Analysis of Rolling of Angles by Energy Method Using Finite Element Division*

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The shape rolling of an angle was analyzed by the energy method, in which the kinematically admissible velocity field was expressed as the sum of the fundamental velocity fields, and finite element division was applied, as reported in the previous paper. Here, most fundamental velocity fields are determined by a method similar to that proposed already, though some special velocity fields are defined concerning the position of a material before rolling and the curling of a material after rolling. Rolling properties are obtained and some discussions are made from a mechanical standpoint. Curling depends on the difference of the circumferential velocities of the upper and lower rolls, and distortion of the cross section of a material is caused by the reduction of the side parts of a material near the upper and lower rolls. Consequently, it is proved that the authors' energy method can be applied to shape rolling successfully.

**Key Words**: Plasticity, Forming, Energy Method, FEM, Rolling, Three-dimensional Deformation

1. Introduction

In the field of shape rolling of an angle or a channel, much experimental research has been performed using plasticine and lead\(^{20}\). However, theoretical approaches are necessary for understanding the influence of rolling conditions on rolling properties clearly.

Recently, we have proposed a new energy method using finite element division\(^{21}\). In this paper, this method is applied to shape rolling. In shape rolling, the shape of the cross section is not always symmetrical toward the direction of thickness, so the position of the material at the entrance toward the direction of thickness cannot be determined beforehand. Further, the material at the exit may curl toward the direction of thickness if it is not guided.

In this paper, the boundary conditions are as follows: At the entrance, the material moves along a pass line without lateral force. At the exit, the material moves without lateral force or moment.

An analysis of the shape rolling of an angle for large width/height ratio is done by the usual energy method\(^{20}\), because the flow of material is simple; hence we analyze the flow of material for small width/height ratio.

2. Method of Analysis

2.1 Fundamental equations

The shape rolling of an angle is considered. Coordinate axes are taken with the origin at the midpoint of the exit cross section, as shown in Fig.1 (a). The axes x, y and z are the directions of the length, width and thickness, respectively. Half of the material is taken because of its symmetricity.

The variational principle about rigid plastic solids is expressed as follows\(^{30}\):
\[ T_0 \omega_u + T_1 \omega_l \leq \Phi \]
\[ \Phi = \int \sqrt{3} k \varepsilon_{u,\text{eq}} dV + \int \sum_{s} k \Delta \varepsilon_{\text{eq}} dT^s \]
\[ + \int \tau r \Delta \varepsilon_{\text{eq}} dS + Q_v u_0 \]
\[ (1) \]

where

- \( T_0, T_1 \): true torque of upper and lower rolls
- \( \omega_u, \omega_l \): angular velocity of upper and lower rolls
- \( Q_v, Q_0 \): back and front tension
- \( u_0, v_0 \): velocities of material at entry and exit cross sections
- \( \tau_r \): frictional stress on roll surface
- \( V \): volume of material in plastic solid
- \( S_t \): contact surface with rolls
- \( T^s \): interface on which discontinuity of tangential velocity arises
- \( k \): suffix indicating kinematically admissible velocity fields
- \( k \): shearing yield stress
- \( \Delta \varepsilon_{\text{eq}} \): amount of discontinuity of velocity
- \( \varepsilon_{\text{eq}} \): equivalent plastic strain rate

The material is assumed to be a rigid plastic Mises' solid.

### 2.2 Method of analysis

The method of analysis was explained in the previous paper\(^{19}\), so only a brief description will be given in this paper. The plastic deformation region is divided into finite elements in a way similar to that of FEM. And the actual velocity field is expressed as the summation of several admissible fundamental velocity fields, which are given beforehand.

\[ \mathbf{v} = \sum_{i=1}^{N} a_i \mathbf{v}^{(i)} \quad (a_i \text{: unknown coefficient}) \quad (2) \]

Here, \( \mathbf{v}^{(i)} \) is a velocity vector, which is composed of \( x \), \( y \), \( z \) components of velocities at nodal points, so the number of the total components is \( 3N(N \text{ : number of nodes}) \). \( x \) is the number of fundamental velocity fields. Then \( \Phi \) in Eq. (1) is expressed as \( \Phi(\mathbf{v}) \).

The procedure of analysis is as follows. First, the shape of the material is assumed, and coefficients \( a_i \) in Eq. (2) are determined by minimizing the functional \( \Phi \) in Eq. (1). Next, the shape of the material is updated by the obtained velocity field. This procedure is iterated until the obtained velocity field becomes compatible with the assumed shape of the surface.

