Modal Parameter Estimation with Multi-reference Curve Fitting

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Many kinds of curve fit techniques have been developed for the estimation of the modal parameters of a mechanical structure, using the theory of modal analysis. As one of the most popular curve fit techniques which correspond to the multiple input vibration test, Vold's method in the time domain (Poly-reference) is well-known and used extensively. In this paper, a new method which results from improving Vold's method in the time domain is proposed to obtain global modal parameters. In Vold's method, residues are determined by using every column of the transfer function matrix. Then, any two columns of residue matrix contradict each other. In the proposed method, the residues are obtained at the same time by using all of the transfer function matrix. And such contradiction can be solved. With this method, modal parameters of the fly wheel of an automobile are identified. Comparing this new method with former methods, the validity of the new method is confirmed.

Key Words: Vibration, Modal Analysis, Curve Fitting, Vibration Test, Transfer Function

1. Introduction

To identify the dynamic characteristics of a mechanical structure by using experimental data, curve-fit techniques have been developed. In these methods, dynamic characteristics are represented by modal parameters such as natural frequencies, modal damping ratios, natural modes, and so on.

In the last report, six methods of multi-reference curve-fits which identify multiple eigenvalues of multiple transfer functions at the same time are examined, and the accuracy and the calculating time are compared with one another.

As one of the multi-reference curve-fit techniques which correspond to the multiple input vibration test, Vold's method in the time domain (Poly-reference) is well-known and used extensively. In this paper, a new method resulting from improving Vold's method in the time domain is proposed to obtain global modal parameters. Modal parameters of the fly wheel of an automobile are identified, and the accuracy of this method and that of former methods are compared with each other.

2. Analysis method

2.1 Vold's method

The analysis method of Vold's method in the time domain (Poly-reference) is explained briefly here. Assuming that the number of exciting points is \( f \) and the number of measurement points is \( l \), \( f \times l \) transfer functions (TFs) are obtained. Then, unit impulse response data in the time domain are also obtained through the Inverse Fast Fourier Transform.

When the point \( b \) is excited, displacement responses on \( l \) measured points at time \( t=kT \) are represented by a vector \( \{ H_s(kT) \} \):

\[
[H_s(kT)]=\{ \phi \}[\beta]^n(\xi_n)
\]  

(1)

where

\[
[\beta]=\begin{bmatrix}
e^{s_1T} \\
e^{s_2T} \\
\vdots \\
e^{s_nT}
\end{bmatrix}
\]

\[
[\phi]=\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_l
\end{bmatrix}
\]

(2)

and \( n, s, \) and \( T \) represent the degree of freedom, the
eigenvalue of order $r$ and the time interval, respectively. Furthermore, $\{\phi_i\}$ and $\{\xi_i\}$ are respectively, the eigenvector and the modal participation factor.

Changing the parameter $k$ in Eq. (1) from 0 to $p$, we get

$$
\begin{align*}
\begin{bmatrix}
(H_s(0 \cdot T)) \\
(H_s(1 \cdot T)) \\
\vdots \\
(H_s(p \cdot T))
\end{bmatrix} \\
\begin{bmatrix}
[\phi] \\
[\phi][\beta] \\
\vdots \\
[\phi][\beta]^p
\end{bmatrix}
\begin{bmatrix}
(\xi_0) \\
(\xi_1) \\
\vdots \\
(\xi_n)
\end{bmatrix}
= [\phi] \begin{bmatrix}
(\xi_0) \\
(\xi_1) \\
\vdots \\
(\xi_n)
\end{bmatrix}
\end{align*}
$$

(3)

If $p$ satisfies $2n = lp$, matrix $[\phi]$ has $(p+1)l = 2n + l$ rows and $2n$ columns. Because the row is larger than the line by $l$, the following equation can be given.

$$
\sum_{k=0}^{2n}[B_s][\phi][\beta]^{k-i} = [0]
$$

(4)

Assuming that $[B_s]$ is a unit matrix, Eq. (5) is obtained for any $k$.

$$
\sum_{k=0}^{2n}[B_s][H_s(\tau + k \cdot T)] = -[H_s((p+k)T)]
$$

(5)

Varying the parameter $k$ in Eq. (5) from 0 to $m$, we have

$$
[B][T_s] = -[D_s]
$$

(6)

For other excited points, matrix $[B]$ is invariable. Hence, the following equation can be given.

$$
[B][T_s] = -[D_s]
$$

(7)

where $[B]$ can be obtained by the least squares method.

2.2 Introduction of proposed method

In a typical vibration test, the number of measured points is greater than that of excited points. Therefore, in Vold's method, excited points and measured points are exchanged using the theorem of Maxwell's reciprocity. However, in this method, the natural mode matrix $[\phi]$ in Eq.(10) has $f$ rows, representing only the elements of excited points.

