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Elasto-Plastic Bending of Thin Annular Plates*

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This paper is concerned with the analysis of the elasto-plastic bending of thin annular plates based on Mises' yield condition and Hencky's stress-strain relations. The fundamental equations for the purely elastic range are obtained from the usual ones by introducing a new variable representing the appearance of plastic state on the surface of plate, and the elasto-plastic range is analysed by using the corresponding V. V. Sokolovsky's equations.

Numerical calculation is performed for several combinations of edge conditions, loading conditions and the ratios of inner and outer radii. Propagation of the plastic range in the meridian section and distributions of the deflection and the components of bending moments for various magnitudes of load intensity are obtained for each case.

1. Introduction

The bending problem of annular plates subject to lateral loads is one of the most important problems of structural elements. The behaviour of the annular plates in elastic state has been investigated considerably well in many papers⁽¹⁾. It is necessary, however, to investigate the behaviour in elasto-plastic state to obtain the fundamental data for plastic design of structures. Although several investigations concerned with the limit analysis of the annular plates have been reported⁽²⁾⁻⁽⁶⁾ hitherto, the theory of limit analysis which is based on a number of simplified assumptions furnishes no detailed informations about the process of elasto-plastic bending. Accordingly, to obtain the detailed data about elasto-plastic bending, we have to rely upon the more elaborate theory of elasto-plastic deformation which would reveal the processes of the appearance of plastic regions in the purely elastic state and their propagation according to the increase of load.

Extending V. V. Sokolovsky's method for a simply supported circular plate subject to uniform lateral load⁽⁷⁾, the authors already studied the analytical solution of elasto-plastic bending of circular plates under arbitrary axisymmetric edge and loading conditions⁽⁸⁾. In the present paper, those methods are applied to investigate the elasto-plastic bending of annular plates. Numerical calculation is performed for annular plates with free

inner edge together with simply supported or clamped outer edge and subject to the load which is uniformly distributed or concentrated along inner edge. The developments of plastic range and the distributions of bending moments and deflection at given magnitudes of loads for each case are obtained as results of calculation.

2. Fundamental equations

As shown in Fig. 1, a , b and $2h$ denote the inner and outer radii and the thickness of an annular plate, respectively. By taking cylindrical coordinates r , θ , z , the $z=0$ plane of which coincides with the middle plane of the plate and the positive direction of z -axis of which is assigned downward, when the components of membrane strain are assumed to be negligibly small, the components of strain are expressed as

$$\varepsilon_r = z\alpha_r, \quad \varepsilon_\theta = z\alpha_\theta \quad \dots\dots\dots (1)$$

where

$$\alpha_r = -\frac{d^2w}{dr^2}, \quad \alpha_\theta = -\frac{1}{r} \frac{dw}{dr} \quad \dots\dots\dots (2)$$

denote the principal curvatures of the middle layer of the deformed plate, and w denotes the deflection. By expressing the elasto-plastic interfaces in the meridian section of annular plates with a dimensionless variable χ (cf. Fig. 1) and using Mises' yield condition and Hencky's stress-strain relations, the principal curvatures and components of the bending moment in the elasto-plastic range are expressed as follows⁽⁷⁾⁽⁸⁾:

$$\alpha_r = \mp \frac{k}{2Gh\chi} \frac{\sin(\phi \mp \mu)}{\cos \mu} \quad \dots\dots\dots (3)$$

and

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$$\left. \begin{aligned} M_r \\ M_\theta \end{aligned} \right\} = \frac{4}{3} k h^2 \left\{ \chi^2 \cos \left(\phi \pm \frac{\pi}{6} \right) + \frac{3}{2} (1 - \chi^2) \cos \left(\omega \pm \frac{\pi}{6} \right) \right\} \dots (4)$$

where k and G denote yield stress in pure shear and modulus of rigidity, and μ is obtained from Poisson's ratio ν by the relation

$$\tan \mu = \frac{\sqrt{3}(1-\nu)}{1+\nu},$$

where ϕ or ω is a variable representing the state of strain in the elastic or plastic region. They are related to each other by the relation

$$\tan \phi = \sqrt{3} \tan \mu \tan \omega \dots (5)$$

In the range of purely elastic bending, we can introduce such a variable s that is related to the effective stress $\bar{\sigma}_h$ on the surface of the plate $|z|=h$ by the equation