### 2.3 Finite element division

In the rolling, the deformation between the roll contact surface and the deformation near the free surface differ widely from each other, so the interface between elements should be taken to coincide with the boundary of the contact surface.

Thus, the interface between elements (\( \Delta A \) and \( \Delta B \) in Figs. 1(a) and (d)) are set at the boundary of the upper and lower roll contact surfaces, respectively. In the roll gap, we first divide the deformation region as shown by the dotted line in Fig. 1(b) and then the dotted lines are moved to the position of the solid lines. It is expected that the singular deformation in the neighborhood of the boundary of the contact surface is approximated well by such element divisions.

If \( \theta_{cl} = \theta_{co} = 0^\circ \) in Fig. 4, this mesh system coincides with that of the flat rolling of a bar reported in the previous paper\(^{19}\).

### 2.4 Procedure of introducing admissible velocity fields

We must provide many kinematically admissible velocity fields as fundamental velocity fields, which is not an easy task. So two components of velocity (\( x \) and \( z \) components) are assumed and the third component (\( y \) component) is calculated numerically by the equation of volume constancy.

That is, the velocity component toward the direction of width at nodes \( 1-5 \) in Fig. 2 is expected by a polynomial of \( z \) (\( j \) : the number of elements toward the direction of thickness).

\[ v_y = A_0 + A_1 2 + \cdots + A_{j-1} \]
\[ (3) \]

And \( A_i \) is determined by the condition of volume constancy at each element with marks \( \bigcirc \) in Fig. 2.

\( v_z \) on the roll contact surface is determined by using the boundary condition and the equation of volume constancy.

### 2.5 Elimination of singularity in the functional

If we minimize the functional (Eq. (1)) by the Newton-Raphson method, we cannot obtain useful
results because of the small strain rate at the entrance. This was pointed out by Morii. We have already solved the problem of the singularity of the functional due to the discontinuity of frictional stress at the neutral point and the velocity discontinuity at the entrance. For these reasons, this method was used.

If we look at one integration point (P) of one element and let \( \dot{\varepsilon}_{eq} \) be the equivalent strain rate, \( \Phi_{r} \) which is the functional concerning P is expressed as:

\[
\Phi_{r} = A \dot{\varepsilon}_{eq} (A: \text{constant}) \quad (4)
\]

We introduce a non-dimensional variable,

\[
\dot{\varepsilon}_{eq} = \dot{\varepsilon}_{eq} \cdot (h_{r}/U) \quad (U, h_{r}: \text{see Section 3.1}) \quad (5)
\]

and we will modify \( \Phi_{r} \) in the neighborhood of \( P \) as follows:

\[
\Phi_{r} = \frac{A}{h_{r}/U} \dot{\varepsilon}_{eq} \quad (\dot{\varepsilon}_{eq} \geq \delta) \quad (6)
\]

\[
\Phi_{r} = \frac{A}{h_{r}/U} \left( \frac{(\dot{\varepsilon}_{eq})^{2}}{2.5} + \frac{\delta}{2} \right) \quad (\dot{\varepsilon}_{eq} < \delta)
\]

The relationship between \( \Phi_{r} \) and \( \dot{\varepsilon}_{eq} \) is shown in Fig. 3(a). The solid line is Eq. (6) and the dotted line is Eq. (4). The results of the numerical calculation for Case 1 in Table 1 are shown in Fig. 3(b), where \( \delta \) is changed. From Fig. 3, we found that the rolling properties change little and that convergence is rather good, if we set \( \delta = 0.01 \). Hence, we choose this value for the following calculations.

