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Approximate Solution of Elastic and Damping Properties of Oil-Film in Journal Bearings*

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In this paper, an analytical approximate solution for journal bearings of finite lengths is presented. The characteristic performances of steady loaded bearings, elastic and damping properties of oil-film, stability criteria and vibrations of a rotor-bearing system are studied. The results well agree with the Lund-Sternlicht's computer solutions and the experimental results.

1. Introduction

Vibrations of rotors supported by journal bearings are influenced by lubricant films. In designing a rotor-bearing system, critical speeds, amplitudes, and stability of oil-whirl or oil-whip must be predicted. Many practical bearings will have the length-diameter ratio of one or thereabouts. Accordingly, the design must be based on the theory for finite journal bearings. For steady loaded finite bearings, the simple and accurate analytical solution has been already obtained by Aoki⁽¹⁾. For dynamically loaded finite bearing, several numerical solutions are reported, but simple analytical solutions are not yet developed.

In this paper, the approximate analytical solution is developed by introducing the correction factor of finite lengths. The results well agree with the analytical solutions and the computer solutions⁽²⁾ over a wide range of length-diameter ratios. Then, the approximate solution is extended to a fluctuating case and the analytical expressions of the spring and damping coefficients, well agreeing with the computer solutions by Lund and Sternlicht⁽³⁾, are derived. And the vibration analysis of a rotor-bearing system shows good agreement between the theoretical and experimental results in amplitudes at critical speed and stability boundary.

2. Nomenclature

$b_{xx}, b_{xy}, b_{yx}, b_{yy}$ = damping coefficients (first index is force direction, second is velocity direction) kg sec/cm

$B_{xx}, B_{xy}, B_{yx}, B_{yy}$ = dimensionless damping coefficients $b_{ij} C\omega/SW$

C = radial clearance cm

D = bearing diameter cm

e = journal eccentricity cm

F = oil-film force kg

f = dimensionless oil-film force F/SW

h = film thickness cm

$k_{xx}, k_{xy}, k_{yx}, k_{yy}$ = spring coefficients (first index is force direction, second is displacement direction) kg/cm

$K_{xx}, K_{xy}, K_{yx}, K_{yy}$ = dimensionless spring coefficients $k_{ij}C/SW$

k = rotor stiffness kg/cm

L = bearing length cm

m = rotor mass kg sec²/cm

n = correction factor of finite lengths

N = rotational speed of journal rps

p = pressure in oil-film kg/cm²

P = dimensionless pressure $pC^2/\mu UR$

p_m = load per unit projected bearing area W/DL kg/cm²

R = bearing radius cm

S = Sommerfeld number $\left(\frac{\mu N}{p_m}\right)\left(\frac{R}{C}\right)^2$

t = time sec

U = velocity cm/sec (see Fig. 1)

V = velocity cm/sec (see Fig. 1)

W = constant bearing load kg

$x_a, y_a; X_a, Y_a$ = amplitudes of journal and rotor, respectively (x, X are horizontal components, y, Y are vertical downward components) cm

$x, y; X, Y$ = co-ordinate system for rotor-journal motion cm

Z = co-ordinate in axial direction for oil-film cm

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- α, φ, ϕ = angle rad (see Fig. 1)
 β = dimensionless parameter $2W/Ck$
 δ = distance between center of gravity of rotor and shaft center at midspan cm
 ε = eccentricity ratio
 η = coefficient of friction of journal or bearing
 θ = angular polar co-ordinate for oil-film rad
 μ = oil viscosity kg sec/cm²
 τ = dimensionless time ωt
 $\phi_x, \phi_y, \Psi_x, \Psi_y$ = phase angles between unbalance and amplitude of journal and rotor, respectively (index x is horizontal direction, y is vertical direction) rad
 ω = angular velocity of rotor rad/sec
 ω_c = critical speed of rotor on rigid supports $\sqrt{k/m}$ rad/sec
 Ω = speed ratio ω/ω_c

Subscripts

- c = critical speed
 o = equilibrium position of journal, resultant oil-film force
 r = radial direction
 t = tangential direction
 x = x -direction
 y = y -direction
 ∞ = infinitely long bearings
 $\cdot$ = time derivative
 $'$ = dimensionless time derivative

3. Basic equations

The characteristic performance of dynamically loaded journal bearings is reduced from the generalized Reynolds differential equation (Fig. 1):

$$\frac{1}{R^2} \frac{\partial}{\partial \theta} \left(h^3 \frac{\partial p}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(h^3 \frac{\partial p}{\partial z} \right) = \frac{6\mu U}{R} \frac{\partial h}{\partial \theta} + 12\mu V \quad \dots\dots\dots (1)$$

Fig. 1

where,

$$U = R\omega, \quad V = C \frac{d\varepsilon}{dt} \cos \theta + C\varepsilon \frac{d}{dt} (\varphi + \phi) \sin \theta$$

This assumes the laminar incompressible flow in the lubricant film and constant viscosity. Substituting the relations $Z = z/L$, $h = C(1 + \varepsilon \cos \theta)$, the Reynolds equation in dimensionless form becomes:

$$\begin{aligned} & \frac{\partial}{\partial \theta} \left[(1 + \varepsilon \cos \theta)^3 \frac{\partial P}{\partial \theta} \right] \\ & + \frac{1}{4} \left(\frac{D}{L} \right)^2 (1 + \varepsilon \cos \theta)^3 \frac{\partial^2 P}{\partial Z^2} \\ & = -6\varepsilon [1 - 2(\varphi' + \phi')] \sin \theta + 12\varepsilon' \cos \theta \quad \dots (2) \end{aligned}$$