$$\bar{\sigma}_h = \sqrt{3} k s, \quad (0 \leq s \leq 1)$$

According to this variable, the principal curvatures and the components of bending moment can be expressed as follows⁽⁸⁾:

$$\left. \begin{aligned} \alpha_r \\ \alpha_\theta \end{aligned} \right\} = \mp \frac{k s}{2 G h} \frac{\sin(\phi \pm \mu)}{\cos \mu} \dots (6)$$

and

$$\left. \begin{aligned} M_r \\ M_\theta \end{aligned} \right\} = \frac{4}{3} k h^2 s \cos \left(\phi \pm \frac{\pi}{6} \right) \dots (7)$$

If we consider two typical load-distributions as shown in Fig. 1, the corresponding equation of

$$2 \left\{ \chi \cos \left(\phi + \frac{\pi}{6} \right) - \frac{3}{2} \chi \cos \left(\omega + \frac{\pi}{6} \right) \right\} \frac{d\chi}{dR} - \left\{ \chi^2 \sin \left(\phi + \frac{\pi}{6} \right) + \frac{\sqrt{3}(1-\chi^2) \sin \left(\omega + \frac{\pi}{6} \right) \cos^2 \omega}{2 \tan \mu \cos^2 \phi} \right\} \frac{d\phi}{dR} = \frac{1}{R} \left\{ \chi^2 \sin \phi + \frac{3}{2} (1-\chi^2) \sin \omega \right\} + \frac{3}{2R} I \dots (12)$$

where

$$I = A^2 - R^2 \quad \text{for uniform load}$$

(a) Uniform load, simply supported edge

(c) Concentrated load, simply supported edge

(b) Uniform load, built-in edge

(d) Concentrated load, built-in edge

Fig. 1 Annular plates with nomenclatures

equilibrium is expressed as follows:

$$\frac{dM_r}{dr} + \frac{M_r - M_\theta}{r} = -Q_r \dots (8)$$

where

$$Q_r = \frac{p}{2} \frac{(r^2 - a^2)}{r} \dots (9.a)$$

for a load p distributed uniformly [Fig. 1 (a), (b)] and

$$Q_r = \frac{p}{2} \frac{a^2}{r}, \quad p = \frac{P}{\pi a^2} \dots (9.b)$$

for a load concentrated along inner edge [Fig. 1 (c), (d)]. In the 2nd of the above representations of Q_r , the concentrated load P was replaced by the fictitious uniform load p distributed over an fictitious part of radius a . The equations of compatibility and deflection are

$$\frac{d\alpha_\theta}{dr} + \frac{\alpha_\theta - \alpha_r}{r} = 0 \dots (10)$$

and

$$\frac{dw}{dr} = -r \alpha_\theta \dots (11)$$

In the range of elasto-plastic bending, the fundamental equations are obtained by substituting Eqs. (3), (4) and (5) into Eqs. (8), (10) and (11), and by changing variables r and a into R and A according to the relation $R = \frac{r}{2h} \sqrt{\frac{p}{k}}$ and $A = \frac{a}{2h} \sqrt{\frac{p}{k}}$.

Then, the equation of equilibrium becomes as follows:

and

$$I = -A^2 \quad \text{for concentrated load.}$$

The equations of compatibility and deflection become

$$\begin{aligned} \sin(\psi + \mu) \frac{d\chi}{dR} - \chi \cos(\psi + \mu) \frac{d\psi}{dR} \\ = 2 \cos \mu \frac{\chi}{R} \sin \psi \dots \dots \dots (13) \end{aligned}$$

and

$$\frac{dW}{dR} = - \frac{2}{\cos \mu} \frac{R}{\chi} \sin(\psi + \mu) \dots \dots \dots (14)$$

where $W = Gpw/k^2h$ is a dimensionless representation of the deflection.

