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Studies on Vortex Tubes*

(2nd Report, Reynolds Number. The Effects of the Cold Air Rate and the Partial Admission of Nozzle on the Energy Separation)

By Heishichiro TAKAHAMA** and Nobuo SOGA***

The authors tried to explain the effects of the dimensions of a vortex tube (especially the rate of partial admission of nozzle) on the energy separation of gas flowing through it. They tried also to explain the influence of the cold air rate on the flow and thermal fields in a vortex tube with the optimum proportion of the total area of nozzle openings (F_n) to the cross-sectional area of main tube (F_t).

From these researches the following results have been obtained:

- (1) The maximum efficiency of energy separation is obtained when $F_n \doteq 0.17F_t$.
- (2) The values of turbulent diffusivity calculated from the tangential velocity profiles nearly coincide with the values computed from Keyes' empirical formula. Reynolds number of jet from the nozzle, Re_t has no influence on the flow and thermal fields in the vortex tube when $Re_t = (0.5 \sim 3.0) \times 10^6$.
- (3) For the case of large axial velocity, the modified formulae including the axial velocity terms (36) and (37) should be used to predict the stagnation temperature profile in the cross-section at the cold end of the tube.

1. Introduction

Hartnett and Eckert proposed a theory of energy separation in the Hilsch's vortex tube, which states that the stagnation temperature gradient in the radial direction increases in proportion to the angular velocity of vortex flow and an increase in axial velocity causes a rapid decrease of the above-mentioned radial temperature gradient⁽¹⁾. Accordingly, for a certain angular velocity the heat drop of gas leaving the cold-end of the vortex chamber ($i_{tn1} - i_{tc}$) may be considered to decrease in inverse proportion to the ratio of F_n/F_t , where i_{tn1} and i_{tc} : the stagnation enthalpies of gas at the entrance of nozzle and outside the cold-end orifice respectively, F_n and F_t : the total nozzle opening area and the cross-sectional area of the vortex chamber respectively. However, Westley investigated the size effect on the heat drop of cold gas ($i_{tn1} - i_{tc}$) and found that for a certain pressure at the entrance of nozzle, an

increase in the ratio of F_n/F_t caused an increase in the maximum heat drop for the case of $F_n/F_t < 0.2$ ⁽²⁾. By measurements of the internal flow Takahama found that for a certain tangential velocity near the tube wall, the heat drop of cold gas ($i_{tn1} - i_{tc}$) was independent of the ratio of F_n/F_t in the case of $0.20 \geq F_n/F_t \geq 0.08$ ⁽³⁾⁽⁴⁾.

The purpose of this study, therefore, is to explain the effects of the dimension of a vortex tube (especially the rate of partial admission of nozzle) on the energy separation of gas flowing through it, and to explain the influence of the cold air rate on the flow and thermal fields in a vortex tube with the optimum proportion of F_n to F_t . In addition the values of turbulent diffusivity will be calculated and compared with Keyes' empirical formula⁽⁵⁾. Furthermore modified formulae of the stagnation temperature profile including factors of dimension of the chamber and axial velocity term will be derived based on the two dimensional analysis proposed by one of the authors in the previous report⁽⁴⁾.

Nomenclature

G_c, G_h : mass flow leaving cold-end and hot-end, respectively

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** Assistant Professor, Faculty of Engineering, Nagoya University, Chikusa-ku, Nagoya.

*** Engineer, Nagoya Motor Vehicle Works, Mitsubishi Heavy Industries, Ltd.

- G_t : total mass flow entering the system,
 $G_c + G_h$
 ξ : cold air ratio, G_c/G_t
 D, d_n : inside dia. of vortex chamber and nozzle
 opening, respectively
 N, A : number of nozzles and rate of partial
 admission of nozzle, respectively
 F_t, F_n : cross-sectional area of vortex chamber
 and total nozzle opening area, respec-
 tively
 p_t, p : total and static pressure, respectively
 V : velocity
 u, v, w : radial, tangential and axial component
 of velocity, respectively
 T_t, T : stagnation and static temperature, re-
 spectively
 φ : velocity coefficient of nozzle
 ϕ : coefficient of deceleration of jet just
 behind nozzle
 M, Pr : Mach number and Prandtl number,
 respectively
 $Re_t = V_n D / \nu$: tangential Reynolds number
 $Re_r = u_i r_i / \nu$, $Re_r^* = u_i r_i / (\varepsilon + \nu)$: radial Reynolds
 number
 ν, k : kinematic viscosity and thermal con-

ductivity, respectively

$\varepsilon, \varepsilon_h$: turbulent exchange coefficient for mo-
 mentum and heat transfer

$\eta_i, \eta_i', \eta_i''$: efficiency of energy separation defined
 by Eqs. (5), (11) and (12), respectively

U : reference velocity, $\phi \varphi V_{ns}$ or ϕV_n

$t_t', \Delta t_d', t'$: stagnation, dynamic and static temper-
 ature in dimensionless form defined by
 Eqs. (14), (15) and (16), respectively

r', z' : nondimensional coordinate defined by
 Eq. (20)