If we define

$$P = \sum_{i=0}^{\infty} P_i \sin (2i+1)\pi Z \quad \dots\dots\dots (3)$$

using the relation

$$1 = \frac{4}{\pi} \sum_{i=0}^{\infty} \frac{\sin (2i+1)\pi Z}{(2i+1)}, \quad (0 \leq Z \leq 1) \quad \dots\dots (4)$$

In Eq. (2), the series expansion for P is strongly convergent, since we can replace the first term only. Then Eq. (2) becomes:

$$\begin{aligned} & \frac{d}{d\theta} \left[(1 + \varepsilon \cos \theta)^3 \frac{dP_0}{d\theta} \right] - \frac{\pi^2}{4} \left(\frac{D}{L} \right)^2 (1 + \varepsilon \cos \theta)^3 P_0 \\ & = \frac{24}{\pi} [-\varepsilon \{1 - 2(\phi' + \varphi')\} \sin \theta + 2\varepsilon' \cos \theta] \quad \dots (5) \end{aligned}$$

In the left-hand side of Eq. (5), if we neglect the first or second term, we obtain the equation for infinitely short or long bearing.

In finite bearing analysis, we define the correction factor n :

$$\begin{aligned} n = & \frac{d}{d\theta} \left[(1 + \varepsilon \cos \theta)^3 \frac{dP_0}{d\theta} \right] / \\ & \left[-\frac{\pi^2}{4} \left(\frac{D}{L} \right)^2 (1 + \varepsilon \cos \theta)^3 P_0 \right] \quad \dots\dots\dots (6) \end{aligned}$$

then Eq. (5) becomes:

$$\begin{aligned} P_0 = & \frac{96}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{1}{(1 + \varepsilon \cos \theta)^3} \\ & \times [\varepsilon \{1 - 2(\varphi' + \phi')\} \sin \theta - 2\varepsilon' \cos \theta] \quad \dots (7) \end{aligned}$$

$$\begin{aligned} & \frac{d}{d\theta} \left[(1 + \varepsilon \cos \theta)^3 \frac{dP_0}{d\theta} \right] = -\frac{24}{\pi} \frac{n}{1+n} \\ & \times [\varepsilon \{1 - 2(\varphi' + \phi')\} \sin \theta - 2\varepsilon' \cos \theta] \quad \dots\dots (8) \end{aligned}$$

Therefore, if the function n is determined, the pressure distribution is given by:

$$\begin{aligned} P = & \frac{96}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\sin \pi Z}{(1 + \varepsilon \cos \theta)^3} \\ & \times [\varepsilon \{1 - 2(\varphi' + \phi')\} \sin \theta - 2\varepsilon' \cos \theta] \quad \dots\dots (9) \end{aligned}$$

4. Correction factor

P_0 and n are only functions of θ . If n is determined in a series of θ , by substituting n into Eq. (8) and integrating with adequate boundary conditions, P_0 is obtained in a series with undetermined coefficients. P_0 in Eq. (7) must be equal

identically to P_0 obtained from Eq. (8) with respect to θ . By equating the expressions for P_0 according to Eqs. (7) and (8), undetermined coefficients are successively determined. In Eq. (7), substituting $\varepsilon' = \varphi' = \phi' = 0$, $\varepsilon = \varepsilon_0$, the pressure distribution under steady load is obtained.

$$P = \frac{96}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D}\right)^2 \frac{\varepsilon_0 \sin \theta}{(1+\varepsilon_0 \cos \theta)^3} \sin \pi Z \dots (10)$$

This satisfies the following boundary conditions.

$$\theta = 0, \pi : P = 0, Z = 0, 1 : P = 0 \dots (11)$$

The Aoki's analytical solution for finite bearings under steady load is as follows:

$$\begin{aligned} \frac{pC^2}{24\mu UR} = & \frac{\varepsilon_0 \sin \theta}{2(1+\varepsilon_0 \cos \theta)^3} \frac{1}{A^2} \\ & \times \left[Z(1-Z) - \frac{128}{\pi^3} \frac{1}{4(4+A^2\pi^2) + \varepsilon_0^2(8-A^2\pi^2)} \right. \\ & \times \left\{ 1 + \frac{(8-A^2\pi^2)(144+13A^2\pi^2)}{4(16+A^2\pi^2)(36+A^2\pi^2)} \varepsilon_0^2 \right. \\ & - \frac{6A^2\pi^2}{16+A^2\pi^2} \varepsilon_0 \cos \theta \\ & \left. \left. - \frac{9(8-A^2\pi^2)A^2\pi^2}{(16+A^2\pi^2)(36+A^2\pi^2)} \varepsilon_0^2 \cos^2 \theta \right\} \sin \pi Z \right] \end{aligned} \dots (12)$$

where, $A = D/L$.

