frac{E\alpha}{\pi(1-\nu)} \frac{[I_0(p)K_1(p) + I_1(p)K_0(p)][(2-2\nu+p^2)I_1(p) + 2(1-\nu)pI_0(p)]}{p^2 I_1(p)[p^2 I_0^2(p) - (2-2\nu+p^2)I_1^2(p)]} \sum_{s=0}^{\infty} \frac{(-1)^s a_s p^{2s+1}}{(2s+1)!}, \\
\phi_2(p) &= -\frac{E\alpha}{\pi(1-\nu)} \frac{[I_0(p)K_1(p) + I_1(p)K_0(p)]I_0(p)}{p I_1(p)[p^2 I_0^2(p) - (2-2\nu+p^2)I_1^2(p)]} \sum_{s=0}^{\infty} \frac{(-1)^s a_s p^{2s+1}}{(2s+1)!}
\end{aligned} \quad \dots\dots\dots (23)$$

With these values of $\phi_1(p)$ and $\phi_2(p)$, it appears that the integral in Eq. (21) becomes divergent at the lower limit, while the divergence can be removed by modifying the integrals as follows:

$$\begin{aligned}
\chi_1 &= a^3 \int_0^{\infty} \{ \phi_1(p) I_0(p\rho) + \phi_2(p) p \rho I_1(p\rho) \} \cos p z dp \\
&\quad - \frac{E\alpha a^3}{\pi(1-\nu)} \int_0^{\infty} \frac{[2K_1(p) + pK_0(p)]}{(1+\nu)p} \left[\left\{ \frac{6(1-\nu)}{p^2} \left(1 - \frac{p^2 z^2}{2} + \frac{p^2 \rho^2}{4} \right) - \rho^2 \right\} a_0 - (1-\nu)a_1 \right] dp \quad \dots\dots\dots (24)
\end{aligned}$$

This modification is permissible for it produces no effect on the stresses in the cylinder as well as on the convergence at the upper limit. Then χ_1 thus obtained gives no stress at z infinity or on the surface of the cylinder.

In expressing the function in terms of spherical coordinates, the following relations are useful.

$$\begin{aligned}
p \rho I_1(p\rho) \cos p z &= \sum_{n=1}^{\infty} \frac{(-1)^n (pr)^{2n}}{(2n-2)!(4n-1)} [P_{2n}(\mu) - P_{2n-2}(\mu)], \\
I_0(p\rho) \cos p z &= \sum_{n=0}^{\infty} \frac{(-1)^n (pr)^{2n}}{(2n)!} p_{2n}(\mu)
\end{aligned} \quad \dots\dots\dots (25)$$

Consequently, we have

$$\chi_1 = a^3 \sum_{n=1}^{\infty} (\Omega_{2n} + K_{2n} r^2) r^{2n} P_{2n}(\mu) + a^3 \{ g_1(p) + g_2(p) r^2 + g_3(p) r^2 P_2(\mu) \} \quad \dots\dots\dots (26)$$

where

$$\left. \begin{aligned} \Omega_{2n} &= \int_0^\infty \frac{(-1)^n}{(2n)!} \left[\frac{(4n-1)\phi_1(p) + 2n(2n-1)\phi_2(p)}{(4n-1)} \right] p^{2n} dp, \\ K_{2n} &= \int_0^\infty \frac{(-1)^n \phi_2(p)}{(2n)!(4n+3)} p^{2n+2} dp \end{aligned} \right\} \dots\dots\dots (27)$$

and

$$\left. \begin{aligned} g_1(p) &= -\frac{E\alpha}{\pi} \int_0^\infty \frac{[2K_1(p) + pK_0(p)]}{(1+\nu)p} \left(\frac{6a_0}{p^2} - a_1 \right) dp, \\ g_2(p) &= \frac{E\alpha}{\pi(1-\nu)} \int_0^\infty \frac{[2K_1(p) + pK_0(p)]}{(1+\nu)p} \frac{2}{3} a_0 dp, \\ g_3(p) &= \frac{E\alpha}{\pi(1-\nu)} \int_0^\infty \frac{[2K_1(p) + pK_0(p)]}{(1+\nu)p} \frac{(7-9\nu)}{3} a_0 dp \end{aligned} \right\} \dots\dots\dots (28)$$

χ_1 gives the following normal and tangential stresses

$$\left. \begin{aligned} \sigma_{r,z_1} &= -\sum_{n=1}^\infty \left[2n(2n+1) \left\{ (2n+2)\Omega_{2n+2} + \frac{(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \right\} r^{2n-1} P_{2n+1}(\mu) \right. \\ &\quad \left. + \frac{2n(4n+3)(4n^2-6n-2\nu)}{4n+1} K_{2n} r^{2n-1} P_{2n-1}(\mu) \right], \\ \tau_{r\theta,z_1} &= (1-\mu^2)^{1/2} \sum_{n=1}^\infty \left[2n \left\{ (2n+2)\Omega_{2n+2} + \frac{(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \right\} P_{2n+1}'(\mu) \right. \\ &\quad \left. - \frac{(4n+3)(2-2\nu-4n^2)}{(4n+1)} K_{2n} P_{2n-1}'(\mu) \right] r^{2n-1} \end{aligned} \right\} \dots\dots\dots (29)$$