Subscripts

n_1, n_2 : entrance and exit of nozzle, respectively

c, h : cold-end and hot-end of vortex chamber,
 respectively

i : boundary of back flow region

w : tube wall

s : isentropic change

2. Experimental apparatus and procedure

Fig. 1 shows the schematic view of apparatus. Compressed air is cooled to constant temperature through an after-cooler, then passed through a separator in order to remove drain and dust. Then the air enters into a straight tube through several tangential nozzles fixed near one end. A high velocity vortex flow is set up in the straight tube. The present study covers the range of Mach numbers at the exit of nozzles from 0.5 to 1. The total mass flow rate entering the system G_t kg/sec and the mass flow rate leaving the hot-end and the cold-end, G_h and G_c , are measured by standard pipe orifices. Both the hot air and the cold one have peripheral velocity components, so they are diminished by the rectifiers. ξ is the cold air ratio i.e. G_c/G_t . The value of ξ was changed by the cone valve at the hot-end of the tube. Table 1 shows the dimensions of the vortex chamber used.

A cylinder type Pitot tube with a stagnation temperature probe, the outside diameter of which was 3 mm, was used to measure the flow angle α° , the total pressure p_t kg/m², the static pressure p kg/m² and the stagnation temperature T_t °K of the air flowing in the vortex chamber. These measurements were made in a few cross-sections of the tube. These values in the cross-section including the nozzle center were measured by a Pitot tube with stagnation temperature probe inserted axially through the cold-end orifice. The

Table 1 Dimensions of vortex chamber

Test No.	Inside dia. of chamber D mm	Nozzle				
		Dia. d_n mm	Num- bers N	$N d_n^2 / D^2$	Velocity coef. φ	Rate of partial admission A
1	78.0	9.03	2	2.68×10^{-2}	0.980	0.221
2	78.0	11.02	2	3.99	0.972	0.245
3	52.8	6.52	2	3.05	0.981	0.229
4	52.8	9.02	2	5.84	0.874	0.271
5	52.8	11.01	2	8.70	0.913	0.302
6	52.8	10.91	4	17.08	0.840	0.601
7	52.8	15.60	2	17.46	0.894	0.366
8	27.6	4.53	4	10.78	0.861	0.531
9	27.6	3.23	8	10.96	0.874	0.889
10	27.6	6.48	2	11.02	0.865	0.322
11	27.6	5.54	4	16.12	0.853	0.591

Fig. 1 Schematic diagram of test apparatus.

static temperature $T^\circ\text{K}$, the velocity $V\text{ m/sec}$, the tangential velocity $v\text{ m/sec}$ and the axial velocity $w\text{ m/sec}$ of the air flow were calculated by the following equations:

$$M = \sqrt{\frac{2}{\kappa - 1} \left\{ \left(\frac{p_t}{p} \right)^{(\kappa - 1)/\kappa} - 1 \right\}} \quad \dots\dots\dots (1)$$

$$T = T_t / \{1 + (\kappa - 1) M^2 / 2\} \quad \dots\dots\dots (2)$$

$$a = \sqrt{\kappa g R T} \quad \dots\dots\dots (3)$$

$$V = Ma, \quad v = V \cos \alpha, \quad w = V \sin \alpha \quad \dots\dots\dots (4)$$

where M is Mach number, a is sound velocity m/sec , κ is the ratio of specific heats of air c_p/c_v .

The efficiency of energy separation η_i was introduced to compare the performances of vortex tubes. It is defined as follows:

$$\eta_i = (i_{tn1} - i_{tc}) / \{(\Delta i)_{ns} + (\Delta i)_{cs}\} = (T_{tn1} - T_{tc}) / \{(\Delta T)_{ns} + (\Delta T)_{cs}\} \quad \dots\dots\dots (5)$$

where i_{tn1} and i_{tc} : total enthalpies of air at the entrance of the nozzle and the cold-end orifice respectively, $(\Delta i)_{ns} = c_p(\Delta T)_{ns}$: the reversible adiabatic heat drop of air expanding through the nozzle, $(\Delta i)_{cs} = c_p(\Delta T)_{cs}$: the reversible adiabatic heat drop of air expanding from the state at the exit of the nozzle to the atmospheric pressure.

3. Experimental results and discussion

3.1 Velocity coefficient of nozzle, φ

The static pressure $p_{n1}\text{ kg/m}^2$ and the specific weight $\gamma_{n1}\text{ kg/m}^3$ of the air at the entrance of nozzle, the wall pressure at the exit of nozzle $p_{n2}\text{ kg/m}^2$ and the total mass flow entering the nozzles $G_t\text{ kg/sec}$ were measured. Then the mean value of discharge coefficients of nozzles α was calculated by the following equation:

$$\alpha = G_t / \sqrt{\frac{2g\kappa}{\kappa - 1} p_{n1} \gamma_{n1} \left\{ \left(\frac{p_{n2}}{p_{n1}} \right)^{2/\kappa} - \left(\frac{p_{n2}}{p_{n1}} \right)^{(\kappa + 1)/\kappa} \right\} \left(\frac{\pi N d_n^2}{4} \right)} \quad \dots\dots\dots (6)$$

where κ : the ratio of specific heats of air c_p/c_v , N : number of nozzles, d_n : dia. of nozzle opening. Next an index n for the polytropic expansion in these nozzles which satisfies the following equation was obtained by a graphical method.