In the range of purely elastic bending, the corresponding equations are obtained by a similar procedure and become as follows:

$$\begin{aligned} \cos\left(\psi + \frac{\pi}{6}\right) \frac{ds}{dR} - s \sin\left(\psi + \frac{\pi}{6}\right) \frac{d\psi}{dR} \\ = \frac{s}{R} \sin \psi + \frac{3}{2R} I \dots \dots \dots (15) \end{aligned}$$

$$\begin{aligned} \sin(\psi + \mu) \frac{ds}{dR} + s \cos(\psi + \mu) \frac{d\psi}{dR} \\ = -2 \cos \mu \frac{s}{R} \sin \psi \dots \dots \dots (16) \end{aligned}$$

and

$$\frac{dW}{dR} = - \frac{2}{\cos \mu} s R \sin(\psi + \mu) \dots \dots \dots (17)$$

3. Numerical calculations

Calculations can be performed in a similar way to that employed in the previous paper⁽⁸⁾.

The values of parameter χ_a (the value of χ at the inner edge $R=A$) are selected as $\chi_a = 1.0, 0.8, 0.6, 0.4, 0.2$, and 0 for the plate simply supported at the periphery. For the clamped plate, besides these six values of χ_a and a value of 0.8 of another parameter s_a (the value of s at $R=A$), the value of s_a at which plastic yielding takes place at the surface of periphery is also considered.

As the inner edge of the plate is free, the corresponding boundary condition is $M_r = 0$. Accordingly, ϕ_a (the value of ϕ at $R=A$) must be so determined by Eq. (4) or (7) as to satisfy the above condition for given values of χ_a or s_a . We can put $W=0$ at $R=A$ for the time being, because the integration constant of Eq. (14) or (17) can be determined afterward so that the deflection at the periphery may become zero.

When the inner edge is in elasto-plastic state, Eqs. (12), (13) and (14) are integrated numerically under the boundary conditions described above, starting from $R=A$ and increasing R by a suitable increment. In this range, the value of χ increases with the increase of R . The value of R at which χ becomes unity corresponds to the boundary

between ranges of elasto-plastic and purely elastic bending. At the boundary, the value of s also becomes equal to unity. In the subsequent range, the fundamental equations to be integrated are replaced by Eqs. (15), (16) and (17). The value of s decreases gradually according to the increase of R . The calculation goes on to the periphery in this way when the plate is simply supported. In the case of a clamped plate, however, s decreases in the beginning, but it begins to increase again on its way. Where the value of s attains unity again, the value of R corresponds to the boundary between the range of purely elastic bending and another range of elasto-plastic bending. Accordingly, using the variable χ which is equal to unity there in place of s , Eqs. (12), (13) and (14) are calculated in the subsequent range.

On the other hand, when the inner edge is still in purely elastic state, Eqs. (15), (16) and (17) are calculated with s_a as a parameter, and the method of subsequent calculation is the same as described above.

At the periphery, the condition of a simply supported or clamped edge corresponds to $M_r = 0$ or $dW/dR = 0$, respectively. The calculations are stopped as soon as either of these conditions is satisfied. By denoting B as the value of R there, it is related to the outer radius of annular plate b with the relation $B = (b/2h)\sqrt{p/k}$. Then, the intensity of load p corresponding to the parameter χ_a or s_a is determined immediately, and the results of calculation are re-expressed as functions of r .

Since the value of B varies depending on the value of χ_a or s_a , the value of A cannot be determined in advance. Consequently, the assumed value of A must be determined by trial method so that the value of ratio A/B may coincide with the prescribed value of a/b . These calculations are all carried out by means of an electronic computer NEAC-2203 B, and the numerical integration is performed based on the Runge-Kutta-Gill method.

4. Results of calculations

The calculations are performed for four kinds of annular plates with free inner edge as shown in Fig. 1; i.e., annular plates with (a) uniform load and simply supported outer edge, (b) uniform load and built-in outer edge, (c) concentrated load and simply supported outer edge and (d) concentrated load and built-in outer edge. For each sort of annular plates, three kinds of a/b , $a/b = 0.25, 0.50$ and 0.75 are considered. Poisson's ratio of the material is assumed as $\nu = 0.3$.

The results of calculations are shown in Tables 1~4 and Figs. 2~21, where P is designated as

total load [for uniform load p , $P = \pi(b^2 - a^2)p$], and $M_0 = \sqrt{3}kh^2$ and E denote fully plastic bending moment and Young's modulus.

