Heat Transfer of Slurry Flow with Internal Heat Generation

(4th Report, Laminar Heat Transfer in Thermal Entrance Region between Parallel Plates for Pseudo-Plastic and Dilatant Fluids)

By Itaru Michiyoshi** and Ryūichi Matsumoto***

This paper deals with an analysis to determine the heat transfer characteristics in the thermal entrance region for the established laminar flow of a heat generating pseudo-plastic or dilatant fluid between two parallel plates with wall heat transfer. The thermal conditions at wall treated in this paper are for both cases of uniform wall heat flux and uniform wall temperature. In this analysis, the numerical calculations were performed by means of a digital computer KDC-1.

As the result of calculations it is shown that the relationship between the Nusselt number and the distance from inlet in the presence of internal heat generation is different from that in the absence of internal heat source.

1. Introduction

In the previous report, the authors analyzed the laminar heat transfer from a pseudo-plastic fluid with internal heat generation in a circular tube\(^{(1)}\). But other different shapes of channels are often encountered in the practical engineering systems. Therefore, the heat transfer from heat generating pseudo-plastic fluid in thermal entrance region for established laminar flow is analyzed in this paper. If the shape of a rectangular channel is flat, the flow is like the stream between two parallel plates, and the results of the present analysis are applicable to the problem in the rectangular channel flow.

Nomenclature

- \( a \): one half of channel width
- \( C_0 \): coefficient in series expansion of temperature
- \( c \): constant
- \( D_0 \): coefficient in series expansion of temperature
- \( D_0 ' \): constant
- \( E_0 \): coefficient in series expansion of temper-
- \( F \): parameter defined by Eq. (32)
- \( F_a \): coefficient in series expansion of temperature
- \( f(y) \): function for temperature distribution in fully developed region
- \( g \): acceleration of gravity
- \( k \): thermal diffusivity
- \( N \): rheology constant
- \( N_0 \): Nusselt number
- \( P \): symbol defined by Eq. (17)
- \( q \): function for temperature distribution in fully developed region
- \( Q \): rate of heat generation per unit volume of fluid
- \( q_r \): heat transfer rate per unit surface area
- \( R \): symbol defined by Eq. (16)
- \( T \): temperature
- \( T_m \): mixed-mean temperature
- \( T_0 \): temperature at \( z=0 \)
- \( T_w \): wall temperature
- \( u \): velocity
- \( u_m \): mean velocity
- \( u_{max} \): velocity at \( y=0 \)
- \( x \): axial co-ordinate
- \( Y_a \): eigenfunction
- \( Y_a ' \): derivative of eigenfunction \((=dY_a/d\eta)\)
- \( \gamma \): co-ordinate

Greek

- \( \alpha \): heat transfer coefficient

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2. Velocity distribution

In Fig. 1, a pseudo-plastic fluid or dilatant fluid flows with fully developed laminar flow pattern through a slit formed by two parallel plates in the direction of \(x\). In this case, the relation between the derivative of velocity component \(du/dy\) and the shear force per unit area \(\tau\) is

\[
du/dy = -\tau N / \mu_p \tag{1}
\]

In this equation, the case of \(N=1\) corresponds to the Newtonian fluid, \(N<1\) to the dilatant fluid and \(N>1\) to the pseudo-plastic fluid.

Using Eq. (1) and the balance equation of force, \(u\), \(u_m\) and \(u_{\text{max}}\) are given as follows:

\[
\begin{align*}
\rho & : \text{density} \\
\tau & : \text{shear stress} \\
\mathcal{U} & : \text{eigenfunction} \\
\mathcal{U}' & : \text{derivative of eigenfunction (}=d\mathcal{U}/d\eta) \\
\end{align*}
\]

\[
\begin{align*}
\dot{\beta}_n & : \text{eigenvalue} \\
\dot{\gamma}_n & : \text{eigenvalue} \\
\gamma & : \text{non-dimensional co-ordinate (}=y/a) \\
\lambda & : \text{thermal conductivity} \\
\mu_p & : \text{viscosity of fluid} \\
\xi & : \text{non-dimensional axial co-ordinate} \\
& = \frac{1}{R_0 P_r} \frac{x}{a} \\
\end{align*}
\]

\[
\begin{align*}
u &= \frac{\tau N a}{(N+1) \mu_p} \left[1 - \left(\frac{y}{a}\right)^{N+1}\right] \tag{2} \\
u_m &= \frac{\tau N a}{(N+2) \mu_p} \tag{3} \\
u_{\text{max}} &= \frac{\tau N a}{(N+1) \mu_p} \tag{4}
\end{align*}
\]

where \(a\) is one half of the channel width and \(\tau_r\) is the shear force per unit area at wall. Fig. 2 shows the velocity profiles between parallel plates for several values of \(N\).