$$\sqrt{\left(\frac{p_{n2}}{p_{n1}} \right)^{2/n} - \left(\frac{p_{n2}}{p_{n1}} \right)^{(n+1)/n}} = \alpha \sqrt{\left(\frac{p_{n2}}{p_{n1}} \right)^{2/\kappa} - \left(\frac{p_{n2}}{p_{n1}} \right)^{(\kappa+1)/\kappa}} \quad \dots\dots\dots (7)$$

And finally a velocity coefficient of the nozzle was calculated as follows:

$$\varphi = \sqrt{\left\{ 1 - \left(\frac{p_{n2}}{p_{n1}} \right)^{(n-1)/n} \right\} / \left\{ 1 - \left(\frac{p_{n2}}{p_{n1}} \right)^{(\kappa-1)/\kappa} \right\}} = \alpha \left(\frac{p_{n2}}{p_{n1}} \right)^{1/\kappa - 1/n} \quad \dots\dots\dots (8)$$

Fig. 2 Distributions of absolute velocity in cross-section including nozzle center ($z'=0$), where A : rate of partial admission of nozzle as shown in Fig. 3

These values of φ are shown in Table 1.

3.2 Effect of the partial admission of nozzle on velocity field

In the cross-section of the vortex chamber including the nozzle center ($z'=0$) the vortex flow is axially asymmetric as suggested by Fig. 2. A correction factor ψ is defined as a ratio of the mean velocity at the radius including the nozzle center ($r_w - d_n/2$) in that cross-section ($z'=0$) to the jet velocity at the exit of

nozzle. Fig. 3 represents values of ψ obtained by experiments. The empirical relationship which fits these data is

$$\psi = (1.038A - 0.1730) / (A - 0.141) \quad \dots\dots\dots (9)$$

$$A = (N/2\pi) \cos^{-1}(1 - d_n/r_w) \quad \dots\dots\dots (10)$$

where A : rate of partial admission as illustrated in Fig. 3. In all these chambers it was found that the vortex flow became nearly axially symmetric in the cross-section which was at a distance of $z=r_w$ from the nozzle center towards the hot-end of the tube i.e. $z'=1$, and also that the velocity V was $\psi\phi V_{ns}$ at a radius $r=r_w-d_n/2$ in that cross-section, where $V_{ns}=\sqrt{2Jg(\Delta i)_{ns}}$. $U=\psi\phi V_{ns}$ will be used in nondimensional expression of velocity and temperature in the following sections. The correction factor ψ increases in proportion to the ratio of the total nozzle opening area F_n to the cross-sectional area of chamber F_t and it may be considered that $\psi \approx 1$ for $F_n/F_t \geq 0.16$.

3.3 Efficiency of energy separation η_i

Einstein and Li derived equations which give the distribution of tangential velocities for a single vortex in real fluids as a function of radial Reynolds number $R_{er} = -ur/\nu$ (for a laminar flow) or $R_{er}^* = -ur/(\varepsilon + \nu)$ (for a turbulent flow), where u : radial velocity at radius r , ν : kinematic viscosity, ε : turbulent exchange coefficient⁽⁶⁾. As reported in the previous paper the flow in the vortex chamber is nearly of forced vortex type⁽⁴⁾, then it corresponds to the velocity profile for the case of $R_{er} < 1$ in Einstein and Li's theory. Although the value of u is smaller than 1% of the jet velocity at the exit of nozzle as will be shown later, the value of R_{er} calculated using the value of u obtained in the present experiment is in the range from 100 to 1000. Thus the flow in the vortex

chamber should be considered as a turbulent vortex flow in which u is very small compared with the tangential velocity component v and ε is very large compared with ν , therefore the value of R_{er}^* corresponding to these values of u and ε becomes very small. The value of ε/ν obtained by Keyes' method for the flow in vortex chambers is in the range from 500 to 1000 and also nearly coincides with the value calculated by Keyes' empirical formula⁽⁵⁾:

$$\varepsilon = 2.03 \times 10^{-3} \nu R_{et}^{0.86}$$

where $R_{et} = V_n D / \nu$: tangential Reynolds number, V_n : jet velocity at the exit of nozzle. Moreover it was found that the value of radial velocity u varied with the number of nozzles, the size of nozzle and chamber, the jet velocity at the exit of nozzle V_n and the cold air ratio ξ .

As the result the radial Reynolds number R_{er}^* for a turbulent flow may be considered as a main factor which influences the flow field in the vortex chamber and u and ε included in R_{er}^* may be considered to vary with the tangential Reynolds number of jet from the nozzle R_{et} . Consequently the values of R_{et} and ξ should be considered to determine the flow field in a certain vortex chamber.