Accordingly, the first assumption in the proposed method is that excited points and measured points are not exchanged. Therefore, the natural mode matrix $[\phi]$ in Eq.(10) has the elements of all the measured points.

Another assumption is that the driving point's TFs are also measured. Hence, the natural mode matrix $[\phi]$ includes the elements of all the measured and excited points.

2.3 Proposed method

If point $i$ is an excited point and point $j$ is a measured point, the TF (compliance) of $n$ degrees of freedom under the assumption of general viscous damping is represented as follows.

$$
G_{ij}(\omega) = \sum_{\tau=0}^{2n} \left( \frac{\phi_i \phi_j}{d_r(j\omega - s_r)} + \frac{\phi_i \phi_j}{d_r(j\omega - s_r)^2} \right)
$$

(12)

where $\phi_i$ is the element of the $i$-th point of the $r$-th natural mode, and $s_r$ is the $r$-th eigenvalue. Furthermore, $d_r$ is defined as follows. The equation of motion of a general viscous damping system becomes

$$
\begin{bmatrix}
C & M \end{bmatrix}\begin{bmatrix}
\dot{x} \\
x
\end{bmatrix} + \begin{bmatrix}
0 & K \\
-M & 0
\end{bmatrix}\begin{bmatrix}
\ddot{x} \\
\dot{x}
\end{bmatrix} = \begin{bmatrix}
f \\
0
\end{bmatrix}
$$

(13)

If the right side of Eq.(13) is set to 0, it can be expressed as

$$
[D](y) = [E](y) = 0
$$

(14)

Then, assuming $[\Phi_i]$ to be the $r$-th eigenvalue of Eq. (14), $d_r$ is defined as follows.

$$
[\Phi_i]'[D][\Phi_i] = d_r
$$

(15)

The eigenvalue $s_r$ and natural mode $[\phi_i]$ can be obtained by solving Eq.(11). Due to the assumptions (Section 2.2), natural mode $[\phi_i]$ includes all the elements of the measured points. As a result, the basis of the proposed method is that TFs can be specified by determining the modal parameter $d_r$. This parameter can be determined using either time domain or frequency domain data.

2.4 Time domain method

The unit impulse response function of a general viscous damping system at time $t = kT$ is represented as

$$
\begin{bmatrix}
h_{ij}(kT) \\
h_{ij}(kT) \\
\vdots \\
h_{ij}(kT)
\end{bmatrix}
= \sum_{\tau=0}^{2n} \left( \frac{\phi_i \phi_j}{d_r} e^{n \tau} \right)
$$

(16)

When point $i$ is excited, the responses of $l$ measured points are
\begin{equation}
\{H_b(kT)\} = [\phi] \begin{bmatrix} \varphi_b \beta \end{bmatrix} \{1/d\}
\end{equation}

where
\begin{equation}
[\varphi_b] = \begin{bmatrix} \varphi_{1b} \\
\vdots \\
\varphi_{nb} \end{bmatrix}
\end{equation}

\begin{equation}
\{1/d\} = \begin{bmatrix} 1/d_1 \\
\vdots \\
1/d_n \end{bmatrix}
\end{equation}

and \([\phi]\) and \([\beta]\) are given by Eq.(2). The relation between Eq.(1) and Eq.(17) is expressed as
\begin{equation}
\{z_b\} = [\phi] \begin{bmatrix} \varphi_{1b} \\
\vdots \\
\varphi_{nb} \end{bmatrix}
\end{equation}

If the number of time domain data of each measured point is \(q\), Eq.(17) becomes
\begin{equation}
\begin{bmatrix}
(H_b(0 \cdot T)) \\
(H_b(1 \cdot T)) \\
\vdots \\
(H_b((q-1) \cdot T))
\end{bmatrix} = [\phi] [\varphi_b] [\beta] \{1/d\}
\end{equation}

Equation (20) is rewritten as
\begin{equation}
\{H_b\} = [A_b] [1/d]
\end{equation}

Vector \([1/d]\) is common to all the excited points. Hence, Eq.(21) can be expressed for \(b = 1 \sim f\) as
\begin{equation}
\begin{bmatrix}
H_1 \\
H_2 \\
\vdots \\
H_f
\end{bmatrix} = [A] [1/d]
\end{equation}

Equation (22) can be solved by the least squares method.