In Eq. (12), by using the approximation $4Z(1-Z) \doteq \sin \pi Z$ and comparing Eq. (10), with Eq. (12), the following equation is obtained.

$$\begin{aligned} \frac{1}{1+n} = & \frac{\pi^3}{32} \left[1 - \frac{512}{\pi^3} \frac{1}{4(4+A^2\pi^2) + \varepsilon_0^2(8-A^2\pi^2)} \right. \\ & \times \left\{ 1 + \frac{(8-A^2\pi^2)(144+13A^2\pi^2)}{4(16+A^2\pi^2)(36+A^2\pi^2)} \varepsilon_0^2 \right. \\ & - \frac{6A^2\pi^2}{(16+A^2\pi^2)} \varepsilon_0 \cos \theta \\ & \left. \left. - \frac{9(8-A^2\pi^2)A^2\pi^2}{(16+A^2\pi^2)(36+A^2\pi^2)} \varepsilon_0^2 \cos^2 \theta \right\} \right] \end{aligned} \dots (13)$$

In Fig. 2, $1/(1+n)$ and $n/(1+n)$ are plotted for $L/D=1$. For small values of eccentricity ratio, these may be independent constants for θ . Even

Fig. 2 Correction factor

if the eccentricity ratio is not small, the pressure in the region of $\theta = 0 \sim \pi/2$ is usually low. Therefore, in calculating the load of bearings, the values of $1/(1+n)$ and $n/(1+n)$ are treated as constants equal to the mean values in the region $\theta = \pi/2 \sim \pi$. This approximation will not greatly affect the estimation of the load carrying capacity. Accordingly, if we assume that n is a constant determined by ε_0 and D/L , the analysis becomes very simple. Such treatment means that the determination of n is performed at a point of θ . Now, equating the oil-film forces (bearing load) derived from Eqs. (7) and (8), we determine the constant value of n . As mentioned later, this approximate method is valid, so far as the bearing load is evaluated.

5. Solution for steady loaded bearings

5.1 Pressure distribution

The pressure distribution under steady load is given by Eq. (10). From Eq. (8), using the boundary conditions in Eq. (11) and treating n as a constant, the pressure distribution under steady load is also given by:

$$P = \frac{24}{\pi} \frac{n}{1+n} \frac{(2+\varepsilon_0 \cos \theta) \varepsilon_0 \sin \theta}{(2+\varepsilon_0^2)(1+\varepsilon_0 \cos \theta)^2} \sin \pi Z \dots (14)$$

5.2 Oil-film forces and load capacity

Oil-film forces F_{r0} and F_{t0} in equilibrium with the bearing load are:

$$\left. \begin{aligned} F_{t0} = W \sin \phi_0 &= \int_0^L \int_0^\pi p R \sin \theta d\theta dz \\ F_{r0} = W \cos \phi_0 &= - \int_0^L \int_0^\pi p R \cos \theta d\theta dz \end{aligned} \right\} \dots (15)$$

Substituting Eq. (10) or (14) and carrying out the integration, the dimensionless oil-film forces are as follows:

$$f_{t0} = \frac{96}{\pi^2} \frac{1}{1+n} \left(\frac{L}{D}\right)^2 \frac{\varepsilon_0}{(1-\varepsilon_0^2)^{3/2}} \dots (16)$$

$$f_{r0} = \frac{384}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D}\right)^2 \frac{\varepsilon_0^2}{(1-\varepsilon_0^2)^2} \dots (17)$$

or

$$f_{t0} = \frac{48n}{1+n} \frac{\varepsilon_0}{(2+\varepsilon_0^2)(1-\varepsilon_0^2)^{1/2}} \dots (18)$$

$$f_{r0} = \frac{96n}{\pi(1+n)} \frac{\varepsilon_0^2}{(2+\varepsilon_0^2)(1-\varepsilon_0^2)} \dots (19)$$

Furthermore, the resultant oil-film forces of Eq. (16) and (17) or (18) and (19) are given respectively by,

$$f_0 = \frac{96}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D}\right)^2 \frac{\varepsilon_0}{(1-\varepsilon_0^2)^2} \sqrt{\pi^2(1-\varepsilon_0^2) + 16\varepsilon_0^2} \dots (20)$$

or

$$f_0 = \frac{48n}{\pi(1+n)} \frac{\varepsilon_0}{(2+\varepsilon_0^2)(1-\varepsilon_0^2)} \sqrt{\pi^2(1-\varepsilon_0^2) + 4\varepsilon_0^2} \dots (21)$$

By equating Eq. (20) to (21), the correction factor n becomes:

$$n = \frac{2}{\pi^2} \left(\frac{L}{D} \right)^2 \frac{(2 + \varepsilon_0^2)}{(1 - \varepsilon_0^2)} \sqrt{\frac{\pi^2(1 - \varepsilon_0^2) + 16\varepsilon_0^2}{\pi^2(1 - \varepsilon_0^2) + 4\varepsilon_0^2}} \dots (22)$$

The Sommerfeld number S denoting the load capacity is:

$$1/S = f_0 \dots (23)$$

as shown in Eqs. (16)~(22), the oil-film forces or the load capacity, modified to the values of finite lengths by the correction factor, are developed.