Replacing the term of kinetic energy of jet from nozzle $(\Delta i)_{ns}$ with $\varphi^2(\Delta i)_{ns}$ in order to eliminate the effect of friction in nozzle, Eq. (5) becomes

$$\eta_i' = (i_{tn1} - i_{tc}) / \{\varphi^2(\Delta i)_{ns} + (\Delta i)_{cs}\} \quad \dots\dots\dots (11)$$

where φ : the velocity coefficient of nozzle. Values for this efficiency η_i' for various chambers are different from one another as shown in Fig. 4 (a).

Replacing $\varphi^2(\Delta i)_{ns}$ with $\psi^2\varphi^2(\Delta i)_{ns}$ in order to eliminate the effect of partial admission of nozzle mentioned previously in the section 3.2, Eq. (11) becomes

$$\eta_i'' = (i_{tn1} - i_{tc}) / \{\psi^2\varphi^2(\Delta i)_{ns} + (\Delta i)_{cs}\} \quad \dots\dots\dots (12)$$

Fig. 3 Correction factor for velocity coefficient of nozzle A , where number near each symbol: test apparatus No. shown in Table 1

Fig. 4 Efficiencies of energy separation for various chambers defined by Eqs. (11) and (12) versus tangential Reynolds number of jet

There is little difference among values for this efficiency η_i'' for various chambers as shown in Fig. 4 (b), except the case of $F_n/F_t < 0.08$ indicated by the symbol $\odot$, where F_n and F_t are the total nozzle opening area and the cross-sectional area of the chamber, respectively. A perfect forced vortex flow is not set up in the central reverse flow region in the chamber of $F_n/F_t < 0.08$ as mentioned in the previous report so that such cases may be considered as exceptions. On the other hand it may be said that as the effect of partial admission of nozzle on the flow field decreases in inverse proportion to the ratio of F_n/F_t , the value of the efficiency of energy separation defined by Eq. (5) or (11) increases in proportion to F_n/F_t so far as $F_n/F_t \leq 0.16$.

Fig. 5 shows values of η_i'' for various values of cold air ratio ξ for the 2 in vortex tube with the optimum dimension i.e. $F_n/F_t = 0.17$. For a certain value of ξ , the value of η_i'' is nearly constant over the range of $Re_t \div 10^5 \sim 10^6$ which corresponds to the range of jet Mach number $M_n \div 0.3 \sim 1$. This may be explained as follows: in a certain chamber an increase in the jet velocity (accordingly an increase in the tangential Reynolds number of jet Re_t) causes an increase in both the radial velocity u and the turbulent exchange coefficient ε so that the value of Re_t does not affect the tangential velocity distribution and consequently the efficiency of energy separation remains constant. An increase in cold air ratio ξ causes an increase in loss of head through the cold-end orifice and consequently a decrease in efficiency of energy separation η_i'' as shown in Fig. 5. This relation can be neglected for chambers of small value of F_n/F_t but is remarkable for chambers of $F_n/F_t \geq 0.17$.

3.4 Effect of cold air ratio ξ on velocity profile

As mentioned in the previous report, the non-dimensional absolute velocity profiles in cross-

Fig. 5 Effect of cold air ratio on efficiency of energy separation

Fig. 6 Effect of cold air ratio on radial velocity profile in cross-section near nozzle ($z'=1$)

Fig. 7 Effect of cold air ratio on tangential velocity profile in cross-section near nozzle ($z'=1$)

Fig. 8 Effect of cold air ratio on axial velocity profile in cross-section near nozzle ($z'=1$)

sections at various distances from nozzle coincide with one another with little difference and the cold air ratio ξ does not affect these profiles even if the value of ξ ranges from 0.25 to 0.75⁽⁴⁾. An increase in ξ causes an increase in both static pressure and radius of the core region and causes a decrease in length of the core. When $\xi=0$, a weak reverse flow is generated under the influence of lower static pressure in the central part of vortex flow near the nozzle. When the mass rate of hot air is decreased by throttling the regulating

valve at the hot-end, a strong reverse flow is generated under the influences of both the lower pressure mentioned above and the increment of mean static pressure near the hot-end. Accordingly the velocity profile may be considered to change in its details.

It is assumed that the flow is steady and axially symmetric.

In this case, from the equation of continuity the radial velocity at a radius r_1 is written as follows:

$$u_{r=r_1} = \left[\frac{\partial}{\partial z} \int_{r_1}^{r_w} \rho w r dr \right] / \rho r_1 \dots \dots \dots (13)$$

Substituting the experimental values of ρ and w into the above equation and calculating by a graphical method, there are obtained the distributions of radial velocity in the cross-section near the nozzle ($z'=1$) for $\xi=0.25, 0.50$ and 0.75 as shown in Fig. 6. Figs. 7 and 8 show the distributions of tangential and axial velocities corresponding to the radial velocity in Fig. 6 respectively.