3. Temperature distribution

In this paper, the following assumptions are made:

1. The heat source per unit volume within the fluid is independent of both time and coordinate.
2. The heat produced by viscous force is neglected.
3. The heat conduction in \(x\)-direction is neglected.
4. The physical properties of fluid are independent of temperature.

In Fig. 1, the fluid flows from left to right with a fully developed laminar velocity profile. And both the internal heat generation and the heat transfer at wall surface start from \(x=0\).

Using the above assumptions, the equation of heat energy becomes

\[
\frac{\partial T}{\partial x} = \frac{1}{\rho c} \frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho c} \tag{5}
\]

Substituting Eqs. (2), (3) and (4) in this equation, the temperature distribution of fluid can be obtained.

3.1 Uniform heat flux at wall

Considering the assumption (1), Eq. (5) is a linear differential equation and one can separate the general problem into the following two simpler cases:

1. A situation where there is internal heat generation within the fluid between two thermally insulated parallel plates.
2. A situation where there is heat transfer \(q\) at wall but no internal heat generation within the fluid. If these temperatures are designated \(T_q\) and \(T_e\) respectively, the solution to the general problem can be obtained in the following form:

\[
T = T_q + T_e \tag{6}
\]

where the fluid temperature is defined as the excess temperature over the value at \(x=0\).

The equation and boundary conditions to be used to determine \(T_q\) and \(T_e\) are as follows, respectively:
Case 1
\[ \frac{\partial T_a}{\partial x} = k \frac{\partial^2 T_a}{\partial y^2} + \frac{Q}{\rho c_p} \]  

Boundary conditions:
\[ y = 0 : \partial T_a / \partial y = 0, \quad y = a : \partial T_a / \partial y = 0, \]
\[ x = 0 : T_a = 0 \]

Case 2
\[ \frac{\partial T_s}{\partial x} = k \frac{\partial^2 T_s}{\partial y^2} \]

Boundary conditions:
\[ y = 0 : \partial T_s / \partial y = 0, \quad y = a : \partial T_s / \partial y = -q / \lambda, \]
\[ x = 0 : T_s = 0 \]

3.1.1 Insulated wall with internal heat source
Consider the fully developed region. Since Eq. (5) must be satisfied in this region, the following equation is obtained:
\[ \frac{u}{\rho c_p u_m} \frac{\partial T_{w0}}{\partial x} = k \frac{\partial^2 T_{w0}}{\partial y^2} + \frac{Q}{\rho c_p} \]

where \( T_{w0} \) is the temperature in the fully developed region. By using the mixed-mean temperature,
\[ T_m = \frac{\int_0^a T_u dy + \int_0^a T_d dy}{\int_0^a u dy} \]

the following relation is obtained,
\[ \frac{\partial T_{w0}}{\partial x} = \frac{\partial T_{w0}}{\partial x} - \frac{Q}{\rho c_p u_m} \]

Hence, \( T_{w0} \) is expressed by
\[ T_{w0} = \frac{Q}{\rho c_p u_m} x + f(y) \]

where \( f(y) \) is a function of \( y \) alone, which shows the temperature profile of a fully developed region.