The remaining boundary conditions to be satisfied are that on the surface of the cavity the normal and tangential stresses vanish identically. Then we next complete the states $(\sigma_{r,\theta}, \sigma_{r,z_1})$ and $(\sigma_{r\theta,\theta}, \sigma_{r\theta,z_1})$ by the stresses σ_{r,z_2} and $\sigma_{r\theta,z_2}$ respectively. The stress function is chosen in such a way that it produces no effect on the stresses on the surface of the cylinder while on the surface of the cavity.

$$(\sigma_{r,\theta} + \sigma_{r,z_1} + \sigma_{r,z_2})_{r=\lambda} = 0, \quad (\tau_{r\theta,\theta} + \tau_{r\theta,z_1} + \tau_{r\theta,z_2})_{r=\lambda} = 0 \dots\dots\dots (30)$$

This stress function, in this paper, is constructed from the same two fundamental biharmonic functions as C. B. Ling presented⁽⁵⁾ in the solution of tension problem for a circular cylinder.

These are

$$\omega_0 = -\frac{a^3}{2} \log \frac{1+\mu}{1-\mu} - a^3 \int_0^\infty \{ [\psi_1(p)I_0(p\rho) + \psi_2(p)p\rho I_1(p\rho)] \sin pz + f_1(p)z \} dp \dots\dots\dots (31)$$

$$\omega_0' = -a^3 r - a^3 \int_0^\infty \{ [\psi_3(p)I_0(p\rho) + \psi_4(p)p\rho I_1(p\rho)] \cos pz + f_2(p)\rho^2 + f_3(p)\left(1 - \frac{1}{2}p^2 z^2\right) \} dp \dots\dots\dots (32)$$

where

$$\left. \begin{aligned} \psi_1(p) &= -\frac{2}{\pi} \frac{pI_0(p)K_0(p) + [p^2 + 2(1-\nu)]I_1(p)K_1(p)/p}{p^2 I_0^2(p) - [p^2 + 2(1-\nu)]I_1^2(p)} - 2(1-\nu)\phi_2(p), \\ \psi_2(p) &= \frac{2}{\pi} \frac{I_0(p)K_1(p) + I_1(p)K_0(p)}{p^2 I_0^2(p) - [p^2 + 2(1-\nu)]I_1^2(p)}, \\ f_1(p) &= \frac{4}{\pi(1+\nu)} \left[(2-\nu)K_0(p) + \frac{3(1-\nu)}{p} K_1(p) \right] \end{aligned} \right\} \dots\dots\dots (33)$$

and

$$\left. \begin{aligned} \psi_3(p) &= [p^2 - 2(1-\nu)(1-2\nu)]\phi_2(p)/p, \quad \psi_4(p) = [\psi_1(p) + 4(1-\nu)\phi_2(p)]/p, \\ f_2(p) &= \frac{2}{\pi(1+\nu)} \left[\nu K_0(p) - \frac{(1-\nu)}{p} K_1(p) \right], \\ f_3(p) &= \frac{4(1-\nu)(1-2\nu)}{\pi(1+\nu)p^2} \left[K_0(p) + \frac{2}{p} K_1(p) \right] \end{aligned} \right\} \dots\dots\dots (34)$$

Each of these functions gives no traction on the surface of the cylinder and gives no stress at z infinity. It is obvious that successive differentiation with respect to z gives functions which produce no effect on the desired properties, but the functions being odd in z can not enter into the required solution. Then we take

$$\left. \begin{aligned} \omega_{2s+1} &= \frac{-1}{(2s)!} \frac{\partial^{2s+1}}{\partial z^{2s+1}} \omega_0 = a^3 \left\{ \frac{P_{2s}(\mu)}{r^{2s+1}} - \sum_{n=0}^{\infty} [{}^{2n}\alpha_{2s} + {}^{2n}\gamma_{2s} r^2] r^{2n} P_{2n}(\mu) \right\}, \quad (s \geq 0) \\ \omega'_{2s+1} &= \frac{-1}{(2s-1)!} \frac{\partial^{2s}}{\partial z^{2s}} \omega'_0 = a^3 \left\{ \frac{(2s+2)}{(4s+3)r^{2s+1}} [P_{2s}(\mu) - P_{2s+2}(\mu)] \right. \\ &\quad \left. - \sum_{n=0}^{\infty} [{}^{2n}\beta_{2s} + {}^{2n}\delta_{2s} r^2] r^{2n} P_{2n}(\mu) \right\}, \quad (s \geq 0) \end{aligned} \right\} \dots\dots\dots (35)$$

where

$$\left. \begin{aligned} {}^{2n}\alpha_{2s} &= (2n+1){}^{2n}\alpha'_{2s} + \frac{4n}{4n-1} {}^{2n-2}\gamma'_{2s}, \quad {}^{2n}\beta_{2s} = (2n+1){}^{2n}\beta'_{2s} + \frac{4n}{4n-1} {}^{2n-2}\delta'_{2s}, \\ {}^{2n}\gamma_{2s} &= \frac{(2n+1)(4n+5)}{4n+3} {}^{2n}\gamma'_{2s}, \quad {}^{2n}\delta_{2s} = \frac{(2n+1)(4n+5)}{4n+3} {}^{2n}\delta'_{2s} \end{aligned} \right\} \dots\dots\dots (36)$$