### 2.5 Frequency domain method

The TF of a general viscous damping system, as represented in Eq.(12), is rewritten as follows.
\begin{equation}
G_b(\omega) = \sum_{r=1}^{2n} \phi_{br} \psi_{rr}
\end{equation}

When point \(b\) is excited, the responses of \(l\) measured points are expressed as
\begin{equation}
\{G_b(\omega)\} = \sum_{r=1}^{2n} [\phi] \psi_{br} \{1\}
\end{equation}

\begin{equation}
[\Gamma_b] = \begin{bmatrix} 1/\omega_b - s \end{bmatrix}
\end{equation}

where \([\phi]\) and \([\psi_{br}]\) are given by Eq.(2) and Eq.(18), respectively. If the number of frequency domain data is \(q\), Eq.(18) becomes
\begin{equation}
\begin{bmatrix}
\{G_b(\omega)\} \\
\{G_b(\omega)\} \\
\vdots \\
\{G_b(\omega)\}
\end{bmatrix} = [\phi] [\varphi_b] [\Gamma_b] \{1/d\}
\end{equation}

\begin{equation}
\{G_b(\omega)\} = [\phi] [\varphi_b] [\Gamma_b] \{1/d\}
\end{equation}

Equation (23) is rewritten as
\begin{equation}
\{G_b\} = [H_b] [1/d]
\end{equation}

As in the time domain method, \([1/d]\) is common to all the excited points. Hence, Eq.(25) can be represented for \(b = 1 \sim f\) as
\begin{equation}
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots \\
G_f
\end{bmatrix} = [H] \begin{bmatrix} I_1 \\
I_2 \\
\vdots \\
I_f
\end{bmatrix}
\end{equation}

and \([1/d]\) can be determined using the least squares method. As in the usual frequency domain method, a weighting function can be applied.

### 2.6 Estimation of residual terms

In the two methods explained above, residual terms are not taken into account. Including residual terms, Eq.(12) is rewritten as follows.
\begin{equation}
G_{\mu}(\omega) = \sum_{r=1}^{2n} \phi_{br}^s \psi_{rr} + \frac{\varphi_{bs} \varphi_{s}^t}{\omega_b^2 + \omega_s^2}
\end{equation}

\begin{equation}
Y_{Y \mu} = Y_{Y \mu} + Y_{a \mu}
\end{equation}

where \(Y_{\mu}\) and \(Z_{\mu}\) are, respectively, the residual mass and residual flexibility when point \(i\) is an excited and point \(j\) is a measured point. Based on Maxwell's reciprocity, the following equation can be given.
\begin{equation}
Y_{Y \mu} = Y_{Y \mu} + Y_{a \mu}
\end{equation}

Including residual terms, Eq.(24) is rewritten as
\begin{equation}
\{G_{\mu}(\omega)\} = [\phi] [\varphi_b] [\Gamma_b], \{X\}, \{I\} \{D\}
\end{equation}

where \([X]\) represents a unit matrix of order \(l\) and
\begin{equation}
\begin{bmatrix}
1/\omega_b \\
\vdots \\
1/\omega_b
\end{bmatrix}
\end{equation}

Equation (30) can be rewritten for all the excited points \((b = 1 \sim f)\).
\begin{equation}
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots \\
G_f
\end{bmatrix} = [I_b] \begin{bmatrix} X & I \\
\vdots & \vdots \\
I_b & X \\
\end{bmatrix} \{D\}
\end{equation}

where
\begin{equation}
\begin{bmatrix}
X_1 \\
\vdots \\
X_q
\end{bmatrix}, \begin{bmatrix}
I_1 \\
\vdots \\
I_l
\end{bmatrix}
\end{equation}

\begin{equation}
\{D\} = [1/d_1, \cdots, 1/d_n, Y_{1b}, \cdots, Y_{fb}, Z_{1b}, \cdots, Z_{fb}]^{(2n+2l)}
\end{equation}

and \([D]\) is an unknown vector which becomes very large as the number of excited or measured points become large. In such a case, by utilizing Eq.(29) and considering \(Y_{\mu}\) and \(Z_{\mu}\) as real numbers, \([D]\) can be reduced. Then, Eq.(32) can be solved by the least squares method.
2.7 Comparison with former methods

Here, the proposed method and former methods are compared and discussed.

The TF, as expressed in Eq.(12), can also be expressed as a matrix.

\[ [G(\omega)] = \begin{bmatrix} G_{ll} & G_{l2} & \cdots & G_{l1} \\ G_{2l} & G_{22} & \cdots & G_{21} \\ \vdots & \vdots & \ddots & \vdots \\ G_{11} & G_{12} & \cdots & G_{11} \end{bmatrix} \]

\[ [U_r] = \frac{1}{d_r} [\phi_1][\phi_i]^T \begin{bmatrix} U_{l1} \\ U_{l2} \\ \vdots \\ U_{ll} \end{bmatrix} \]

(34)

where \( f \) is the number of measured points, and \([G_{\omega}]\) and \([U_r]\) represent a TF matrix and a residue matrix, respectively. If \( f \) excited points are chosen from \( f \) measured points, experimental data will correspond to \( f \) columns of \([G_{\omega}]\).