Substituting Eq. (12) into Eq. (9) and considering the velocity profile, \( f(y) \) can be given by
\[ f(y) = \frac{Q a}{\lambda} \left[ \frac{1}{2(N+1)} \frac{y^2}{a} - \frac{1}{(N+1)(N+3)} \frac{y^{N+3}}{a} + C_2 \right] \]

In order to determine the solution in the thermal entrance region, it is convenient to introduce a temperature \( T_q^* \). Hence, \( T_q^* \) is written as follows:
\[ T_q = T_q^* + T_{w0} \]

Substituting this equation into Eq. (7) and applying Eq. (9), the following equation can be obtained,
\[ \frac{u}{\rho c_p u_m} \frac{\partial T_q^*}{\partial x} = k \frac{\partial^2 T_q^*}{\partial y^2} \]

Boundary conditions:
\[ y = 0 \quad \text{and} \quad y = a : \partial T_q^* / \partial y = 0 \]

Using the following symbols corresponding to the formula of Reynolds number and Prandtl number, respectively,
\[ R_e = u_m (2a) / \nu \]
\[ Pr = \nu / (\rho k) \]

the solution of Eq. (15) is given by
\[ \frac{T_q^*}{Q a^2 / \lambda} = \sum_{n=1}^{\infty} C_n Y_n \exp \left( -2 \beta_n \frac{y}{u_{\text{max}}} \right) \]

where
\[ \xi = \frac{x}{R_e P_r a} \]

And, \( \beta_n \) and \( Y_n \) are the eigenvalue and eigenfunction of the following Sturm-Liouville type differential equation, respectively,
\[ \frac{d^2 Y_n}{dy^2} + \beta_n \frac{u}{u_{\text{max}}} Y_n = 0 \]

Boundary conditions:
\[ \eta = y / a \quad \text{and} \quad \eta = 1 : d Y_n / d \eta = 0 \]

Taking into account \( T_q = 0 \) at \( x = 0 \) and using Eqs. (13) and (18), the following relation is obtained:
\[ T_q^* = \frac{Q a^2}{\lambda} \sum_{n=1}^{\infty} C_n Y_n = -f(y) \]

From the orthogonality of Sturm-Liouville system, the coefficients \( C_n \) can be written as follows:
\[ C_n = -\int_0^1 \int_0^a \frac{d Y_n}{u \partial^2 / \partial y^2} \frac{d Y_n}{u \partial \partial y} \]

Table 1: Eigenvalues \( \beta_n \) and eigenfunctions \( Y_n(\xi) \)

<table>
<thead>
<tr>
<th>( N = 0.5 )</th>
<th>( N = 1 )</th>
<th>( N = 2 )</th>
<th>( N = 3 )</th>
<th>( N = 5 )</th>
<th>( N = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>( \beta_n )</td>
<td>( Y_n(\xi) )</td>
<td>( \beta_n )</td>
<td>( Y_n(\xi) )</td>
<td>( \beta_n )</td>
</tr>
<tr>
<td>1</td>
<td>21.125 5</td>
<td>-1.337 8</td>
<td>18.920 8</td>
<td>-1.271 9</td>
<td>14.667 6</td>
</tr>
<tr>
<td>2</td>
<td>79.137 8</td>
<td>1.480 5</td>
<td>69.902 7</td>
<td>1.403 5</td>
<td>55.896 9</td>
</tr>
<tr>
<td>3</td>
<td>173.792 2</td>
<td>-1.576 3</td>
<td>153.803 6</td>
<td>-1.493 1</td>
<td>135.351 9</td>
</tr>
<tr>
<td>4</td>
<td>305.923 5</td>
<td>1.651 5</td>
<td>277.527 8</td>
<td>1.561 4</td>
<td>246.165 3</td>
</tr>
<tr>
<td>5</td>
<td>475.442 1</td>
<td>-1.673 4</td>
<td>420.280 5</td>
<td>-1.619 8</td>
<td>377.762 3</td>
</tr>
</tbody>
</table>
These calculations were performed by means of a digital computer (KDC-1) with the same mathematical manipulation as in the previous reports\(^{12}\)\(^{12}\). Eigenvalue \(\beta_s\) and eigenfunction at wall \(Y_s(1)\) are tabulated in Table 1. In order to compare the results obtained with values by Siegel in the case of Newtonian fluid \((N = 1)\), Siegel’s values are also given in the parentheses\(^{12}\).

The solid lines in Fig. 3 show the distributions of temperature difference between \(T_{Qw}\) and \(T_{Qw}\) in the case of \(N = 3\). In order to show the change in temperature profile due to \(N\), the temperature distributions in the case of \(N = 1\) (Newtonian fluid) and \(N = 10\) at \(\xi = 0.03\) are illustrated by the dotted lines.