From Eqs. (16) and (17), the attitude angle is given by:

$$\tan \phi_0 = \frac{\pi(1 - \varepsilon_0^2)^{1/2}}{4\varepsilon_0} \dots (24)$$

and from Eqs. (18) and (19)

$$\tan \phi_0 = \frac{\pi(1 - \varepsilon_0^2)^{1/2}}{2\varepsilon_0} \dots (25)$$

Eq. (24) is the same expression for the attitude angle of the infinitely short bearings and Eq. (25) is that for the infinitely long bearings.

5.3 Coefficient of friction

For simplicity, if we neglect the occurrence of oil-film cavity, and assume $p = \partial p / \partial \theta = 0$ as well as the unruptured oil-film in the region $\pi \leq \theta \leq 2\pi$, the coefficient of friction is given by:

$$\left(\frac{R}{C} \right) \eta = \frac{2\pi^2 S}{(1 - \varepsilon_0^2)^{1/2}} \left[1 \pm \frac{24}{\pi^4} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\varepsilon_0^2}{(1 - \varepsilon_0^2)} \right] \dots (26)$$

where the plus sign applies to the journal and the minus sign to the bearing.

6. Solution for fluctuating journal, spring and damping coefficients of oil-film

6.1 Oil-film forces

The pressure distribution for a fluctuating case is given by Eq. (9). In this case, the positive pressure region is π radians in extent beginning at $\theta = \theta_1$, where θ_1 is given by the following equation from Eq. (9).

$$\tan \theta_1 = 2 \left(\frac{\varepsilon'}{\varepsilon} \right) / \{1 - 2(\varphi' + \phi')\} \dots (27)$$

θ_1 is a complicated variable with respect to time. Furthermore, the correction factor must be calculated over again. However, as described in the next section, if the displacement of journal is small, the values ε'/ε and α' are much smaller than unity. Accordingly, θ_1 is nearly equal to zero, and the pressure due to the radial velocity ε' is much lower than that due to the journal rotation. Then, to simplify the analysis, we apply the correction factor given by Eq. (22) for the fluctuating case and determine the oil-film forces by integrating

from 0 to π for θ . The oil-film forces for this case are given by:

$$f_t = \frac{192}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \times \left[\frac{\pi}{2} \{1 - 2(\varphi' + \phi')\} \frac{\varepsilon}{(1 - \varepsilon^2)^{3/2}} + \frac{4\varepsilon\varepsilon'}{(1 - \varepsilon^2)^2} \right] \dots (28)$$

$$f_r = \frac{384}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \times \left[\{1 - 2(\varphi' + \phi')\} \frac{\varepsilon^2}{(1 - \varepsilon^2)^2} + \frac{\pi(1 + 2\varepsilon^2)\varepsilon'}{2(1 - \varepsilon^2)^{5/2}} \right] \dots (29)$$

6.2 Assumption of small displacement of journal

We take rectangular co-ordinates with the x -axis horizontal and the y -axis vertical downward (Fig. 3). Denoting the journal displacements as x and y , the following equations are obtained.

$$C_\varepsilon \cos \alpha = C_{\varepsilon_0} \cos \alpha_0 + x, \quad C_\varepsilon \sin \alpha = C_{\varepsilon_0} \sin \alpha_0 + y \dots (30)$$

Carrying out the differentiation, ε'/ε and α' become:

$$\frac{\varepsilon'}{\varepsilon} = \left(\frac{x'}{C_\varepsilon} \right) \cos \alpha + \left(\frac{y'}{C_\varepsilon} \right) \sin \alpha \dots (31)$$

$$\alpha' = - \left(\frac{x'}{C_\varepsilon} \right) \sin \alpha + \left(\frac{y'}{C_\varepsilon} \right) \cos \alpha \dots (32)$$

Generally, small displacements of the journal may be expressed by the following equation,

$$x = x_a \cos \omega t, \quad y = y_a \sin \omega t \dots (33)$$

Assuming small displacements of the journal, i.e. x/C_ε and $y/C_\varepsilon \ll 1$, the derivatives x'/C_ε and y'/C_ε become very small. Therefore, the following relations are satisfied with the assumption of small displacements.

$$\varepsilon'/\varepsilon \text{ and } \alpha' \ll 1 \dots (34)$$

6.3 Spring and damping coefficients of oil-film

Consider a small deviation $\Delta\alpha$, $\Delta\varepsilon$ of the journal center from O' to O'' . Then, the oil-film forces are determined by substituting $\varepsilon = \varepsilon_0 + \Delta\varepsilon$

O: Bearing center
O': Journal center in equilibrium point
O'': Journal center

Fig. 3

and $\alpha = \alpha_0 + \Delta\alpha$. Let F_x, F_y be the forces acting on the journal in the x, y direction resulting from this displacement, and these are given by:

$$F_x = F_r \cos \alpha - F_t \sin \alpha, \quad F_y = F_r \sin \alpha + F_t \cos \alpha \quad \dots\dots\dots (35)$$

We linearize these forces with respect to displacement and velocity by assuming small displacements of journal, i. e.

$$F_x = k_{xx}x + k_{xy}y + b_{xx}\dot{x} + b_{xy}\dot{y}, \quad F_y = k_{yx}x + k_{yy}y + b_{yx}\dot{x} + b_{yy}\dot{y} + F_0 \quad \dots\dots\dots (36)$$