Table 1 Relations between χ_a and $P/\pi M_0$, $\{E(2h)^3/M_0b^2\}w_{\max}$, c/b (Uniform load, simply supported edge)
 $a/b=0.25$

χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	1.80	2.16	2.72	3.51	4.67	5.54
$\{E(2h)^3/M_0b^2\}w_{\max}$	1.59	1.92	2.44	3.35	5.96	∞
c/b	0.25	0.29	0.36	0.55	0.93	—

$a/b=0.50$

χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	2.08	2.50	3.11	3.88	4.57	4.77
$\{E(2h)^3/M_0b^2\}w_{\max}$	1.89	2.28	2.92	4.19	8.16	∞
c/b	0.50	0.57	0.71	0.97	—	—

$a/b=0.75$

χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	2.37	2.85	3.44	3.92	4.21	4.30
$\{E(2h)^3/M_0b^2\}w_{\max}$	1.45	1.75	2.26	3.33	6.61	∞
c/b	0.75	0.86	—	—	—	—

(a)

(b)

(c)

Fig. 2 Propagation of plastic region (Uniform load, simply supported edge)

5. Discussion

5.1 Uniform load, simply supported plate

Dimensionless total load $P/\pi M_0$, dimensionless maximum deflection $\{E(2h)^3/M_0b^2\}w_{\max}$ and specific radius of boundary between elasto-plastic and purely elastic bending ranges c/b are summarized in Table 1 as functions of parameter χ_a . Figs. 2~5 show the distributions of χ , M_r/M_0 , M_θ/M_0 and $\{E(2h)^3/M_0b^2\}w$. The relation between χ_a and $P/\pi M_0$ is shown in Fig. 6.

As seen in Fig. 2, the plastic range, in this

(a) $a/b=0.25$

(b) $a/b=0.50$

(c) $a/b=0.75$

Fig. 3 Distribution of M_r/M_0 (Uniform load, simply supported edge)

case, always appears first at the surface of inner edge. Especially when $a/b=0.25$, the outer edge remains elastic until the most of the meridian section near the inner edge is occupied by plastic range (i.e. $\chi_a \approx 0.2$). Since χ becomes zero all over the plate for $\chi_a=0$, the plastic region which has appeared at the surface of the outer edge progresses rapidly to the middle surface while χ_a decreases to zero from the value described above. For $a/b=0.75$, on the other hand, the plastic region on the surface reaches the outer edge when χ_a

Fig. 4 Distribution of M_θ/M_0 (Uniform load, simply supported edge)

becomes about 0.6. Accordingly, the larger the value of a/b is, the smaller the difference of χ in the radial direction is.

By comparing Fig. 3 with Fig. 4, it is clear that the maximum values of M_r/M_0 are considerably

Fig. 5 Deflection of central layer (Uniform load, simply supported edge)

smaller than those of M_θ/M_0 and the ratios between them are about 1/3, 1/5 and 1/14 for $a/b=0.25$, 0.50 and 0.75, respectively.

In Fig. 5, by comparing the deflections at inner edge for given values of χ_a larger than 0.4, it is shown that the largest deflection is found when $a/b=0.50$ and the next largest is obtained when $a/b=0.25$. The angle of deflection is, however, largest when $a/b=0.75$ and diminishes toward the smaller values of a/b .

The relations between χ_a and $P/\pi M_0$ are shown

Table 2 Relations between s_a or χ_a and $P/\pi M_0$, $\{E(2h)^3/M_0b^2\}w_{\max}$, c_1/b , c_2/b (Uniform load, built-in edge)
 $a/b=0.25$

	$s_a=0.8$	0.9*	$\chi_a=1.0$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	5.29	5.94	6.61	7.87	9.24	10.1	11.3	12.1
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.91	1.03	1.14	1.38	1.83	2.73	5.27	∞
c_1/b	—	—	0.25	0.29	0.36	0.50	—	—
c_2/b	—	1.0	0.97	0.92	0.88	0.86	—	—

$a/b=0.50$

	$s_a=0.382^*$	0.8	$\chi_a=1.0$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	7.05	11.2	11.6	12.0	12.6	13.2	13.7	14.0
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.54	1.23	1.61	2.00	2.64	3.90	7.74	∞
c_1/b	—	—	0.50	0.58	0.72	—	—	—
c_2/b	1.0	0.89	0.88	0.87	0.81	—	—	—