3.1.2 Uniform wall heat flux without internal heat source Let us consider the fully developed heat transfer situation. Denoting the temperature of fluid in this region as \(T_{Qw}\), Eq. (8) becomes

\[
\frac{\partial T_{Qw}}{\partial x} = k \frac{\partial^2 T_{Qw}}{\partial y^2} \quad \text{...(22)}
\]

Boundary conditions:
\[y = 0 : \frac{\partial T_{Qw}}{\partial y} = 0, \quad y = a : \frac{\partial T_{Qw}}{\partial y} = -q/\lambda\]

If the heat flux \(q\) at wall is constant in the direction of \(z\), and we define that \(q\) is positive when heat transfers from fluid to wall, the following equation can be obtained:

\[
\frac{\partial T_{Qw}}{\partial x} = \frac{\partial T_{Qw}}{\partial x} = -\frac{q}{a \rho c u_w} \quad \text{...(23)}
\]

Hence, \(T_{Qw}\) is given by

\[
T_{Qw} = \frac{-q}{a \rho c u_w} x + p(y) \quad \text{...(24)}
\]

Substituting this into Eq. (22), the following relation can be derived,

\[
\frac{u}{u_w} \frac{q}{a \rho c} = k \frac{dp(y)}{dy} \quad \text{...(25)}
\]

Applying Eqs. (2) and (3), \(p(y)\) becomes

\[
p(y) = -\frac{q a}{\lambda} \left( \frac{N+2}{2(N+1)} \right) \left( \frac{y}{a} \right)^2
- \frac{1}{(N+1)(N+3)} \left( \frac{y}{a} \right)^{N+3} + D_0 \quad \text{...(26)}
\]

In order to determine the temperature \(T_{Q}\) in the thermal entrance region, it is convenient to introduce a temperature \(T_{Q}^{+}\), and

\[
T_{Q} = T_{Q}^{+} + T_{Qw}\]

Substituting this into Eq. (8) and applying Eq. (22), the following equation is obtained,

\[
\frac{u}{\lambda} \frac{\partial T_{Q}^{+}}{\partial x} = k \frac{\partial^2 T_{Q}^{+}}{\partial y^2} \quad \text{...(28)}
\]

Boundary conditions:
\[y = 0 \quad \text{and} \quad y = a : \frac{\partial T_{Q}^{+}}{\partial y} = 0\]

Since this equation has the same type of differential equation and the same boundary conditions as Eq. (15), the same eigenvalues and eigenfunctions as in the case of Eq. (15) are available. Thus \(T_{Q}^{+}\) becomes

\[
T_{Q}^{+} = \sum_{n=1}^{\infty} D_n Y_n \exp \left( -2\beta_n \frac{u_n}{u_{max}} \xi \right) \quad \text{...(29)}
\]

Considering the following relation at \(x = 0\),

\[
T_{Q}^{+} = \frac{q a}{\lambda} \sum_{n=1}^{\infty} D_n Y_n = -p(y)
\]

the coefficients \(D_n\) can be calculated by

\[
D_n = \int_0^1 \left[ \int_0^{1} \frac{p(y)}{\lambda} - D_n \right] Y_n d\eta \int_0^1 u Y_n^2 d\eta
\]

\[
D_0 = \int_0^1 \left[ \int_0^{1} \frac{p(y)}{\lambda} - D_0 \right] d\eta \int_0^1 u d\eta
\]

Fig. 3 Temperature distribution of fluid with internal heat generation and thermally insulated wall (solid lines show \(N = 3\))

Fig. 4 Temperature distribution of fluid with no internal heat generation and uniform heat flux at wall (solid lines show \(N = 3\))
The solid lines in Fig.4 show the results calculated in the case of \( N=3 \). Moreover, in this figure, the dotted lines show the case of \( N=1 \) and \( N=10 \).