3. Fundamental Velocity Fields in the Shape Rolling of Angles

3.1 Definitions of rolling conditions and rolling properties

Non-dimensional variables are defined in the rolling of angles as follows (see Fig. 4). The suffix B denotes entrance, and the suffix F denotes exit. In Fig. 4, the dotted lines denote material shape before rolling, solid lines denote material shape after rolling, and fine lines denote caliber shape. We define the material height and roll radius at \( y = w_{2}/2 \) in Fig. 4. These variables change along the \( y \) axis. Some other variables are defined:

(i) Variables concerning rolling conditions

\[
\text{Reduction} : r = (2h_{r} - 2h_{f})/2h_{r}
\]

Upper roll radius : \( R_{u}/h_{r} \)

Lower roll radius : \( R_{l}/h_{r} \)

Friction ratio : \( m = \eta/h \)

(ii) Variables concerning rolling properties

Width spread : \( w = (b_{r} - b_{h})/b_{h} \)

\( (2b_{r} : \text{mean width at the exit}) \)

Mean gradient of free side surface : \( \theta_{f} \)

Forward slip : \( \xi = (v_{f} - U)/U \)

\( (v_{f} : \text{material velocity at the exit}) \)

\( U : \text{mean rolling velocity} \)

\( (R_{u}w_{2} + R_{l}w_{1})/2 \)

The warping of a cross section after rolling, which is perpendicular to the \( x \) axis before rolling, is expressed as follows:

\[
\text{Warping of section along thickness} : u_{2}/(2h_{r})
\]

\[
\text{Warping of section along width (upper roll)} : u_{5}/b_{r}
\]

\[
\text{Warping of section along width (lower roll)} : u_{5}/b_{f}
\]

Torque : \( T = \Phi(U \cdot 2S_{l}/3h) \)

\( (2S_{d} : \text{cross section at the entrance}) \)

Material position : \( z_{m}/(2h_{r}) \)

\( (z_{m} : \text{mean} z \text{ value of material in } xy \text{ plane at the entrance}) \)

Curvature : \( h_{r}/\rho \)

(when material curls toward the upper roll at the exit \( \rho > 0 \))

3.2 Definition of fundamental velocity fields

Considering the results in the previous paper, the finite element division as shown in Fig. 1 is used.

Table 1

<table>
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<td>b_{1}(deg)</td>
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Fig. 3

Fig. 4

Table 1

<table>
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</table>
Fundamental velocity fields will be described in the following. The number of fundamental velocity fields is 56.

3.2.1 Fundamental velocity fields for whole region

The following three velocity fields are introduced with reference to the velocity field in the flat rolling of a bar:

\[ v^{(1)} = \frac{2h_0}{(h_0 - h_r)(y = 0)} U \]  
\[ v^{(2)} = U \]  
\[ v^{(3)} = \left( \frac{x}{l} \right) \left( \frac{2h_0}{(h_0 - h_r)(y = 0)} - 1 \right) U \]

\( h_0 = h_0(x, y) \) : z value on upper roll 
\( h_r = h_r(x, y) \) : z value on lower roll 
\( l' \) : maximum projected contact length

\( v^{(1)} \) and \( v^{(2)} \) are determined under the following conditions: For any cross section perpendicular to the x-axis, \( v^{(1)} \) is independent of \( z \), \( v^{(2)} \) changes linearly in the direction of thickness.

3.2.2 Fundamental velocity fields for region before boundary of contact surface

In Fig. 5(a), the interface between elements, which is nearly parallel to the yz plane is shown by planes 1, 2, ..., We call these “cross sections” hereafter. 1 denotes the entrance cross section, and 2 denotes the exit cross section. If we project the cross section to the yz plane, we obtain Fig. 5(b).

Now, if we assume that rectangular ABCD is composed of one element (it is actually composed of sixteen elements), and that \( v_x \) is equal to \( U \) at node A, velocity fields in rectangular ABCD are expressed using the isoparametric element and normalized coordinates (\( \eta \) is taken toward the BC direction and \( \xi \) is taken toward the BA direction),

\[ v_x = \frac{1}{4} (1 - \eta)(1 + \xi) U, \quad v_z = 0 \]  

This is a velocity field concerning \( v_x \) at node A.

Velocity fields concerning \( v_x \) at node B, C, and D are obtained in the same way. So, we can make velocity fields concerning at nodes A, B, C, and D on cross section 2 and 3, respectively (\( v^{(4)} \sim v^{(11)} \)). We also consider a velocity field concerning \( v_x \) at node A with reference to Eq. (10). And we also make velocity fields concerning \( v_x \) at nodes A, B, C, and D on cross section 1, 2 and 3 respectively (\( v^{(12)} \sim v^{(23)} \)).