$a/b=0.75$

	$s_a=0.124^*$	0.8	$\chi_a=1.0$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	12.5	20.6	21.0	21.4	21.9	22.2	22.4	22.5
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.14	1.08	1.38	1.67	2.19	3.25	6.45	∞
c_1/b	—	—	0.75	—	—	—	—	—
c_2/b	1.00	0.92	0.90	—	—	—	—	—

* Denotes the value of s_a at which plastic region appears in annular plate for the first time

Fig. 6 Relation between χ_a and $P/\pi M_0$ (Uniform load, simply supported edge)

in Fig. 6. It is worth noting that, for $\chi_a < 0.4$, while the rate of increase of $P/\pi M_0$ decreases on the curve for $a/b=0.75$ the curve for $a/b=0.25$ shows a steeper increase of $P/\pi M_0$, and for $\chi_a \leq 0.2$ the value of $P/\pi M_0$ becomes less as the value of a/b becomes larger. This is due to the fact that the total load P has been taken as a measure of load. If the intensity of distributed load p is adopted in place of P , it increases with the value of a/b .

5.2 Uniform load, built-in edge

The results of calculation are shown in Table 2 and Figs. 7~11. Fig. 7 shows that the plastic range always appears first on the surface of outer edge and the difference of the development of plastic region over the plate increases toward the larger value of a/b . That is to say, when $\chi_a=1.0$, almost entire meridian section near the periphery has been in plastic state in the case of $a/b=0.75$, while the development of the corresponding plastic region in the case of $a/b=0.25$ is far less for the

(a)

(b)

(c)

Fig. 7 Propagation of plastic region (Uniform load, built-in edge)

(a) $a/b=0.25$ (a) $a/b=0.25$ (b) $a/b=0.50$ (b) $a/b=0.50$ (c) $a/b=0.75$ (c) $a/b=0.75$ Fig. 8 Distribution of M_r/M_0 (Uniform load, built-in edge)Fig. 9 Distribution of M_θ/M_0 (Uniform load, built-in edge)

same value of χ_a .

As shown in Fig. 8, the maximum values of M_r/M_0 decrease with the larger value of a/b . The values at the outer edge decrease according to the increase of load and gradually approach $-2/\sqrt{3}$,

(a) $a/b=0.25$

(b) $a/b=0.50$

(c) $a/b=0.75$

Fig. 10 Deflection of central layer (Uniform load, built-in edge)

which is the minimum value obtained by Mises' yield condition for plane stress state. The corresponding minimum value of M_0/M_0 in Fig. 9 is $-1/\sqrt{3}$. It is shown from these figures that, for larger values of a/b , the plastic region at the outer edge develops at a considerably early stage of loading, which corresponds to the characteristics of development of the plastic range shown in Fig. 7.

The angle of deflection for $\chi_a \geq 0.4$ in Fig. 10 (a) decreases continuously in the radial direction and attains zero at the outer edge so that it corresponds to the condition of clamped edge. The deflection curve for $\chi_a = 0.2$ in Fig. 10 (a), however, is similar to a truncated cone as in the case of supported plates, and its slope changes abruptly at the outer edge. This is due to the fact that the outer edge of the plate, as seen in Fig. 7 (a), has already yielded almost completely, and corroborates the concept of hinge circle in the theory of limit analysis. For $a/b = 0.50$ and 0.75 , these features are more remarkable, hence the deflection curves shown in Fig. 10 (b), (c) are similar to those of supported plates, Fig. 5 (b), (c).

Fig. 11 shows that the total load P for a given value of χ_a increases rapidly as a/b increases, which is different from the previous situation of simply supported plates.

5.3 Concentrated load, simply supported edge

The relevant results are shown in Table 3 and Figs. 12~16. By comparing Table 3 with Table 1, it is clear that, while the ratios of the total load $P/\pi M_0$ in Table 3 to that of Table 1 for a given value of χ_a are about $1/3$ (for $a/b = 0.25$)~ $1/2$ (for $a/b = 0.75$), the maximum deflections for a given χ_a in Table 3 have almost the same magnitudes as those of Table 1 for each value of a/b . Consequently, it may be said that, according to the value of a/b , if one-third to half of the total

Fig. 11 Relation between s_a or χ_a and $P/\pi M_0$ (Uniform load, built-in edge)

magnitude of uniform load is concentrated along the inner edge of annular plate, almost the same