3-1-3 With internal heat source and wall heat transfer (uniform heat flux) Using \( T_Q \) and \( T_v \) calculated in the preceding sections, the resultant temperature \( T \) is given by

\[
T = (\frac{Q}{\rho c m u_a} - \frac{q}{\rho c m u_a}) x + f(y) + p(y)
\]

\[
+ \frac{Q a^2}{\lambda} \sum_{n=1}^{\infty} C_n Y_n \exp \left(-2 \beta_n \frac{u_m}{u_{\text{max}}} \xi \right)
\]

\[
+ \frac{q a}{\lambda} \sum_{n=1}^{\infty} D_n Y_n \exp \left(-2 \beta_n \frac{u_m}{u_{\text{max}}} \xi \right)
\]

As an example, the resultant temperature \( T \) is shown in Fig. 5. In this figure, the temperature difference between the fluid and the wall at \( \xi = 0.03 \) is illustrated and the parameter \( F \) is the ratio of heat contributing to the temperature rise of fluid to heat due to internal generation. And \( F \) is expressed by

\[
F = 1 - \frac{q}{Q a}
\]

3-2 Uniform wall temperature

If the temperature of fluid at \( x=0 \) is \( T_0 \) and the temperature at wall is kept at zero, one can separate the general problem into two simpler cases as follows:

(3) A situation where there is internal heat generation within the fluid and the inlet temperature \( T_0 \) is zero.

(4) A situation where there is no internal heat generation but the inlet temperature \( T_0 \) has an arbitrary value.

If the solutions of these problems are denoted by \( T_H \) and \( T_L \) respectively, the solution \( T \) of general problem can be obtained as follows:

\[
T = T_H + T_L
\]

The equation and boundary conditions to be used to determine \( T_H \) and \( T_L \) are given respectively as follows:

Case 3

\[
\frac{\partial T_H}{\partial x} = k \frac{\partial^2 T_H}{\partial y^2} + \frac{Q}{\rho c m} \quad \text{Boundary conditions:} \quad y=0: \frac{\partial T_H}{\partial y}=0, \quad y=a: T_H=0, \quad x=0: T_H=0
\]

Case 4

\[
\frac{\partial T_L}{\partial x} = k \frac{\partial^2 T_L}{\partial y^2} \quad \text{Boundary conditions:} \quad y=0: \frac{\partial T_L}{\partial y}=0, \quad y=a: T_L=0, \quad x=0: T_L=0
\]

3-2-1 Zero inlet temperature and with internal heat source Let us consider the fully developed region. The temperature profile is kept constant and the temperature of wall is equal to zero. Therefore, the fluid flows isothermally in the direction of \( x \). Then Eq. (34) can be transformed into

\[
k \frac{\partial^2 T_{H_0}}{\partial y^2} + \frac{Q}{\rho c m} = 0 \quad \text{Boundary conditions:} \quad y=0: \frac{\partial T_{H_0}}{\partial y}=0, \quad y=a: T_{H_0}=0
\]

where \( T_{H_0} \) is the temperature in the fully developed region. This equation yields

\[
T_{H_0} = \frac{1}{2} \frac{Q a^2}{\lambda} \left[ \frac{1 - (x/a)^2}{1 - (x/a)} \right]
\]

Introducing a temperature \( T_{H^+} \) to determine the solution in the thermal entrance region, \( T_H \) can be expressed by

\[
T_H = T_{H^+} + T_{H_0}
\]

Substituting this into Eq. (34) and applying Eq. (36), the following equation can be obtained,

\[
\frac{\partial T_{H^+}}{\partial x} = k \frac{\partial^2 T_{H^+}}{\partial y^2}
\]

Boundary conditions:

\[
y=0: \frac{\partial T_{H^+}}{\partial y}=0, \quad y=a: T_{H^+}=0
\]

Applying the same mathematical treatments as in the preceding sections, we have

\[
\frac{T_{H^+}}{Q a^2/\lambda} = \sum_{n=1}^{\infty} E_n \gamma_n \exp \left(-2 \beta_n \frac{u_m}{u_{\text{max}}} \xi \right)
\]

In this equation, \( \gamma_n \) and \( \gamma_n \) are the eigenvalue and eigenfunction of the following Sturm-Liouville type differential equation, respectively,

\[
\frac{\partial^2 \gamma_n}{\partial \eta^2} + \frac{\beta_n}{u_{\text{max}}} \frac{\gamma_n}{u_m} = 0
\]

Boundary conditions:

\[
\eta=0: \frac{\partial \gamma_n}{\partial \eta}=0, \quad \eta=1: \gamma_n=0
\]