Then the proposed method (A) can be expressed as follows:

(A) First, eigenvalues and natural modes are determined by referring to \( f \) columns of \([G(\omega)]\).

Next, \( \{1/d\} \) or \( \{D\} \) is also determined by referring to \( f \) columns of \([G(\omega)]\). Then \( f \) columns of the residue matrix and residual terms can be determined at the same time.

In the past, two methods were used to determine residues. These methods, (B) and (C) are explained as follows:

(B) Modal participation factors \( \{\xi_2\} \) (Eq.(3)) are determined by using eigenvalues and natural modes.

(C) The least squares method is iterated by each transfer function using only eigenvalues.

Now, it is assumed that when eigenvalues are estimated, Maxwell’s reciprocity is not applied.

In method (B), an unknown vector \( \{\xi_3\} \) is dependent on the excited point, so the least squares method is iterated \( f \) times by each column of the TF matrix \([G(\omega)]\). Columns of the residue matrix will contradict one another due to variance of the data. This also occurs in method (C).

Method (A), however, doesn’t contain such an inconsistency.

3. Experimental verification

3.1 Multi-point excitation

The TFs of the fly wheel of an automobile (Fig.1) are measured by an impulse hammer excitation.

The excited points are numbered 1~8 in Fig. 1. The measured points also are numbered 1~8. Hence, a TF matrix of \( 8 \times 8 \) is now measured.

3.2 Pole estimation

With reference to all the TFs, the eigenvalues and natural modes are determined by solving Eq.(11). Five modes are estimated, as shown in Table 1.

3.3 Residue estimation

As explained in Section 2.7, residues can be determined by the three methods (A)~(C). Method (A) is the proposed method and (B) and (C) are methods previously established.

Table 1 Estimated undamped natural frequencies and modal damping ratios

<table>
<thead>
<tr>
<th>Order</th>
<th>Freq. (Hz)</th>
<th>Modal damping(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1511</td>
<td>0.31</td>
</tr>
<tr>
<td>2</td>
<td>1512</td>
<td>0.30</td>
</tr>
<tr>
<td>3</td>
<td>3006</td>
<td>0.76</td>
</tr>
<tr>
<td>4</td>
<td>3758</td>
<td>0.18</td>
</tr>
<tr>
<td>5</td>
<td>3781</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Fig. 1 Fly wheel of an automobile

Fig. 2 Distribution of modal parameter 1/d_

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3.4 Comparison of the results

The results of the three methods (A)~(C) are compared from the point of view of which result represents a global modal parameter.

In method (A), if modal parameter (1/d) is determined with reference to only a column of [G(u)], this method has the same result as method (B). Hence, it is interesting to see how variant modal parameter (1/d) is related to a referring column of [G(u)].

Modal parameter (1/d) is determined with reference to each column where natural modes are normalized, by summation of the square of the real and imaginary parts to one. Figure 2 shows the distribution of 1/d, for each excited point. Point A represents the result of method (A) when all the columns are used at the same time. Hence, the variance of 1/d, implies the inconsistency of method (B).

Consistency of the results is also examined in the next comparison. In methods (A)~(C), the residues are first determined without referring to the 5th and 8th columns. Next, the TFs between points 5 and 8 are calculated and compared with the experimental data which was not referred. Figures (3)~(5) show the results of the three methods and a measured TF between points 5 and 8. The result of method (A) is obtained by referring to six columns other the 5th and 8th columns, and those of method (B),(C) are obtained by referring to the 2nd column only. These three figures indicate that method (C) results in a serious error, and method (A) is the most accurate.

Finally, estimated natural modes are examined. As shown in Table 1, the 1st and the 2nd modes have almost equal natural frequencies. The mode shapes determined by method (A) are shown in Fig. 6. Judging from their shapes, these two modes are repeated roots. The same mode shapes are determined by method (B).

Figure 7 also shows the 1st and 2nd modes deter-
4. Conclusions

A new method which results from improving Vold's method in the time domain has been proposed to obtain global modal parameters. In this method, modal parameter \(1/d\) was determined by referring to all the measured columns of a TF matrix, and all the residues were obtained at the same time. Modal parameters of the fly wheel of an automobile were estimated, and the accuracy of the new method with that of the former methods was compared. Then, the validity of the new method was confirmed.

References