As shown in Fig. 6 an increase in cold air ratio ξ causes an increase in radial velocity $|u|$ but the value of $|u|$ is so small compared with the tangential velocity v that a free vortex type flow is not set up in the chamber even for the case of $\xi=0.75$. Absolute value of the axial velocity $|w|$ increases with increasing cold air ratio ξ as shown in Fig. 8. It may be concluded that both $|u|$ and $|w|$ increase with increasing ξ but V and v are not affected by ξ , and $|u|$ is so small compared with both v and $|w|$ that the term of u may be neglected

Fig. 9 Dynamic temperature and stagnation temperature profiles; in cross-section $z'=1$, test No.6, comparison between experiment and theory

in the flow field. Moreover, as the value of $|w|$ increases with increasing F_n/F_t , for the larger value of F_n/F_t i.e. $F_n/F_t \geq 0.16$ the term of w can not be neglected for larger values of ξ e.g. $\xi \geq 0.75$.

3.5 Effect of cold air ratio ξ on distributions of stagnation, dynamic and static temperature

Non-dimensional formulae of stagnation, dynamic and static temperature were defined as follows:

$$t'_t = (T_t - T_{tw}) / (U^2 / 2c_p) \quad \dots\dots\dots (14)$$

$$\Delta t'_d = (\Delta T_d - \Delta T_{dw}) / (U^2 / 2c_p) = (V^2 - U^2) / U^2 \quad \dots\dots\dots (15)$$

$$t' = (T - T_w) / (U^2 / 2c_p) \quad \dots\dots\dots (16)$$

where T_t : stagnation temperature, V : absolute velocity, $\Delta T_d = V^2 / 2c_p$: dynamic temperature, T : static temperature, U : max. absolute velocity in the cross-section (for $z'=1$, U is the absolute velocity near the tube wall), subscript w indicates a value near the tube wall in the cross-section.

Figs. 9 (a), (b) and (c) show the experimental values of t'_t and $\Delta t'_d$ in the cross-section near the nozzle ($z'=1$) for $\xi=0.25, 0.50$ and 0.75 respectively, where $r'=r/r_w$, r_w : radius of chamber, symbols $\circ$ and $\bullet$: experimental values of t'_t and $\Delta t'_d$ respectively. Fig. 10 shows the experimental values of t' in the same cross-section for $\xi=0.25, 0.50$ and 0.75 . As there is a relation $T_t = T + \Delta T_d$, each numerical difference between values indicated by symbols $\circ$ and $\bullet$ in Fig. 9 corresponds to the value of t' at the same radius in Fig. 10, and it is found from Fig. 10 that the experimental values of t' are almost independent of the value of ξ .

Next let us consider the effect of cold air ratio ξ on the distribution of dynamic or stagnation temperature. Among the velocity components of the vortex flow a relation $v > w \gg u$ may be observed in the data of Figs. 6, 7 and 8. Therefore it may be considered that only the tangential velocity profile is dominant to the dynamic temperature distribution. But in a chamber with a large value of F_n/F_t the axial velocity is considerably large for the case of a large value of ξ . (e.g. for the case of $F_n/F_t=0.17$ and $\xi=0.75$, $|w/V_n|_{\max} \approx 0.45$). Then the effect of the term of w on the flow and energy field should be considered for larger values

Fig. 10 Static temperature profiles; in cross-section $z'=1$, test No.6, comparison between experiment and theory

of ξ when F_n/F_t is large. It may be observed in the data of Fig. 9 (a), (b) and (c) that in the core regions ($r' \leq 0.587$) the value of $\Delta t'_d$ is dominant to the value of t'_t and an increase in ξ causes an increase in $\Delta t'_d$, consequently in t'_t . Thus it may be considered that an increase in ξ causes a decrease in the heat drop of air flowing out of the cold-end orifice. This corresponds to the fact that an increase in ξ causes a decrease in the efficiency of energy separation η_i'' as shown already in Fig. 5.

4. Analysis

The following general characteristics of the flow in the vortex chamber are evident in the authors' experiments:

$$1. \quad v > w \gg u, \quad \frac{\partial(\quad)}{\partial r} \gg \frac{\partial(\quad)}{\partial z}$$

2. The flow is axially symmetric and steady.

3. The inertia force terms are negligibly small compared with the apparent viscous force terms because of high turbulent diffusivity.

Under the assumptions of the three items mentioned above the differential equations for the velocity and temperature fields in cylindrical coordinates are

$$\frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} - \frac{v}{r} \right) + \frac{2}{r} \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) = 0 \quad \dots\dots\dots (17)$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) = 0 \quad \dots\dots\dots (18)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\mu}{k} \left\{ \left(\frac{\partial w}{\partial r} \right)^2 + \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 \right\} = 0 \quad \dots\dots\dots (19)$$

The Eqs. (17)~(19) can be adopted for turbulent flow by replacing μ and k by $\rho \varepsilon$ and $\rho c_p \varepsilon_h$, respectively⁽¹⁾⁽⁷⁾⁽⁸⁾. The quantities ε and ε_h are the turbulent exchange coefficients for momentum and heat transfer. It is convenient to define the following dimensionless variables.