where ${}^{2n}\alpha'_{2s}$, ${}^{2n}\beta'_{2s}$, ${}^{2n}\gamma'_{2s}$ and ${}^{2n}\delta'_{2s}$ are identical with Eqs. (21) in Ling's paper provided that K is replaced by p in these equations. That is

$$\left. \begin{aligned} {}^{2n}\alpha'_{2s} &= -\frac{(-1)^{n+s}}{(2n)!(2s)!} \int_0^\infty \left[\frac{\phi_1(p)}{(2n+1)} + \frac{2n\phi_2(p)}{(4n+1)} \right] p^{2n+2s+1} dp, \\ {}^{2n}\beta'_{2s} &= \frac{(-1)^{n+s}}{(2n)!(2s+1)!} \int_0^\infty \left[\frac{\phi_3(p)}{(2n+1)} + \frac{2n\phi_4(p)}{(4n+1)} \right] p^{2n+2s+2} dp, \\ {}^{2n}\gamma'_{2s} &= \frac{(-1)^{n+s}}{(4n+5)(2n+1)!(2s)!} \int_0^\infty \phi_2(p) p^{2n+2s+3} dp, \\ {}^{2n}\delta'_{2s} &= \frac{(-1)^{n+s}}{(4n+5)(2n+1)!(2s+1)!} \int_0^\infty \phi_4(p) p^{2n+2s+4} dp \end{aligned} \right\} \dots\dots\dots (37)$$

in particular,

$${}^0\alpha'_0 = -\int_0^\infty [p\phi_1(p) + f_1(p)] dp, \quad {}^0\beta'_0 = \int_0^\infty p^2[\phi_3(p) + f_3(p)] dp \quad \dots\dots\dots (38)$$

Hence the required stress function is constructed as follows:

$$\begin{aligned} \chi_2 &= a^3 \sum_{s=0}^{\infty} \left[A_{2s+1} \left\{ \frac{P_{2s}(\mu)}{r^{2s+1}} - \sum_{n=0}^{\infty} ({}^{2n}\alpha_{2s} + {}^{2n}\gamma_{2s} r^2) r^{2n} P_{2n}(\mu) \right\} \right. \\ &\quad \left. + B_{2s+1} \left\{ \frac{(2s+2)}{(4s+3)r^{2s+1}} (P_{2s}(\mu) - P_{2s+2}(\mu)) - \sum_{n=0}^{\infty} ({}^{2n}\beta_{2s} + {}^{2n}\delta_{2s} r^2) r^{2n} P_{2n}(\mu) \right\} \right] \quad \dots\dots\dots (39) \end{aligned}$$

This function gives the following normal and tangential stresses

$$\left. \begin{aligned} \sigma_{r, z_2} &= \sum_{s=0}^{\infty} \left\{ \frac{(2s+1)(2s+2)(2s+3)A_{2s+1}}{r^{2s+4}} P_{2s+1}(\mu) + \frac{(2s+2)(2s+3)B_{2s+1}}{(4s+5)} \right. \\ &\quad \times \left. \frac{[(2s+2)(2s+4\nu-1)P_{2s+1}(\mu) - (4s^2+18s+18-2\nu)P_{2s+3}(\mu)]}{r^{2s+4}} \right\} \\ &\quad + \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} 2n(2n+1)(2n+2)({}^{2n+2}\alpha_{2s}A_{2s+1} + {}^{2n+2}\beta_{2s}B_{2s+1})r^{2n-1}P_{2n+1}(\mu) \\ &\quad + \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \frac{2n}{(4n+1)} \{ (4n+3)(4n^2-6n-2\nu)P_{2n-1}(\mu) + (4n+2)(8n\nu-6n+6\nu-5) \\ &\quad \times P_{2n+1}(\mu) \} ({}^{2n}\gamma_{2s}A_{2s+1} + {}^{2n}\delta_{2s}B_{2s+1})r^{2n-1} \\ \tau_{r\theta, z_2} &= (1-\mu^2)^{1/2} \sum_{s=0}^{\infty} \left[\frac{(2s+1)(2s+3)A_{2s+1}}{r^{2s+4}} P_{2s+1}'(\mu) \right. \\ &\quad + \frac{(2s+2)B_{2s+1}}{(4s+5)} \left\{ \frac{(2s+3)(2s-1+4\nu)P_{2s+1}'(\mu) - (4s^2+12s+2\nu+7)P_{2s+3}'(\mu)}{r^{2s+4}} \right\} \\ &\quad - (1-\mu^2)^{1/2} \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} 2n(2n+2)({}^{2n+2}\alpha_{2s}A_{2s+1} + {}^{2n+2}\beta_{2s}B_{2s+1})r^{2n-1}P_{2n+1}'(\mu) \\ &\quad - (1-\mu^2)^{1/2} \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(4n+1)} \{ (4n+3)(4n^2+2\nu-2)P_{2n-1}'(\mu) \\ &\quad + 4n(8n\nu-6n+6\nu-5)P_{2n+1}'(\mu) \} ({}^{2n}\gamma_{2s}A_{2s+1} + {}^{2n}\delta_{2s}B_{2s+1})r^{2n-1} \end{aligned} \right\} \dots\dots\dots (40)$$