And considering the following relation at \( x=0 \),

\[
T_{H^+} = \frac{Q a^2}{\lambda} \sum_{n=1}^{\infty} E_n \gamma_n = -\frac{1}{2} \frac{Q a^2}{\lambda} \left[ 1 - \frac{(x/a)^2}{1} \right]
\]

the coefficients \( E_n \) can be calculated by
\[ E_\infty = \left. \int_0^1 u \left( \frac{1}{2} (1 - \eta^2) \right) \eta \, d\eta \right|_0^1 \]

The calculations were performed by means of KDC-1. The eigenvalues estimated are given in Table 2. Fig. 6 shows the distribution of temperature within the fluid in the case of \( N=0.5 \). In order to show the change in temperature profile due to \( N \), the temperature distributions in the case of \( N=1 \) and 10 at \( \xi=0.05 \) are also illustrated in the same figure.

3.2.2 Arbitrary inlet temperature \( T_0 \) and without internal heat source. In the fully developed region, the temperature profile of fluid is perfectly developed and it does not change its shape any more. Further, since heat is transferred from fluid to wall and the excess temperature over the value at wall is discussed, the fluid temperature is the same as wall temperature and \( T_{kw}=0 \).

Introducing the fluid temperature \( T_k^* \) in the thermal entrance region, \( T_k \) can be expressed by \( T_k = T_k^* + T_{kw} \).

Substituting this into Eq. (35), the following equation is obtained:

\[ u \frac{\partial T_k^*}{\partial x} = \frac{\partial^2 T_k^*}{\partial y^2} \]

Boundary conditions:
- \( y=0 : \frac{\partial T_k^*}{\partial y} = 0 \)
- \( y=a : T_k^* = T_0 \)
- \( x=0 : T_k^* = T_0 \)

An example of the temperature distributions calculated by these formulae is shown in Fig. 7 in the case of \( N=0.5 \) and 10, when the non-dimensional inlet temperature \( [T_0/(\eta^2/\lambda)] \) is...
unity.

3.2.3 With internal heat source and wall heat transfer (Uniform wall temperature) From Eq. (33), the resultant temperature $T$ is given by

$$T = \frac{1}{2} \frac{Qa}{\lambda} \left\{ 1 - \left( \frac{V}{a} \right)^2 \right\} + \frac{Qa}{\lambda} \sum_{n=1}^\infty E_n \frac{F_n}{r_n} \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) + T_0 \sum_{n=1}^\infty F_n \frac{F_n'}{r_n} \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \quad (48)$$

Fig. 8 shows the temperature distributions of fluid having the rheology constant $N=0.5, 1$ and $10$ in the case of inlet temperature of $T_0/(Qa^2/\lambda)=0$ and $1$. In this figure, the temperature distribution at $\xi=\infty$ has a profile shown by the chain line regardless of values of $N$ and $T_0/(Qa^2/\lambda)$.

4. Nusselt number

Using the following heat transfer coefficient $\alpha$,

$$\alpha = q \left( T_m - T_w \right) = -\lambda \left( \frac{\partial T}{\partial y} \right)_{y=a} \left( T_m - T_w \right) \quad (49)$$

the Nusselt number is defined as follows:

$$N_u = \alpha (2a)/\lambda \quad (50)$$

In the case of uniform wall heat flux, the difference between mixed mean temperature $T_m$ and wall temperature $T_w$ is given by

$$T_m - T_w = -f(a) - p(a) - \frac{Qa}{\lambda} \sum_{n=1}^\infty C_n Y_n(1) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) - \frac{Qa}{\lambda} \sum_{n=1}^\infty D_n Y_n(1) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \quad (51)$$

Hence, the Nusselt number yields

$$N_u = -\frac{2qa/\lambda}{f(a) + p(a) + \frac{Qa}{\lambda} \sum_{n=1}^\infty C_n Y_n(1) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) - \frac{Qa}{\lambda} \sum_{n=1}^\infty D_n Y_n(1) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right)} \quad (52)$$

In the case of no internal heat source, the Nusselt number is given by the following formula,

$$N_u = -\frac{2qa/\lambda}{p(a) + \frac{Qa}{\lambda} \sum_{n=1}^\infty D_n Y_n(1) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right)} \quad (53)$$