$$\left. \begin{aligned} r' &= r/r_w, \quad z' = z/r_w, \quad v' = v/U, \quad w' = w/U, \\ P_r &= \mu c_p / k = \varepsilon / \varepsilon_h, \quad t' = (T - T_w) / (U^2 / 2c_p), \\ \tau' &= \frac{\partial v'}{\partial r'} - \frac{v'}{r'} \end{aligned} \right\} \quad \dots\dots\dots (20)$$

where U : the reference velocity i.e. $\phi \varphi V_{ns}$ as mentioned in paragraph 3.2, T_w : the reference temperature i.e. the static temperature near the tube wall. By substituting Eq. (20) into Eqs. (17), (18) and (19) we obtain:

$$\frac{\partial \tau'}{\partial r'} + \frac{2\tau'}{r'} = 0 \quad \dots\dots\dots (21)$$

$$\frac{\partial}{\partial r'} \left(r' \frac{\partial w'}{\partial r'} \right) = 0 \quad \dots\dots\dots (22)$$

$$\frac{1}{r'} \frac{\partial}{\partial r'} \left(r' \frac{\partial t'}{\partial r'} \right) + 2P_r \left\{ \left(\frac{\partial w'}{\partial r'} \right)^2 + \tau'^2 \right\} = 0 \quad \dots\dots\dots (23)$$

If we next assume that the tangential velocity v'

can be represented by

$$v' \propto e^{-\lambda z'}$$

where λ is a constant, the equation $v'=1$ which was presented in the previous report for the annular region based on the experimental results becomes

$$1 \geq r' \geq r_i'; v' = \beta e^{-\lambda z'} \quad (24)$$

where $\beta = v'_{z'=0} = \text{constant}$. As it may be considered that τ' is finite at $r'=0$, the solution of equation (21) is

$$\tau' = 0 \quad \text{or} \quad \frac{\partial v'}{\partial r'} - \frac{v'}{r'} = 0$$

By integrating this equation we obtain

$$v' = c(z')r' \quad (25)$$

By assuming that at the core boundary the values of v' by Eqs. (24) and (25) are the same, the tangential velocity distribution in the core region is obtained as follows:

$$r_i' \geq r' \geq 0; v' = (\beta/r_i')e^{-\lambda z'}r' \quad (26)$$

The solution of Eq. (22) is

$$w' = f_1(z') \log r' + f_2(z')$$

If we assume that $w' \neq \infty$ at $r'=0$, then $f_1(z')=0$. In the present problem the continuity equation is reduced to

$$\partial w' / \partial z' = 0$$

with the following assumptions

$$\frac{\partial(\quad)}{\partial \theta} = 0, \quad u = 0$$

So we can write $f_2(z') = w'_0 = \text{constant}$. Thus the above equation reduces to

$$w' = w'_0 \quad (27)$$

The values of w'_0 can be estimated by the following equations which represent the continuity relation for both annular region and core region in the cross-section including the nozzle opening ($z'=0$).

$$\left. \begin{aligned} \int_{r_i}^{r_w} 2\pi w r dr &= \frac{N\pi d_n^2}{4} V_n \\ \int_0^{r_i} 2\pi |w| r dr &= \frac{\xi N\pi d_n^2}{4} V_n \end{aligned} \right\} \quad (28)$$

where N : number of nozzles, d_n : diameter of nozzle opening, V_n : jet velocity at exit of nozzle, ξ : cold air ratio. Now the mean absolute velocity in the annular region in the cross-section at $z'=0$ is defined as the reference velocity U . $U = \phi \varphi V_{ns} = \phi V_n$ as mentioned in paragraph 3.2. Then Eq. (28) becomes

$$\left. \begin{aligned} \int_{r_i}^{r_w} w' r' dr' &= \frac{N}{8\phi} d_n'^2 \\ \int_0^{r_i} |w'| r' dr' &= \frac{\xi N d_n'^2}{8\phi} \end{aligned} \right\} \quad (29)$$

where $d_n' = d_n/r_w$. By substituting Eq. (27) into Eq. (29), we obtain

$$1 \geq r' \geq r_i'; w'_0 = w'_a = N d_n'^2 / 4 \phi (1 - r_i'^2) \quad (30)$$

$$r_i' \geq r' \geq 0; w'_0 = w'_c = -\xi N d_n'^2 / 4 \phi r_i'^2 \quad (31)$$

For the annular flow in the cross-section at $z'=0$

$$U^2 = (v^2 + w^2)_{z'=0}$$

and substitution of Eqs. (24) and (30) into the above equation gives

$$\beta^2 = 1 - \{N^2 d_n'^4 / 16 \phi^2 (1 - r_i'^2)^2\} \quad (32)$$

Substitution of Eqs. (24), (26) and (27) into Eq. (23) gives

$$1 \geq r' \geq r_i'; \frac{\partial}{\partial r'} \left(r' \frac{\partial t'}{\partial r'} \right) + 2 P_r \beta^2 e^{-2\lambda z'} / r' = 0 \quad (23 \cdot a)$$

$$r_i' \geq r' \geq 0; \frac{\partial}{\partial r'} \left(r' \frac{\partial t'}{\partial r'} \right) = 0 \quad (23 \cdot b)$$