The coefficients $k_{xx}, k_{xy}, \dots, b_{xx}, b_{xy}, \dots$ in Eq. (36) denote the spring and the damping coefficients of oil-film, respectively. The dimensionless spring and damping coefficients are as follows:

$$\left. \begin{aligned} K_{xx} &= \left(\frac{\partial f_r}{\partial \varepsilon} \cos \alpha_0 - \frac{\partial f_t}{\partial \varepsilon} \sin \alpha_0 \right) \cos \alpha_0 + \frac{1}{\varepsilon_0} f_{r0} \\ K_{xy} &= \left(\frac{\partial f_r}{\partial \varepsilon} \cos \alpha_0 - \frac{\partial f_t}{\partial \varepsilon} \sin \alpha_0 \right) \sin \alpha_0 - \frac{1}{\varepsilon_0} f_{t0} \\ K_{yx} &= \left(\frac{\partial f_r}{\partial \varepsilon} \sin \alpha_0 + \frac{\partial f_t}{\partial \varepsilon} \cos \alpha_0 \right) \cos \alpha_0 \\ K_{yy} &= \left(\frac{\partial f_r}{\partial \varepsilon} \sin \alpha_0 + \frac{\partial f_t}{\partial \varepsilon} \cos \alpha_0 \right) \sin \alpha_0 \end{aligned} \right\} \quad \dots\dots\dots (37)$$

$$\left. \begin{aligned} B_{xx} &= \left(\frac{\partial f_r}{\partial \varepsilon'} \cos \alpha_0 - \frac{\partial f_t}{\partial \varepsilon'} \sin \alpha_0 \right) \cos \alpha_0 - \left(\frac{\partial f_r}{\partial \alpha'} \cos \alpha_0 - \frac{\partial f_t}{\partial \alpha'} \sin \alpha_0 \right) \frac{\sin \alpha_0}{\varepsilon_0} \\ B_{xy} &= \left(\frac{\partial f_r}{\partial \varepsilon'} \cos \alpha_0 - \frac{\partial f_t}{\partial \varepsilon'} \sin \alpha_0 \right) \sin \alpha_0 + \left(\frac{\partial f_r}{\partial \alpha'} \cos \alpha_0 - \frac{\partial f_t}{\partial \alpha'} \sin \alpha_0 \right) \frac{\cos \alpha_0}{\varepsilon_0} \\ B_{yx} &= \left(\frac{\partial f_r}{\partial \varepsilon'} \sin \alpha_0 + \frac{\partial f_t}{\partial \varepsilon'} \cos \alpha_0 \right) \cos \alpha_0 - \left(\frac{\partial f_r}{\partial \alpha'} \sin \alpha_0 + \frac{\partial f_t}{\partial \alpha'} \cos \alpha_0 \right) \frac{\sin \alpha_0}{\varepsilon_0} \\ B_{yy} &= \left(\frac{\partial f_r}{\partial \varepsilon'} \sin \alpha_0 + \frac{\partial f_t}{\partial \varepsilon'} \cos \alpha_0 \right) \sin \alpha_0 + \left(\frac{\partial f_r}{\partial \alpha'} \sin \alpha_0 + \frac{\partial f_t}{\partial \alpha'} \cos \alpha_0 \right) \frac{\cos \alpha_0}{\varepsilon_0} \end{aligned} \right\} \quad \dots\dots\dots (38)$$

where the partial differential coefficients are evaluated at $\varepsilon = \varepsilon_0, \varepsilon' = \alpha' = 0$. From Eqs. (28) and (29), these are determined as follows:

$$\frac{\partial f_t}{\partial \varepsilon} = \frac{96}{\pi^2} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{1+2\varepsilon_0^2}{(1-\varepsilon_0^2)^{5/2}} \times \left[1 - \frac{1}{1+n} \left(\frac{dn}{d\varepsilon} \right)_{\varepsilon=\varepsilon_0} \frac{\varepsilon_0(1-\varepsilon_0^2)}{1+2\varepsilon_0^2} \right] \quad \dots\dots (39)$$

$$\frac{\partial f_r}{\partial \varepsilon} = \frac{768}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\varepsilon_0(1+\varepsilon_0^2)}{(1-\varepsilon_0^2)^3} \times \left[1 - \frac{1}{1+n} \left(\frac{dn}{d\varepsilon} \right)_{\varepsilon=\varepsilon_0} \frac{\varepsilon_0(1-\varepsilon_0^2)}{1+\varepsilon_0^2} \right] \quad \dots\dots (40)$$

$$\frac{\partial f_t}{\partial \varepsilon'} = \frac{768}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\varepsilon_0}{(1-\varepsilon_0^2)^2} \quad \dots\dots\dots (41)$$

$$\frac{\partial f_r}{\partial \varepsilon'} = \frac{192}{\pi^2} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{1+2\varepsilon_0^2}{(1-\varepsilon_0^2)^{5/2}} \quad \dots\dots\dots (42)$$

$$\frac{\partial f_t}{\partial \alpha'} = \frac{192}{\pi^2} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\varepsilon_0}{(1-\varepsilon_0^2)^{3/2}} \quad \dots\dots\dots (43)$$

$$\frac{\partial f_r}{\partial \alpha'} = \frac{768}{\pi^3} \frac{1}{1+n} \left(\frac{L}{D} \right)^2 \frac{\varepsilon_0^2}{(1-\varepsilon_0^2)^2} \quad \dots\dots\dots (44)$$

and from Eq. (22)

$$\left(\frac{dn}{d\varepsilon} \right)_{\varepsilon=\varepsilon_0} = \frac{6n\varepsilon_0}{(1-\varepsilon_0^2)(2+\varepsilon_0^2)} \times \left[1 + \frac{2\pi^2(1-\varepsilon_0^2)(2+\varepsilon_0^2)}{\{\pi^2(1-\varepsilon_0^2)+4\varepsilon_0^2\} \{\pi^2(1-\varepsilon_0^2)^2+16\varepsilon_0^2\}} \right] \quad \dots\dots\dots (45)$$