Table 3 Relations between χ_a and $P/\pi M_0$, $\{E(2h)^3/M_0 b^2\} w_{\max}$, c/b (Concentrated load, simply supported edge)

$a/b=0.25$						
χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	0.59	0.71	0.89	1.16	1.59	2.00
$\frac{E(2h)^3}{M_0 b^2} w_{\max}$	1.33	1.61	2.05	2.85	5.08	∞
c/b	0.25	0.29	0.35	0.47	0.75	—

$a/b=0.50$						
χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	0.86	1.04	1.29	1.62	1.90	2.00
$\frac{E(2h)^3}{M_0 b^2} w_{\max}$	1.81	2.19	2.81	4.03	7.86	∞
c/b	0.50	0.58	0.70	0.92	—	—

$a/b=0.75$						
χ_a	1.0	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	1.11	1.35	1.60	1.82	1.96	2.00
$\frac{E(2h)^3}{M_0 b^2} w_{\max}$	1.44	1.74	2.25	3.18	6.58	∞
c/b	0.75	0.87	—	—	—	—

(a)

(b)

(c)

Fig. 12 Propagation of plastic region (Concentrated load, simply supported edge)

effect as in the uniform load would be obtained in the same plate.

Fig. 12 shows that the propagation of plastic range in the plate almost coincides with Fig. 2.

In contrast with the fact that M_r/M_0 in the case of uniform load (Fig. 3) increases monotonously with the load, M_r/M_0 in Fig. 13 attains its maximum value at a certain value of load and decreases rapidly thereafter. The difference of M_r/M_0 between Fig. 3 and Fig. 13 is remarkable and the maximum values of M_r/M_0 are $1/4$ ($a/b=0.25$) $\sim 1/10$ ($a/b=0.75$). By comparing Fig. 13 with Fig. 14, M_r/M_0 is much less than M_θ/M_0 and, especially when $\chi_a=0$, $M_\theta/M_0=1.0$ and $M_r/M_0=0$

(a) $a/b=0.25$

(b) $a/b=0.50$

(c) $a/b=0.75$

Fig. 13 Distribution of M_r/M_0 (Concentrated load, simply supported edge)

for every value of a/b .

The deflection curves described in Fig. 15 almost coincide with those of Fig. 5. Fig. 16 shows that $P/\pi M_0$ for $\chi_a=0$ is always 2.0 regardless of a/b . Consequently, we can conclude that the load carrying capacity of a simply supported annular plate subjected to a concentrated load along the free inner edge is independent of the ratio a/b .

5.4 Concentrated load, built-in edge

The corresponding results of calculation are summarized in Table 4 and Figs. 17~21. Table 4

Fig. 14 Distribution of M_θ/M_0 (Concentrated load, simply supported edge)

shows that $P/\pi M_0$ are about $1/4$ (for $a/b=0.25$)~ $1/2$ (for $a/b=0.75$) of the corresponding values shown in Table 2 for a given value of s_a or χ_a , while the maximum deflections are about $1/1.4$

Fig. 15 Deflection of central layer (Concentrated load, simply supported edge)

(for $a/b=0.25$)~1 (for $a/b=0.75$) of the corresponding values in the previous Table. Hence it can also be noted that the difference between these two cases decreases with the increase of a/b .

As shown in Fig. 17 (a), in the case of $a/b=0.25$, the plastic range appears first on the surface of the inner edge far earlier than its appearance on the surface of the outer edge, which is quite different from Fig. 7 (a). For $a/b=0.50$ and 0.75 , plastic regions propagate from the outer edge as in Fig. 7 (b), (c).

Figs. 18 and 19 show that M_r/M_0 and M_θ/M_0

Table 4 Relations between s_a or χ_a and $P/\pi M_0$, $\{E(2h)^3/M_0b^2\}w_{\max}$, c_1/b , c_2/b (Concentrated load, built-in edge)

$a/b=0.25$							
	$s_a=0.8$	$\chi_a=1.0^*$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	1.00	1.25	1.50	1.89	2.52	3.47	3.83
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.680	0.85	1.03	1.31	1.84	3.47	∞
c_1/b	—	0.25	0.29	0.35	0.47	—	—
c_2/b	—	—	—	—	—	—	—