Adopting the static temperature of air flowing near the tube wall as the reference temperature T_w in the cross-section and considering that the vortex chamber is thermally insulated, the boundary conditions of the present problem may be considered to be at

$$r'=1: t'=0 \quad \partial t' / \partial r' = 0$$

$$r'=0: t' \neq \infty$$

$$r'=r_i': t'_{\text{annular}} = t'_{\text{core}}$$

The solutions of Eqs. (23·a) and (23·b) with the above boundary conditions are

$$1 \geq r' \geq r_i'; t' = -P_r \beta^2 e^{-2\lambda z'} (\log r')^2 \quad (33)$$

$$r_i' \geq r' \geq 0; t' = -P_r \beta^2 e^{-2\lambda z'} (\log r_i')^2 \quad (34)$$

respectively. For the present problem the stagnation temperature is written as follows:

$$T_t = T + (v^2 + w^2) / 2c_p$$

Substituting this equation into Eq. (14) and using Eq. (20) the stagnation temperature is written in dimensionless form as follows

$$t_t' = t' + v'^2 + w'^2 - v_w'^2 - w_w'^2 \quad (35)$$

where v_w' and w_w' are the values at the same point as the reference temperature T_w . Substitution of Eqs. (24), (26), (30), (31), (33) and (34) into Eq. (35) gives the distributions of the nondimensional stagnation temperature in any cross-section. Now it may be considered that at the cold-end there is the relation as follows:

$$z'=0: v_w'^2 + w_w'^2 = 1$$

Considering the above equation the distributions of the nondimensional stagnation temperature at the cold-end are obtained as follows:

$$1 \geq r' \geq r_i'; t_t' = -\{1 - N^2 d_n'^4 / 16 \phi^2 (1 - r_i'^2)^2\} P_r (\log r')^2 \quad (36)$$

$$r_i' \geq r' \geq 0; t_t' = \left\{ 1 - \frac{N^2 d_n'^4}{16 \phi^2 (1 - r_i'^2)^2} \right\} \left\{ \frac{r'^2}{r_i'^2} - P_r (\log r_i')^2 \right\} + \frac{\xi^2 N^2 d_n'^4}{16 \phi^2 r_i'^4} - 1 \quad (37)$$

Substituting Eqs. (24), (26), (30), (31) and (32) into Eq. (15), the distributions of the nondimensional

dynamic temperature at the cold-end are obtained as follows:

$$1 \geq r' \geq r_i'; \quad \Delta t_d' = 0 \dots \dots \dots (38)$$

$$r_i' \geq r' \geq 0; \quad \Delta t_d' = \left\{ 1 - \frac{N^2 d_n'^4}{16 \phi^2 (1 - r_i'^2)^2} \right\} \left(\frac{r'}{r_i'} \right)^2 + \frac{\xi^2 N^2 d_n'^4}{16 \phi^2 r_i'^4} - 1 \dots \dots \dots (39)$$

To the No.6 chamber, in which the maximum efficiency of energy separation (η_i') is obtained in the present experiments, the following values may be considered to be applicable from the experimental results.

$$r_i' = (r_w - d_n)/r_w = 0.587, \quad \phi = 0.98$$

Using the above values and $P_r = 0.71$ (for air), t_i' and $\Delta t_d'$ were calculated by the respective equations of (36), (37) and (38), (39). The calculated values of nondimensional stagnation temperature t_i' for $\xi = 0.25, 0.50$ and 0.75 are given by full lines in Fig.9 (a), (b) and (c) respectively and the calculated values of nondimensional dynamic temperature $\Delta t_d'$ for $\xi = 0.25, 0.50$ and 0.75 are given by broken lines in Fig.9 (a), (b) and (c) respectively. The values of nondimensional static temperature t' calculated by Eqs. (33) and (34) are given by a full line in Fig.10. Moreover the values of t_i' and t' are calculated by the equations which have been derived in the previous report without considering the effect of the term of axial velocity $w^{(4)}$.

Although in the core region the theoretical value of the nondimensional static temperature t' calculated by Eqs. (33) and (34) is larger than the experimental value as shown in Fig.10, the difference between the two values is within 17% of the value of the nondimensional dynamic temperature corresponding to the reference velocity U (in the present experiments the maximum value of that difference corresponds to about 3°C). On the other hand, in the core region the theoretical values of the nondimensional dynamic temperature $\Delta t_d'$ are smaller than the experimental values as shown in Fig.9, and the larger the cold air ratio ξ accordingly the axial velocity w in the core region, the larger the difference between the theoretical value and the experimental one is, but the magnitude of that difference is within 15% of the value of the nondimensional dynamic temperature corresponding to the reference velocity U even for $\xi = 0.75$. The former difference and the latter one mentioned above cancel each other, so that the theoretical values of nondimensional stagnation temperature t_i' calculated by Eqs. (36) and (37) coincide well with the experimental

results as shown by the full lines in Fig.9. As is seen in Fig.9, even for a vortex chamber with a ratio $F_n/F_t \doteq 0.17$ the formulae of nondimensional stagnation temperature t_i' derived in the previous report may be used for $\xi \leq 0.5$, but Eqs. (36) and (37) should be used for $\xi \geq 0.75$.