7. Discussion of results

7.1 Static properties

Compare the results with other theories or experiments. The values of $\{1/(1+n)\} (L/D)^2$ and

$n/(1+n)$ estimated from Eq. (22) are plotted in Fig. 4. The pressure distribution can be calculated simply from Eq. (10) or (14). The pressures in the central plane of the bearing are compared with the other experimental and theoretical results in Figs. 5, 6. In the case of $L/D=1$, the calculated values show good agreement with the Aoki's analytical solution⁽¹⁾ for $\varepsilon_0=0.2$, but the value obtained from Eq. (10) deviates from the analytical solution and the experimental results for $\varepsilon_0=0.8$.

The load for infinitely long bearings given by Gumbel is

Fig. 4 Correction factor

$$f_{0\infty} = \frac{6\pi\epsilon_0}{(2+\epsilon_0^2)(1-\epsilon_0^2)} \sqrt{\pi^2(1-\epsilon_0^2)+4\epsilon_0^2} \dots (46)$$

The ratio of Eq. (21) to (46) is

$$\frac{f_0}{f_{0\infty}} = \frac{8}{\pi^2} \frac{n}{1+n} \dots (47)$$

Fig. 7 shows a plot of $f_0/f_{0\infty}$ versus L/D . It is noticed that there is good agreement with reference (1) for $\epsilon_0 < 0.4$. Fig. 8 gives the relation between the eccentricity ratio and the Sommerfeld number,

Table 1 1/S

	$L/D=3/2$		$L/D=1$		$L/D=1/2$	
	$\epsilon_0=0.8$	$\epsilon_0=0.2$	$\epsilon_0=0.8$	$\epsilon_0=0.2$	$\epsilon_0=0.6$	$\epsilon_0=0.2$
Approximate theory	26.52	2.413	21.96	1.482	3.196	0.484 3
Aoki ⁽¹⁾	24.89	2.444	19.47	1.476	2.954	0.484 2
Fedor ⁽³⁾	—	—	16.23	1.471	2.333	0.471 8
DuBois-Ocvirk ⁽⁴⁾	—	—	72.02	2.168	3.998	0.542 1
Gotōda ⁽⁶⁾	23.8	2.41	18.5	1.47	2.87	0.466
Sasaki-Mori ⁽⁷⁾	—	—	—	—	2.759	0.482 0
Cameron-Wood ⁽⁸⁾	31.4	2.42	22.2	1.59	3.13	0.490

Fig. 5 Pressure distribution

Fig. 6 Pressure distribution

and it is observed that there is excellent agreement with the computer solution of reference (2) over a wide range of L/D , ϵ_0 . The theory is also more accurate than the DuBois-Ocvirk's solution ⁽⁴⁾

Fig. 7 Side leakage factor for load capacity

Fig. 8 Load capacity

Fig. 9 Spring coefficients

even in $L/D=1/4, 1/8$.

Next, to investigate the accuracy of the theory. We show the comparison of the value $1/S$ with other solutions in Table (1). The theory agrees well with reference (1) for the small eccentricity ratio, and with reference (8) for the large eccentricity ratio.

7.2 Spring and damping coefficients of oil-film

The spring and damping coefficients are determined from Eqs. (37)~(45). The results are shown

Fig. 10 Damping coefficients

Fig. 11 Spring coefficients

in Figs. 9~12 and Tables 2, 3 for $L/D=1, 1/2$, where the attitude angle is evaluated from Eq. (24). The theory is compared with the computer solution by Lund-Sternlicht⁽⁵⁾. For $L/D=1$, the results agree well with reference (5) up to the eccentricity ratio 0.5 and for $L/D=1/2$, up to the eccentricity ratio 0.7. The experimental values in Table 2 are obtained using the apparatus in Fig. 13.

8. Vibration of rotor-bearing system

8.1 Amplitudes at critical speed

For the symmetrical two-bearing rotor with unbalance in Fig. 14, the equations of motion are:

$$\left. \begin{aligned} m\ddot{X} + k(X-x) &= m\delta\omega^2 \cos \omega t \\ k(X-x) &= 2(k_{xx}x + k_{xy}y + b_{xx}\dot{x} + b_{xy}\dot{y}) \\ m\ddot{Y} + k(Y-y) &= -m\delta\omega^2 \sin \omega t + mg \\ k(Y-y) &= 2(k_{yx}x + k_{yy}y + b_{yx}\dot{x} + b_{yy}\dot{y}) + mg \end{aligned} \right\} \dots\dots\dots (48)$$