Then the solution will be completed by determining the coefficients A_{2s+1} , B_{2s+1} so as to satisfy the condition (30) which gives a system of linear equations obtained by equating separately the coefficient of each Legendre function or its derivative to zero. The system of equations may be replaced by the following

system

$$\begin{aligned}
 & (2n+1)(2n+3)A_{2n+1} \\
 &= \frac{E\alpha}{(1-\nu)} \left[\frac{(2n+1)a_n}{(8n+2)} \lambda^2 - \frac{(2n+3)a_{n+1}}{(8n+10)} - \frac{1}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{n+j} a_j p^{2n+2j} K_0'(p)}{(2j+1)! I_0'(p)} \right. \\
 & \quad \times \left\{ \frac{(2n+2)p^2}{(2n+1)!(4n+5)} \lambda^2 + \frac{2n}{(2n-1)!(4n+1)} \right\} dp \lambda^{4n+3} \Big] - \left[2n \left\{ (2n+2) \Omega_{2n+2} \right. \right. \\
 & \quad + \frac{(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \Big\} \lambda^{4n+3} + \frac{(4n+7)(4n^2+8n+2\nu+2)}{(4n+5)} K_{2n+2} \lambda^{4n+5} \Big] \\
 & \quad - \left\{ \frac{(2n+2)(2n+3)(2n-1+4\nu)B_{2n+1}}{(4n+5)} - \frac{2n(4n^2+4n+2\nu-1)B_{2n-1}\lambda^2}{(4n+1)} \right\} \\
 & \quad + \sum_{s=0}^\infty \left\{ 2n(2n+2)(2n+2)\alpha_{2s}A_{2s+1} + 2n+2\beta_{2s}B_{2s+1} \right\} \lambda^{4n+3} + \frac{(4n+7)}{(4n+5)} (4n^2+8n+2\nu+2) \\
 & \quad \times (2n+2)\gamma_{2s}A_{2s+1} + 2n+2\delta_{2s}B_{2s+1} \lambda^{4n+5} + \frac{4n}{(4n+1)} (8n\nu-6n+6\nu-5) \\
 & \quad \times (2n\gamma_{2s}A_{2s+1} + 2n\delta_{2s}B_{2s+1}) \lambda^{4n+3} \Big\}, \\
 & 2n(8n^2+12n-8n\nu-6\nu+6)B_{2n-1}/(4n+1) \\
 &= -\frac{E\alpha}{(1-\nu)} \left[\frac{2na_n}{(4n+1)} + \frac{1}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{n+j} 2na_j p^{2n+2j} K_0'(p)}{(2j+1)! I_0'(p)} \right. \\
 & \quad \times \left\{ \frac{p^2}{(2n+1)!} \lambda^2 + \frac{(4n+3)}{(2n-1)!(4n+1)} \right\} dp \lambda^{4n+1} \Big] \\
 & \quad - \left[2n \left\{ (2n+2)(4n+3) \Omega_{2n+2} + \frac{(4n+3)(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \right\} \lambda^{4n+1} \right. \\
 & \quad + \left. (2n+2)2n(4n+7)K_{2n+2}\lambda^{4n+3} \right] \\
 & \quad + \sum_{s=0}^\infty \left\{ 2n(2n+2)(4n+3)(2n+2)\alpha_{2s}A_{2s+1} + 2n+2\beta_{2s}B_{2s+1} \right\} \lambda^{4n+1} + 2n(2n+2)(4n+7) \\
 & \quad \times (2n+2)\gamma_{2s}A_{2s+1} + 2n+2\delta_{2s}B_{2s+1} \lambda^{4n+3} + \frac{4n(4n+3)}{(4n+1)} (8n\nu-6n+6\nu-5) \\
 & \quad \times (2n\gamma_{2s}A_{2s+1} + 2n\delta_{2s}B_{2s+1}) \lambda^{4n+1} \Big\}
 \end{aligned} \tag{41}$$

A formal solution by successive approximations is as follows:

Write, for $n \geq 0$,

$$A_{2n+1} = \sum_{r=0}^\infty A_{2n+1}^{(r)}, \quad B_{2n+1} = \sum_{r=0}^\infty B_{2n+1}^{(r)} \tag{42}$$