$a/b=0.50$								
	$s_a=0.668^*$	0.8	$\chi_a=1.0$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	3.15	3.76	4.44	4.81	5.04	5.24	5.42	5.49
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.80	0.96	1.24	1.56	2.16	3.34	6.82	∞
c_1/b	—	—	0.50	—	—	—	—	—
c_2/b	1.00	0.86	0.74	—	—	—	—	—

$a/b=0.75$								
	$s_a=0.194^*$	0.8	$\chi_a=1.0$	0.8	0.6	0.4	0.2	0
$P/\pi M_0$	5.91	9.47	9.64	9.79	9.93	10.1	10.1	10.2
$\frac{E(2h)^3}{M_0b^2}w_{\max}$	0.19	1.00	1.28	1.58	2.09	3.12	6.21	∞
c_1/b	—	—	0.75	—	—	—	—	—
c_2/b	1.00	0.84	0.75	—	—	—	—	—

* Denotes the value of s_a at which plastic region appears in annular plates for the first time

Fig. 16 Relation between χ_a and $P/\pi M_0$ (Concentrated load, simply supported edge)

diminish monotonously toward the outer edge, which is in contrast with the annular plate subject to uniform load where maximum values appear at a certain amount of load.

As already shown in Fig. 17 (a), since the outer edge of the plate of $a/b=0.25$ remains elastic until the plastic region around the inner edge develops considerably, each angle of deflection in Fig. 20 (a) decreases continuously in the radial direction and attains to zero at the outer edge. For $a/b=0.75$, on the other hand, the plastic region of the outer edge develops fast and occupies almost entirely in the early stage of loading. Therefore, the angle of deflection [Fig. 20 (c)] is nearly constant over the whole plate, and the plate deforms into a truncated cone just as Fig. 15.

The above both features are mixed in the case of $a/b=0.50$ [Fig. 20 (b)].

(a)

(b)

(c)

Fig. 17 Propagation of plastic region (Concentrated load, built-in edge)

Fig. 21 shows the relation between $P/\pi M_0$ and s_a or χ_a . In the case of $a/b=0.25$, the value of $P/\pi M_0$ increases more rapidly for $\chi_a=0.4\sim 0.2$, which corresponds with the rapid increase of bending moments shown in Fig. 18 (a) and 19 (a),

Fig. 18 Distribution of M_r/M_0 (Concentrated load, built-in edge)

and accordingly the rapid propagation of plastic region shown in Fig. 17 (a).

Fig. 19 Distribution of M_θ/M_0 (Concentrated load, built-in edge)

Fig. 20 Deflection of central layer (Concentrated load, built-in edge)

6. Conclusion

The elasto-plastic bending of annular plates

Fig. 21 Relation between s_a or χ_a and $P/\pi M_0$ (Concentrated load, built-in edge)

with free inner edge was analysed by using Mises' yield condition and Hencky's stress-strain relation. Numerical calculation was performed for the four combinations of edge and loading conditions: (a) uniform load, simply supported edge, (b) uniform load, built-in edge, (c) concentrated load, simply supported edge and (d) concentrated load, built-in edge. The ratios a/b of inner radius a to the outer one b were selected as 0.25, 0.50 and 0.75. As the results of calculations, the propagation of the plastic range in the meridian section and the distributions of the deflection and the components of bending moments for various amount of load intensity were obtained for each case.

References

- (1) S. Timoshenko and S. Woinowsky-Krieger: *Theory of Plates and Shells*, 2nd ed. (1959), p. 58, McGraw-Hill.
- (2) A.S. Grigoriev: *Inzh. Sborn*, Vol. 16 (1953), p. 177.
- (3) Z. Mroz: *Rozpr. Inz.*, Vol. 6, No. 4 (1953), p. 603.
- (4) V.S. Chernina: *Izb. AN SSSR, OTN Mech. i Mach.*, No. 7 (1958), p. 33.
- (5) Z. Mroz and A. Sawczuk: *Izb. AN SSSR, OTN Mech. i Mach.*, No. 3 (1960), p. 72.
- (6) P.G. Hodge: *Proc. 2nd Symp. on Naval Structural Mechanics*, (1960): p. 297, Pergamon.
- (7) V.V. Sokolovsky: *Theory of Plasticity*, (1960), p. 348 (in Russian).
- (8) Y. Ohashi and S. Murakami: *Bulletin of JSME*, Vol. 7, No. 27 (1964), p. 491.