In the case of uniform wall temperature, $(T_m - T_w)$ is given by

$$T_m - T_w = \frac{\mu_{\text{max}}}{\mu_n} \left[ \frac{Qa}{\lambda} \sum_{n=1}^\infty E_n \frac{F_n}{r_n} \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) - T_0 \sum_{n=1}^\infty F_n \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \right] \quad (54)$$

The heat transferred from fluid to wall $q$ is expressed as follows:

$$q = -\lambda \left( \frac{\partial T}{\partial y} \right)_{y=a} = q \sum_{n=1}^\infty E_n \frac{F_n}{r_n} \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) - \frac{\lambda T_0}{a} \sum_{n=1}^\infty F_n \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \quad (55)$$

Hence, the Nusselt number yields

$$N_u = 2 \frac{\mu_{\text{max}}}{\mu_n} \sum_{n=1}^\infty \left[ E_n \sum_{n=1}^\infty \left( 1 - \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \right) \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \right] \quad (56)$$

In the case of no internal heat source, the Nusselt number becomes

$$N_u = 2 \frac{\mu_{\text{max}}}{\mu_n} \sum_{n=1}^\infty \left[ E_n \exp \left( -2j_n^{*} \frac{\mu_n}{\mu_{\text{max}}} \xi \right) \right] \quad (57)$$

Fig. 9 shows the relation between Nusselt number and $\xi$ in the case of no internal heat source. In this figure, the solid line shows the case of uniform heat flux at wall and the dotted line that of uniform wall temperature. The Nusselt number for $N=1$ in the case of uniform wall temperature coincides well with the value by Sellars et al. Fig. 10 shows the relation between Nusselt number and $\xi$ in the presence of internal heat source and uniform heat flux at wall. Fig. 11 illustrates the case of uniform wall temperature. These heat transfer characteristics have the similar tendencies to those discussed in the previous papers. Namely, the Nusselt number has a minimum value at a particular value of $\xi$ under particular
5. Conclusion

An analysis was made to determine the heat transfer characteristics in the thermal entrance region of established laminar flow of a pseudo-plastic fluid or dilatant fluid with uniform internal heat generation between two parallel plates under thermal conditions at wall, i.e. uniform wall heat flux and uniform wall temperature. The results obtained are as follows:

(1) The temperature distribution of pseudo-plastic fluid and dilatant fluid with internal heat source under uniform heat flux at wall is generally given by Eq. (31).

(2) The temperature distribution of pseudo-plastic fluid and dilatant fluid with internal heat source under uniform temperature at wall is generally given by Eq. (48).

(3) Under uniform heat flux at wall, the length of thermal entrance region is scarcely affected by internal heat generation. But, under uniform wall temperature, the length of thermal entrance region in the presence of internal heat source is larger than that in the absence of internal heat source.

(4) The Nusselt number of heat generating fluid has a minimum value at a particular value of ξ under particular conditions of N and F (for example N=0.5 and F=0.75, and N=1 and F=0.75) in the case of uniform wall heat flux.

(5) When the wall temperature is uniform, the Nusselt number has a minimum value under particular condition of $\frac{T_w}{Qa^2}$.

(6) The Nusselt number of heat generating fluid does not decrease monotonously with increasing ξ but changes discontinuously at a certain value of ξ.
Table 3 Comparison between the authors' values and Schecter's values for $\beta_n$ and $Y_n(1)$ of Newtonian fluid in a circular tube

<table>
<thead>
<tr>
<th>$n$</th>
<th>Authors' values $\beta_n$, $Y_n(1)$</th>
<th>Schecter's values $\beta_n$, $Y_n(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.690 8</td>
<td>25.679 6</td>
</tr>
<tr>
<td>2</td>
<td>83.960 0</td>
<td>83.861 6</td>
</tr>
<tr>
<td>3</td>
<td>174.572 1</td>
<td>174.160 8</td>
</tr>
<tr>
<td>4</td>
<td>297.570 1</td>
<td>296.528 4</td>
</tr>
<tr>
<td>5</td>
<td>455.872 7</td>
<td>450.967 7</td>
</tr>
</tbody>
</table>