where

$$\begin{aligned}
 A_{2n+1}^{(0)} &= \frac{E\alpha}{(1-\nu)(2n+1)(2n+3)} \left[\frac{(2n+1)a_n}{(8n+2)} \lambda^2 - \frac{(2n+3)a_{n+1}}{(8n+10)} - \frac{1}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{n+j} a_j p^{2n+2j} K_0'(p)}{(2j+1)! I_0'(p)} \right. \\
 & \quad \times \left\{ \frac{(2n+2)p^2}{(2n+1)!(4n+5)} \lambda^2 + \frac{2n}{(2n-1)!(4n+1)} \right\} dp \lambda^{4n+3} \Big] - \frac{1}{(2n+1)(2n+3)} \left[2n \left\{ (2n+2) \Omega_{2n+2} \right. \right. \\
 & \quad + \frac{(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \Big\} \lambda^{4n+3} + \frac{(4n+7)(4n^2+8n+2\nu+2)}{(4n+5)} K_{2n+2} \lambda^{4n+5} \Big] \\
 & \quad - \frac{1}{(2n+1)(2n+3)} \left\{ \frac{(2n+2)(2n+3)(2n-1+4\nu)B_{2n+1}^{(0)}}{(4n+5)} - \frac{2n(4n^2+4n+2\nu-1)B_{2n-1}^{(0)}}{(4n+1)} \lambda^2 \right\}, \\
 & \hspace{15em} (n \geq 0) \\
 A_{2n+1}^{(r)} &= -\frac{1}{(2n+1)(2n+3)} \left\{ \frac{(2n+2)(2n+3)(2n-1+4\nu)B_{2n+1}^{(r-1)}}{(4n+5)} - \frac{2n(4n^2+4n+2\nu-1)B_{2n-1}^{(r-1)}}{(4n+1)} \lambda^2 \right\} \\
 & \quad + \frac{1}{(2n+1)(2n+3)} \sum_{s=0}^\infty \left\{ 2n(2n+2)(2n+2)\alpha_{2s}A_{2s+1}^{(r-1)} + 2n+2\beta_{2s}B_{2s+1}^{(r-1)} \right\} \lambda^{4n+3} \\
 & \quad + \frac{(4n+7)}{(4n+5)} (4n^2+8n+2\nu+2)(2n+2)\gamma_{2s}A_{2s+1}^{(r-1)} + 2n+2\delta_{2s}B_{2s+1}^{(r-1)} \lambda^{4n+5} \\
 & \quad + \frac{4n}{(4n+1)} (8n\nu-6n+6\nu-5)(2n\gamma_{2s}A_{2s+1}^{(r-1)} + 2n\delta_{2s}B_{2s+1}^{(r-1)}) \lambda^{4n+3} \Big\}, \quad (n \geq 0)
 \end{aligned} \tag{43}$$

$$\begin{aligned}
B_{2n-1}^{(0)} = & -\frac{E\alpha}{(1-\nu)} \frac{(4n+1)}{2n(8n^2+12n-8\nu-6\nu+6)} \left[\frac{2na_n}{(4n+1)} + \frac{1}{\pi} \int_0^\infty \sum_{j=0}^\infty \frac{(-1)^{n+j} 2na_j p^{2n+2j} K_0'(p)}{(2j+1)! I_0'(p)} \right. \\
& \times \left\{ \frac{p^2 \lambda^2}{(2n+1)!} + \frac{(4n+3)}{(2n-1)!(4n+1)} \right\} \lambda^{4n+1} dp \Big] - \frac{(4n+1)}{2n(8n^2+12n-8\nu-6\nu+6)} \\
& \times \left[2n \left\{ (2n+2)(4n+3) \Omega_{2n+2} + \frac{(4n+3)(16n\nu+12\nu-12n-10)}{(4n+1)} K_{2n} \right\} \lambda^{4n+1} \right. \\
& \left. \left. + (2n+2)2n(4n+7)K_{2n+2}\lambda^{4n+3} \right], \quad (n \geq 1) \right\} \dots (44) \\
B_{2n-1}^{(r)} = & \frac{(4n+1)}{2n(8n^2+12n-8\nu-6\nu+6)} \sum_{s=0}^\infty \left\{ 2n(2n+2)(4n+3) \alpha_{2s} A_{2s+1}^{(r-1)} \right. \\
& + 2^{n+2} \beta_{2s} B_{2s+1}^{(r-1)} \lambda^{4n+1} + 2n(2n+2)(4n+7) \gamma_{2s} A_{2s+1}^{(r-1)} + 2^{n+2} \delta_{2s} B_{2s+1}^{(r-1)} \lambda^{4n+3} \\
& \left. + \frac{4n(4n+3)}{(4n+1)} (8n\nu-6n+6\nu-5) \gamma_{2s} A_{2s+1}^{(r-1)} + 2^n \delta_{2s} B_{2s+1}^{(r-1)} \lambda^{4n+1} \right\}, \quad (n \geq 1)
\end{aligned}$$

The foregoing method of solving the linear equations is valid as long as the two series in Eq. (42) are both convergent. No analytic proof of convergence will be given for the sake of brevity, while the solution will be illustrated by the following numerical examples.