Fig. 12 Damping coefficients

Fig. 13 Experimental apparatus

Fig. 14

The solutions are given by:

$$\left. \begin{aligned} X &= X_a \cos(\omega t - \Psi_x) \\ Y &= Y_a \sin(\omega t - \Psi_y) + \frac{mg}{k} \\ x &= x_a \cos(\omega t - \phi_x) \\ y &= y_a \sin(\omega t - \phi_y) \end{aligned} \right\} \dots\dots\dots (49)$$

For example, the amplitudes at the critical speed are evaluated in Table 4, where $L/D=1$, $R/C=100$, $R=20$ mm, $\mu=5.3 \times 10^{-7}$ kg sec/cm², $k=296$ kg/cm, $mg=20$ kg, $W=12.5$ kg, $\delta=0.016$ mm, and $\omega_c=120$ rad/sec. The results are compared with reference (6) and it is noticed that there is good agreement with the experimental values. As shown

Table 4 Amplitudes at critical speed

	Journal		Rotor	
	x_a/δ	y_a/δ	X_a/δ	Y_a/δ
Approximate theory	1.5	1.0	9.6	12.2
Experiment ⁽⁶⁾	1.3	1.5	11	16.5
Linear calculation ⁽⁶⁾	3.0	2.7	24	26
Nonlinear calculation	2.1	2.2	15.5	19

in the table, the amplitudes can be evaluated by the linearized oil-film forces with no great error.

8.2 Discrimination of stability

The equations of motion for the rotor without unbalance are obtained by substituting $\delta=0$ into

Fig. 15 Stability chart

Fig. 16 Stability chart

Table 2
Spring coefficient $L/D=1$

ϵ	K_{xx}			K_{xy}			K_{yx}			K_{yy}		
	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7
Approximate theory	3.59	8.96	16.4	-6.73	-5.79	-9.81	7.18	13.6	26.9	1.87	9.53	33.6
Experiment	—	8.2	20.0	—	-7.2	-6.4	—	17.9	39.8	—	8.6	28.3
Lund-Sternlicht ⁽⁵⁾	3.34	10.82	24.2	-6.94	-5.68	0.233	8.26	17.09	41.68	1.93	10.89	44.26
Gotōda ⁽⁶⁾	2.6	9.2	21.0	-6.9	-6.3	-2.1	7.9	16	38	1.5	7.9	32

Damping coefficients $L/D=1$

ϵ	B_{xx}			B_{xy}			B_{yx}			B_{yy}		
	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7
Approximate theory	14.2	16.5	20.7	3.70	11.3	25.9	3.70	11.3	24.8	16.3	33.2	91.5
Experiment	—	17.0	—	—	20.0	—	—	16.0	—	—	45.8	—
Lund-Sternlicht ⁽⁵⁾	14.70	18.04	24.74	3.44	11.49	26.28	2.93	10.60	24.02	16.13	31.88	76.93
Gotōda ⁽⁶⁾	5.90	7.5	10	0.94	3.6	8.5	1.3	3.6	8.2	15	24	47

Table 3
Spring coefficients $L/D=1/2$

ϵ	K_{xx}			K_{xy}			K_{yx}			K_{yy}		
	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7
Approximate theory	1.20	3.93	9.32	-2.19	-1.87	-1.15	2.75	6.75	19.5	0.715	4.96	24.4
Lund-Sternlicht ⁽⁵⁾	1.14	4.03	9.97	-2.18	-1.71	0.339	2.64	6.77	20.40	0.651	4.78	23.95
Gotōda ⁽⁶⁾	1.07	3.8	10.2	-2.2	-1.95	0.20	2.7	6.7	21	0.59	4.1	21
Short bearing approximation	1.35	5.39	15.1	-2.45	-1.98	1.57	3.12	9.15	34.6	0.812	6.39	43.2

Damping coefficients $L/D=1/2$

ϵ	B_{xx}			B_{xy}			B_{yx}			B_{yy}		
	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7	0.2	0.5	0.7
Approximate theory	4.65	5.85	9.09	1.21	4.30	11.3	1.21	4.30	11.3	5.33	12.7	39.7
Lund-Sternlicht ⁽⁵⁾	4.51	5.80	9.36	0.999	3.99	10.73	1.11	4.10	10.99	5.20	11.94	35.32
Gotōda ⁽⁶⁾	2.9	3.4	4.6	0.65	2.1	4.6	0.59	2.1	4.6	5.0	10.1	24
Short bearing approximation	5.20	7.33	12.3	1.35	5.12	15.4	1.35	5.30	15.4	5.96	14.8	53.8

Eq. (48). Thus, the characteristic equation is reduced to:

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \dots (50)$$

Accordingly, the condition for stability of Eq. (50) is given as:

$$\frac{\beta}{\Omega^2} > \left(\beta + \frac{c_2}{Sc_1} \right) \frac{c_1^2 + c_2^2 c_3 - c_1 c_2 c_4}{c_0 c_2^2} \dots (51)$$

where,

$$a_0 = c_0 \Omega^4, \quad a_1 = \left(c_1 + \frac{c_2}{S\beta} \right) \Omega^4,$$

$$a_2 = \left\{ c_3 + \frac{c_4}{S\beta} + \frac{1}{(S\beta)^2} \right\} \Omega^4, \quad a_3 = \left(2c_1 + \frac{c_2}{S\beta} \right) \Omega^2,$$

$$a_4 = \left(2c_3 + \frac{c_4}{S\beta} \right) \Omega^2 + c_0, \quad a_5 = c_1, \quad a_6 = c_3$$

$$c_0 = B_{xx}B_{yy} - B_{xy}B_{yx},$$

$$c_1 = K_{xx}B_{yy} + K_{yy}B_{xx} - K_{xy}B_{yx} - K_{yx}B_{xy},$$

$$c_2 = B_{xx} + B_{yy}, \quad c_3 = K_{xx}K_{yy} - K_{xy}K_{yx},$$

$$c_4 = K_{xx} + K_{yy}$$

The stability charts for $L/D=1, 1/2$ are shown in Figs. 15, 16 for the parameter β , where $\beta=0$ means the rigid rotor. The upper and under side of each curve correspond to the stable and unstable running of the rotor, respectively. Also, we plot the stability boundary evaluated from reference (5). The theory agrees well with the values from reference (5) up to the eccentricity ratio 0.5.

Conclusions

(1) The paper presents the analytical solutions of the spring and damping coefficients of oil-film for finite lengths. These results and stability boundary for $L/D=1$ and $1/2$ agree well with the computer solutions⁽⁵⁾ up to the eccentricity ratio 0.5 and 0.7, respectively.

(2) The theory can also be applied to calculating characteristic performances of finite bearings under steady load. Comparing with reference (1) and (2), no great error results for a wide range of eccentricity ratios and length-diameter ratios.

(3) The amplitudes at the critical speed may be evaluated.

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Discussion

J. AKAOKA: (1) The discussor asks the authors about what are the contents of their experiment under fluctuating load.

(2) A new theory for finite bearings should be assessed about the accuracy, simplicity and width of applicable range. The discussor is interested in this paper on these points. However, the possibility of future development and the utility of the theory are also important. What theoretical development based on this theory do the authors expect? How great is the need for this theory? Moreover, how much is the requirement on the design satisfied?

Authors' closure

(1) As shown in Fig. 13, the rotor considered as a rigid shaft is supported by two bearings and vibrates within the bearing clearance owing to an unbalance mass. The bearing diameter is 40 mm, the length is 40 mm and the weight of the rotor including the shaft is 86.6 kg - the load per unit projected bearing area is 2.7 kg/cm². The experiment was carried out using the rotor with the

unbalance 92, 282 g cm and without unbalance in the region of the rotational speed 500~6 000 rpm (the Sommerfeld number 0.059~0.46 or the eccentricity ratio 0.80~0.35). We measured the load capacity and the phase angle of the journal, and determined the spring and damping coefficients.

(2) In this paper, we derived the analytical expressions of the spring and damping coefficients for finite lengths by assuming a small displacement of the journal. However, the displacement of the journal will be generally large. If the correction factor is applicable to such dynamical case, we may further develop this approximate solution. We think that the development of the theory remains to be proved.

Now, limiting the problem to the vibration of the rotor supported by the journal bearings, the amplitude of the journal is very small as compared to the radial clearance except for oil-whip. Thus, many cases may be treated by assuming a small displacement of the journal. For example, the critical speed, amplitude and stability boundary must be estimated by taking into account the

elastic and damping properties of oil-film. In this case, the spring and damping coefficients of oil-film may be simply evaluated by the analytical expressions, since this theory will be useful in the initial stage of design.

We intend to investigate further experimentally and theoretically the accuracy of the solution and the possibility of future development of the theory.

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A Study of Gear Hobbing*

(Chiefly on Cutting Hard Gears)

By Akira WAKURI**, Taku UENO***, and Masando AINOURA****

In making an efficient production of gears of high performance, it is considered best to have them hobbled and finished by shaving, with material of gears as hard as possible provided it can be cut fairly well. In the present paper the authors have made theoretical and experimental studies of a hob with such form and material, as well as cutting oil, as give it the largest durability in cutting large and hard gears. This paper also deals with the form of a hob which reduces the fluctuation of cutting force.

1. Introduction

In order to produce gears of highest accuracy efficiently, it is thought to be best to have them hobbled in the correct basic form and finished by shaving. In case where the size of gears is large, however, the important problem is to consider the durability of hob. If gears of soft material are used, pittings are apt to take place where the load is large.

The authors carried out a study with an aim to hob gears of material as hard as possible so far as it can be shaved. ($H_B=350$, $H_S=50$) The gears of high accuracy, endurance against high contact pressure and good conformability may be rated as best in most cases.

2. The theory of action of the top cutting edges of hob teeth

The mechanism of the cutting action of hob teeth is very complicated. However, since the greater part of the materials of the teeth spaces is removed by the top cutting edge, the wear and failure of this portion are maximum. The authors, therefore, tried to analyze theoretically the cutting

action of top cutting edges of the teeth.

2.1 The traces of the top cutting edges in action and the profile of the generated root surface of the gear at the time of roughing

The hob tooth cuts deepest into the blank on the common perpendicular line of the hob axis and the gear axis. The generating pitch point is located on this line, and the common perpendicular is the center of the generation. The generating portions of hob k_0 lie on both sides of this point.

$$k_0 = \frac{h_{kc} \cos \gamma}{\tan \alpha_s \cos \beta}$$

where,

Fig. 1

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