5. Conclusion

(1) Just behind the exit of nozzle, the jet velocity decreases, and this decrement is in inverse proportion to the ratio of F_n to F_t , where F_n : the total area of the nozzle openings, F_t : the cross-sectional area of the main tube. An increase in the value of F_n/F_t diminishes the decrement of jet velocity mentioned above and consequently causes an increase in the energy separation in the vortex flow for a certain pressure at the entrance of nozzle. If $F_n > 0.16 F_t$, such a decrement of jet velocity is negligibly small. The empirical relationship between the correction factor to the velocity of jet from the nozzle and the rate of partial admission of the nozzle was obtained.

(2) If $F_n < 0.16 F_t$, an increase in the value of F_n/F_t causes an increase in the optimum diameter of the cold-end orifice but the diameter of the cold-end orifice can be enlarged in such cases, so that the loss of head through the cold-end orifice may be kept from increasing. If $F_n \doteq 0.16 F_t$, the optimum diameter of the cold-end orifice is nearly equal to the diameter of the back flow in the central core region. If $F_n > 0.16 F_t$, an increase in the value of F_n/F_t causes a rapid increase in the loss of head through the cold-end orifice because the diameter of the cold-end orifice can not be enlarged over the diameter of the back flow.

(3) Thus the maximum value of the efficiency of energy separation η_i' defined by Eq. (11) is obtained by the chamber of $F_n = 0.16 F_t \sim 0.17 F_t$.

(4) The ratio of the turbulent diffusivity to the molecular kinematic viscosity, ϵ/ν was obtained by comparing the measured tangential velocity profile with the theoretical curves by Einstein and Li⁽⁶⁾. The value of ϵ/ν ranges from 500 to 1000 and nearly coincides with the value computed from Keyes' empirical formula⁽⁶⁾ using the experimental Reynolds number of jet from the nozzle, R_{et} . The jet Reynolds number R_{et} has no influence on the flow and thermal fields in the vortex tube if the value of R_{et} ranges from 0.5×10^6 to 3.0×10^6 , accordingly the jet Mach number M_n ranges from about 0.3 to 1.

(5) In the chamber of $F_n \doteq 0.17 F_t$, the axial velocity is not negligibly small compared with the tangential velocity and so the efficiency of energy separation becomes fairly low for large values of

the cold air ratio ξ . For the case of a large axial velocity, the modified formulae including the axial velocity term were derived. It was found that the modified formulae of nondimensional stagnation temperature (36) and (37) coincide well with the experimental results over a wide range of the cold air ratio i.e. $\xi=0.25\sim 0.75$, then they can be used to predict the performance of energy separation over a wide range of the cold air ratio even in the chambers of $F_n \geq 0.16F_t$.

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References

- (1) J.P. Hartnett and E.R.G. Eckert: *Trans. ASME*, Vol. 79, No.4 (1957-5), p. 751.
- (2) R. Westley: *The College of Aeronautics (Cranfield) Note*, No. 30 (1955-5); No. 67 (1957-7).
- (3) H. Takahama and K. Kawashima: *Memoirs of the Faculty of Engng., Nagoya University*, Vol.12, No.2 (1960-11), p. 227.
- (4) H. Takahama: *Bulletin of JSME*, Vol.8, No.31 (1965-8), p.433.
- (5) J.J.Keyes: *Proc. Heat Transfer and Fluid Mechanics Institute*, (1960), p.31, Stanford University Press.
- (6) H.A.Einstein and H.Li: *Proc. Heat Transfer and Fluid Mechanics Institute*, (1951), p.33, Stanford University Press.
- (7) R. G. Deissler and M. Perlmutter: *Proc. Heat Transfer and Fluid Mechanics Institute*, (1958), p. 40, Stanford University Press.
- (8) M. Suzuki: *Scientific Papers of the Institute of Physical and Chemical Research*, Vol.54, No.1 (1960-3), p. 43.

Errata (Vol. 8, No. 30, May, 1965)

(Author) Kiyoshi OGAWA, (Page) 271, (Column) right, (Line) 3, "Table 1" is as follows.

Table 1

θ_1	60°		
θ_1'	-60°		
θ_2	120°		
θ_2'	-120°		
θ_3	180°		
θ_3'	-180°		
θ_4	0°		
θ_4'	0°		
$\varphi_2 - \varphi_1$	20°		
$\varphi_3 - \varphi_1$	30°		
$\varphi_4 - \varphi_1$	15°		
γ	1		
p	1.5		
γ_2	45°		
R	$R_1 = -0.632\ 837$	$R_2 = 0.234\ 699$	$R_3 = 0.624\ 592$
P_{qux}	1.303 893	0.412 490	1.604 762
P_{quy}	0.803 179	-0.642 802 9	-3.235 738
P_{qsx}	0.613 883	0.613 883	0.613 883
P_{qsy}	-1.285 737	-1.285 737	-1.285 737
$ u $	1.332 780	6.863 087	5.066 046
$ s $	1.088 957	1.088 957	1.088 957
$ t $	0.843 433	1.610 762	3.164 212
$ t_u $	0.537 231	0.435 046	1.828 952
$ t_s $	1.336 573	1.336 573	1.336 573