5. Numerical examples

Numerical examples will be given for the cases $\lambda=0.1, 0.2, 0.3$ and 0.4 . The coefficients a_s are first computed and shown in Table 1, in which the integrals are determined using the results given by W.R. Smythe⁽⁶⁾ shown in Table 2. Next the coefficients Ω_{2n}, K_{2n} in Eq. (27) are computed and shown in Table 3. These integrals are integrated numerically from Simpson's formula within the range of integration $0 < p < 50$ with the aid of the analog computer. The coefficients ${}^{2n}\alpha_{2s}, {}^{2n}\beta_{2s}, {}^{2n}\gamma_{2s}$ and ${}^{2n}\delta_{2s}$ in Eq. (36) are computed from the values of the coefficients in Eq. (37) determined by Ling⁽⁵⁾. These are shown in Table 4. The coefficients A_{2n+1}, B_{2n+1} are then computed by the method of successive approximations for $\lambda=0.1, 0.2, 0.3$ and 0.4 respectively. The results are shown in Table 5. The more rapid the convergence is, the smaller λ is. Here the Poisson's ratio ν is taken as 0.3 .

It may be possible to compute the stresses at any point in the cylinder. The most interesting one, however, is the stress distribution occurring at the surface of the cavity $\rho=\lambda$. This is given by

Table 1 Values of coefficients a_s/τ in Eq. (11)

s	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.4$
$s=0$	$5.003\ 99 \times 10^{-4}$	$4.025\ 65 \times 10^{-3}$	$1.379\ 67 \times 10^{-2}$	$3.371\ 89 \times 10^{-2}$
$s=1$	$-1.501\ 10 \times 10^{-11}$	$-1.545\ 76 \times 10^{-8}$	$-9.051\ 48 \times 10^{-7}$	$-1.658\ 09 \times 10^{-5}$
$s=2$	$5.182\ 91 \times 10^{-16}$	$8.539\ 39 \times 10^{-12}$	$2.531\ 46 \times 10^{-9}$	$1.466\ 40 \times 10^{-7}$
$s=3$	$-1.671\ 81 \times 10^{-20}$	$-4.407\ 21 \times 10^{-15}$	$-6.614\ 16 \times 10^{-12}$	$-1.211\ 77 \times 10^{-9}$
$s=4$		$2.168\ 15 \times 10^{-18}$	$1.647\ 28 \times 10^{-14}$	$9.546\ 67 \times 10^{-12}$
$s=5$			$-3.965\ 20 \times 10^{-17}$	$-7.270\ 45 \times 10^{-14}$

Table 2 Values of $\int_0^\infty \frac{K_1(p)}{I_1(p)} p^n dp$

$n=2$	2.502 02	$n=12$	$2.088\ 06 \times 10^5$
$n=4$	3.769 67	$n=14$	$9.320\ 40 \times 10^6$
$n=6$	$2.342\ 82 \times 10$	$n=16$	$5.514\ 14 \times 10^8$
$n=8$	$3.022\ 83 \times 10^2$	$n=18$	$4.173\ 03 \times 10^{10}$
$n=10$	$6.506\ 02 \times 10^3$	$n=20$	$3.930\ 58 \times 10^{12}$

Table 3 Values of K_{2n}, Ω_{2n} in Eq. (27)

		$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.4$
$K_{2n}/E\alpha\tau$	$n=1$	$1.095\ 61 \times 10^{-4}$	$8.783\ 10 \times 10^{-4}$	$3.008\ 23 \times 10^{-3}$	$7.353\ 60 \times 10^{-3}$
	$n=2$	$-1.677\ 16 \times 10^{-5}$	$-1.366\ 29 \times 10^{-4}$	$-4.682\ 98 \times 10^{-4}$	$-1.145\ 27 \times 10^{-3}$
	$n=3$	$3.821\ 98 \times 10^{-6}$	$3.075\ 74 \times 10^{-5}$	$1.054\ 30 \times 10^{-4}$	$2.579\ 81 \times 10^{-4}$
	$n=4$	$-9.107\ 58 \times 10^{-7}$	$-7.303\ 70 \times 10^{-6}$	$-2.503\ 80 \times 10^{-5}$	$-6.130\ 88 \times 10^{-5}$
	$n=5$	$2.168\ 88 \times 10^{-7}$	$1.760\ 86 \times 10^{-6}$	$6.037\ 18 \times 10^{-6}$	$1.479\ 53 \times 10^{-5}$
	$n=6$	$-5.311\ 13 \times 10^{-8}$	$-4.284\ 81 \times 10^{-7}$	$-1.469\ 32 \times 10^{-6}$	$-3.605\ 92 \times 10^{-6}$
	$n=7$	$1.297\ 00 \times 10^{-8}$	$1.100\ 65 \times 10^{-7}$	$3.779\ 05 \times 10^{-7}$	$9.339\ 71 \times 10^{-7}$
$\Omega_{2n}/E\alpha\tau$	$n=2$	$7.262\ 28 \times 10^{-5}$	$5.844\ 55 \times 10^{-4}$	$2.003\ 16 \times 10^{-3}$	$4.897\ 78 \times 10^{-3}$
	$n=3$	$-8.014\ 23 \times 10^{-6}$	$-6.484\ 15 \times 10^{-5}$	$-2.222\ 62 \times 10^{-4}$	$-5.438\ 45 \times 10^{-4}$
	$n=4$	$1.347\ 34 \times 10^{-6}$	$1.070\ 25 \times 10^{-5}$	$3.669\ 07 \times 10^{-5}$	$8.986\ 32 \times 10^{-5}$
	$n=5$	$-3.142\ 88 \times 10^{-7}$	$-1.966\ 09 \times 10^{-6}$	$-6.741\ 51 \times 10^{-6}$	$-1.653\ 49 \times 10^{-5}$
	$n=6$	$4.743\ 39 \times 10^{-8}$	$4.011\ 85 \times 10^{-7}$	$1.377\ 27 \times 10^{-6}$	$3.417\ 45 \times 10^{-6}$
	$n=7$	$-1.296\ 81 \times 10^{-8}$	$-2.526\ 31 \times 10^{-7}$	$-8.781\ 40 \times 10^{-7}$	$-2.335\ 84 \times 10^{-6}$