Table 4 Comparison between 100 mesh and 200 mesh-dividing (uniform heat flux) $\beta_n$, $Y_n(1)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\beta_n$ 100 mesh</th>
<th>$\beta_n$ 200 mesh</th>
<th>$Y_n(1)$ 100 mesh</th>
<th>$Y_n(1)$ 200 mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25.690 8</td>
<td>25.662 4</td>
<td>0.395 8</td>
<td>0.395 8</td>
</tr>
<tr>
<td>2</td>
<td>83.960 0</td>
<td>83.886 2</td>
<td>0.395 8</td>
<td>0.395 8</td>
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<tr>
<td>3</td>
<td>174.572 1</td>
<td>174.270 0</td>
<td>0.346 5</td>
<td>0.346 5</td>
</tr>
<tr>
<td>4</td>
<td>297.570 1</td>
<td>296.656 1</td>
<td>0.314 5</td>
<td>0.314 5</td>
</tr>
<tr>
<td>5</td>
<td>455.872 7</td>
<td>450.967 7</td>
<td>0.201 3</td>
<td>0.201 3</td>
</tr>
</tbody>
</table>

Nusselt number

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\xi = 0.01$</th>
<th>$\xi = 0.15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isothermal flow (Q=0, F=0) 100 mesh</td>
<td>9.367 661</td>
<td>9.046 030</td>
</tr>
<tr>
<td>200 mesh</td>
<td>9.368 812</td>
<td>9.046 066</td>
</tr>
<tr>
<td>No internal heat source (Q=0) 100 mesh</td>
<td>7.486 290</td>
<td>4.404 066</td>
</tr>
<tr>
<td>200 mesh</td>
<td>7.487 161</td>
<td>4.404 114</td>
</tr>
</tbody>
</table>

Table 5 Comparison between 100 mesh and 200 mesh-dividing (uniform wall temperature) $\gamma_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\gamma_n$ 100 mesh</th>
<th>$\gamma_n$ 200 mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.314 6</td>
<td>7.313 8</td>
</tr>
<tr>
<td>2</td>
<td>44.636 4</td>
<td>44.616 1</td>
</tr>
<tr>
<td>3</td>
<td>114.088 4</td>
<td>113.963 2</td>
</tr>
<tr>
<td>4</td>
<td>215.841 7</td>
<td>215.390 3</td>
</tr>
<tr>
<td>5</td>
<td>349.902 2</td>
<td>348.957 1</td>
</tr>
</tbody>
</table>

Nusselt number

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\xi = 0.01$</th>
<th>$\xi = 0.06$</th>
</tr>
</thead>
<tbody>
<tr>
<td>With internal heat source (Q=0, $T_h=0$) 100 mesh</td>
<td>24.461 3</td>
<td>5.902 3</td>
</tr>
<tr>
<td>200 mesh</td>
<td>23.733 0</td>
<td>6.028 3</td>
</tr>
<tr>
<td>Without internal heat source (Q=0) 100 mesh</td>
<td>5.991 07</td>
<td>3.657 3</td>
</tr>
<tr>
<td>200 mesh</td>
<td>5.990 76</td>
<td>3.656 9</td>
</tr>
</tbody>
</table>

Appendix

As shown in Fig. 12 in the 2nd report (6), the calculations of eigenvalue and eigenfunction were performed by dividing the normalized radius into 100 mesh. And the results obtained were tabulated, for example, in Tables 1 and 2 in the 2nd report, and they agree very well with exact values which can be easily obtained from the table of Bessel function in the case of uniform velocity profile (plug flow). But, in other cases, for example, parabolic velocity profile, the results obtained are slightly different from those of Schecter et al. (7) as shown in Table 3 in this report. In order to clarify the effects of difference of eigenvalue on the Nusselt number, the calculations for Newtonian fluid were performed by dividing the normalized radius into 200 mesh in the same computer program as in the 2nd report. The results are tabulated in Tables 4 and 5. Table 4 shows the results in the case of uniform heat flux and Table 5 shows those of uniform wall temperature. From these tables, it can be found that the Nusselt number obtained by 100 mesh method agree fairly well with that by 200 mesh method. Consequently, it may be concluded that the results obtained by 100 mesh-dividing are sufficiently accurate.

References