$$\begin{aligned}
(\sigma_\theta)_{r=\lambda} &= (\sigma_\theta)_0 + \sigma_{\theta, x_1} + \sigma_{\theta, x_2} \Big|_{r=\lambda} \\
&= \frac{E\alpha}{(1-\nu)} \sum_{s=0}^{\infty} \left[\frac{a_s}{(8s+2)\lambda^{2s+2}} \left\{ (4s^2-2s-1)P_{2s+1}(\mu) - (4s^2-2s+1)P_{2s-1}(\mu) - P_{2s}'(\mu) + P_{2s-2}'(\mu) \right\} \right. \\
&\quad + \frac{1}{\pi} \int_0^\infty \sum_{j=0}^{\infty} \frac{(-1)^{s+j} a_j p^{2s+2j+2} K_0'(p)}{(4s+5)(2s+1)!(2j+1)! I_0'(p)} \left\{ (4s^2+10s+8)P_{2s+1}(\mu) - (2s+2)(2s+3)P_{2s+3}(\mu) - P_{2s}'(\mu) \right. \\
&\quad + P_{2s+2}'(\mu) \Big\} \lambda^{2s+1} + \sum_{s=0}^{\infty} \frac{1}{\lambda^{2s+4}} \left[(2s+2)(2s+3) \left\{ -(2s+1)A_{2s+1} - \frac{(2s+2)(2s+4\nu-1)}{(4s+5)} B_{2s+1} \right\} P_{2s+1}(\mu) \right. \\
&\quad + \frac{(2s+2)(2s+3)(4s^2+10s+10-2\nu)}{(4s+5)} B_{2s+1} P_{2s+3}(\mu) + \left\{ (2s+1)A_{2s+1} + \frac{(2s+2)(2s+3)}{(4s+5)} B_{2s+1} \right\} P_{2s+2}'(\mu) \\
&\quad - \frac{(2s+2)(2s+3)B_{2s+1}}{(4s+5)} P_{2s+4}'(\mu) \Big] - \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(4n+1)} [4n(4n+3)(2n^2+n+2-\nu)P_{2n-1}(\mu) \\
&\quad - 4n(2n+1)(6n-8n\nu-6\nu+5)P_{2n+1}(\mu) - 2n(4n+3)P_{2n-2}'(\mu) - (4n+2)P_{2n}'(\mu)] \\
&\quad \times [A_{2s+1}^{2n} \gamma_{2s} + B_{2s+1}^{2n} \delta_{2s} - K_{2n}] \lambda^{2n-1} - \sum_{s=0}^{\infty} \sum_{n=1}^{\infty} (2n+2)[(2n+1)2nP_{2n+1}(\mu) - P_{2n}'(\mu)] \\
&\quad \times [A_{2s+1}^{2n+2} \alpha_{2s} + B_{2s+1}^{2n+2} \beta_{2s} - Q_{2n+2}] \lambda^{2n-1} \dots \dots \dots (45)
\end{aligned}$$

The results are obtained and shown in Table 6 and Fig. 2. The values in the limiting case $\lambda=0$ are the known results given by Florence and Goodier⁽¹⁾. It may be interesting that the effect of the surface of the cylinder is shown clearly on the stress distributions at the surface of the cavity.

Table 4 Values of $2n\alpha_{2s}$, $2n\beta_{2s}$, $2n\gamma_{2s}$ and $2n\delta_{2s}$ in Eq.(36)

	s	n=0	n=1	n=2	n=3	n=4	n=5
$2n\alpha_{2s}$	0		-1.988 6	0.373 01	-0.088 918	0.021 442	-0.005 185 8
	1		3.231 4	-1.915 7	0.866 54	-0.338 34	0.120 91
	2		-2.517 6	2.836 3	-2.070 1	1.190 8	-0.590 10
	3		1.411 8	-2.564 6	2.754 4	-2.194 2	1.439 3
	4		-0.658 86	1.762 3	-2.619 2	2.761 2	-2.315 0
	5		0.274 49	-1.015 7	1.997 4	-2.691 5	2.810 2
$2n\beta_{2s}$	0		-1.325 5	0.715 16	-0.272 84	0.090 963	-0.028 280
	1		1.852 1	-1.800 5	1.133 2	-0.570 03	0.249 91
	2		-1.351 0	2.128 9	-2.005 5	1.416 6	-0.833 21
	3		0.735 09	-1.726 9	2.279 1	-2.156 5	1.639 9
	4		-0.338 02	1.113 4	-1.966 7	2.405 4	-2.293 4
	5		0.139 93	-0.615 84	1.405 4	-2.153 9	2.521 7
$2n\gamma_{2s}$	0	-0.583 63	0.270 91	-0.104 46	0.035 280	-0.011 013	0.003 281 5
	1	0.632 13	-0.984 84	0.721 62	-0.390 57	0.178 75	-0.073 468
	2	-0.383 00	1.133 9	-1.331 5	1.056 6	-0.667 02	0.362 26
	3	0.176 40	-0.836 92	1.440 9	-1.577 1	1.315 6	-0.913 36
	4	-0.069 75	0.485 19	-1.152 1	1.666 4	-1.776 9	1.532 9
	5	0.025 15	-0.241 38	0.757 49	-1.400 5	1.855 7	-1.950 6
$2n\delta_{2s}$	0	-0.080 607	0.345 30	-0.234 90	0.112 97	-0.045 893	0.016 869
	1	0.268 57	-0.738 30	0.770 31	-0.542 55	0.306 31	-0.150 05
	2	-0.172 27	0.726 26	-1.109 8	1.086 4	-0.817 43	0.515 80
	3	0.080 70	-0.498 27	1.058 1	-1.380 6	1.338 0	-1.057 9
	4	-0.032 30	0.277 14	-0.784 37	1.318 3	-1.600 8	1.551 7
	5	0.011 75	-0.134 46	0.490 27	-1.032 3	1.536 9	-1.792 3

Table 5 Values of coefficients $A_{2n+1}/E\alpha\tau$, $B_{2n+1}/E\alpha\tau$ in Eq.(42)

n	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.4$
n=0	$1.191 6 \times 10^{-6}$	$3.839 1 \times 10^{-5}$	$2.970 4 \times 10^{-4}$	$1.300 2 \times 10^{-3}$
n=1	$-4.880 1 \times 10^{-12}$	$-4.801 2 \times 10^{-9}$	$-2.534 3 \times 10^{-7}$	$-4.368 2 \times 10^{-6}$
n=2	$1.704 8 \times 10^{-16}$	$2.442 8 \times 10^{-12}$	$5.409 4 \times 10^{-10}$	$2.198 8 \times 10^{-9}$
n=3		$-9.902 7 \times 10^{-16}$	$-1.247 1 \times 10^{-12}$	$-1.162 4 \times 10^{-12}$
n=0	$-1.680 0 \times 10^{-9}$	$-4.150 5 \times 10^{-7}$	$-9.652 9 \times 10^{-6}$	$-9.137 2 \times 10^{-5}$
n=1	$4.355 1 \times 10^{-14}$	$1.662 7 \times 10^{-10}$	$1.658 3 \times 10^{-8}$	$5.189 5 \times 10^{-7}$
n=2	$-1.039 4 \times 10^{-18}$	$-6.176 3 \times 10^{-14}$	$-3.479 2 \times 10^{-11}$	$-3.239 8 \times 10^{-10}$
n=3		$2.182 1 \times 10^{-17}$	$5.500 8 \times 10^{-14}$	$1.442 2 \times 10^{-11}$

Table 6 Values of $(\sigma_\theta)_\lambda/E\alpha r\lambda$ obtained by Eq.(45)

θ	$\lambda=0$	$\lambda=0.1$	$\lambda=0.2$	$\lambda=0.3$	$\lambda=0.4$
0°	-0.7143	-0.7118	-0.6924	-0.6856	-0.6459
15°	-0.6900	-0.6879	-0.6709	-0.6685	-0.6419
30°	-0.6186	-0.6176	-0.6066	-0.6155	-0.6181
45°	-0.5050	-0.5052	-0.5013	-0.5223	-0.5496
60°	-0.3571	-0.3579	-0.3589	-0.3848	-0.4183
75°	-0.1849	-0.1855	-0.1876	-0.2055	-0.2270
90°	0	0	0	0	0

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Fig. 2 The values of $(\sigma_\theta)_\lambda/E\alpha r\lambda$

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Empirical Formula and Method of Design for Elliptic Shell Vessel*

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As to a special vessel, whose cross section is elliptic or approximately elliptic, the design formulas are not decided yet. In the previous paper the authors reported the experimental results together with the theoretical analysis on the strength of the vessel of this sort. The analysis, however, is attended with very complicated calculations so that it can not be used for practical design.

Therefore, in this paper, the representation of the maximum stresses of the vessel as the functions of its principal dimensional elements is considered using the results of the previous experiments; and moreover the validity of a practical design formula, which is based on the maximum stresses, is proved by authors' analytical theory for usual design range.

1. Introduction

On a special shaped vessel whose cross section is elliptic or approximately elliptic, any design formula is not clearly decided and not officially published hitherto.

The authors once carried out an experimental research⁽¹⁾ to investigate the strength on the vessel and a theoretical analysis⁽²⁾ was also made by means of the strain-energy method, so that the problem is supposed to have been solved from both theory and experiment. The theoretical analysis, however, is so complicated that it can not be used for practical design. Therefore, in this paper, for the purpose of practical design some formulas are deduced and discussed.

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