3.2.3 Fundamental velocity fields for region at boundary of contact surface

Figure 5(c) is a projected cross section 1 to yz plane. We consider a velocity field concerning \( v_x \) at each of ten nodal points, which is denoted * in Fig. 5(c) (\( v^{(23)} \sim v^{(30)} \)). For example, a velocity field concerning \( v_x \) at node E is as follows. We assume that rectangular AEFD is composed of one element and we apply Eq. (10). Rectangular EBCF is treated in the same way. We also consider a velocity field concerning \( v_x \) at nodal points except A, B, C, and D (\( v^{(34)} \sim v^{(39)} \)).

3.2.4 Fundamental velocity fields for region after boundary of contact surface

Velocity fields in rectangular ABCD in Fig. 5(b) are given as follows.

\[ v_x = \frac{1}{2} (1 - \eta) \xi U, \quad v_z = 0 \]  
\[ v_x = \frac{1}{2} (1 + \eta) \xi U, \quad v_z = 0 \]  
\[ v_x = \frac{1}{2} (1 - \eta) \xi^2 - \frac{1}{3} U, \quad v_z = 0 \]  
\[ v_x = \frac{1}{2} (1 + \eta) \xi^2 - \frac{1}{3} U, \quad v_z = 0 \]

We consider these velocity fields on cross sections 5, 6 and 7 respectively (\( v^{(10)} \sim v^{(17)} \)).

A velocity field concerning barreling of the side free surface is determined as follows. For each node toward the direction of thickness (for example, nodes 1-5 in Fig. 2), \( \xi_x \) is assumed as:

\[ \xi_x = (x - x_{C0})^2 + C \]  
\( x_{C0} \) : z value in the middle of CD in Fig. 5(b), 
\( C \) : constant

So, \( v_x \) is determined numerically, satisfying Eq. (15), considering the boundary condition of the velocity fields.

We consider this velocity field on cross sections 5, 6 and 7 respectively (\( v^{(20)} \sim v^{(27)} \)).

3.2.5 Fundamental velocity fields concerning curling toward the direction of thickness

The following velocity field is given (\( v^{(30)} \)).

\[ v_x = \frac{z + l}{l} \frac{z - z_c}{2h_r} U, \quad v_z = 0 \]

\( z_c \) : z value of centroid of the exit cross section

3.2.6 Fundamental velocity fields concerning material position toward the direction of thickness

The material position is assumed beforehand. If

![Fig. 5](image-url)
this assumption is correct, the material at the entrance does not move toward the direction of thickness in the calculation of minimizing the functional $\Phi$ in Eq.(1). So we introduce a velocity field which makes the material at the entrance move toward the direction of thickness at the speed of $U(e^{i\varphi(t)})$. We modify the material position at the time when the functional are minimized, if the coefficient of this velocity field ($\omega_{a}$ in Eq.(2)) is not zero. But this procedure needs a very long computation time. So we will find the limit value $\omega_{a}$, below which rolling properties change very little.

Rolling properties with fixed material position, whose deviation from the correct position is $\delta_{m}$, are shown in Fig.6(a) for Case 1 in Table 1. Judging from Fig.6(a), rolling properties change little, if $\omega_{a}$ is smaller than 0.005. So we choose this value in the following calculations.

4. Results

The following results are obtained. The standard rolling condition is shown in Table 1. Friction ratio $m$ is equal to 1.

4.1 Effect of roll angle

The effect of the roll angle is shown in Fig.7. We use $\theta_{m}$ instead of $\theta_{a}$ and $\bar{\theta}_{m}$, because $\delta_{m} = \theta_{b}$ in this section. Results are plotted by $\bigcirc$.

As the roll angle increases, the width spread decreases (Fig.(a)). The reason is as follows. As the roll angle increases, the roll-material contact width increases. The mean gradient of the free side surface decreases as the roll angle increases (Fig.(b)). The reason is that the reduction of the side parts of the material is large near the upper roll, and is small near the lower roll. As the roll angle increases, the warping of the section along the thickness and the width increases (Figs.(c),(d)). The shape of the contact surface and neutral line is shown in Figs.(e) and (f), respectively. This tendency concerning warping is expected from Figs.(c) and (f). Figure (g) shows the material position, and as the roll angle increases, the material moves downward. The material cross section at the entrance and roll shape at the exit are supplemented, and the dotted line is obtained by $S_{U} = S_{L}$. The material position obtained by these two lines are slightly different. Torque is shown in Fig.(h).

4.2 Effect of material angle

Effect of material angle is shown in Figs.8 and 9. When the material angle changes, the material cross section at the entrance does not change (see Fig.4). The tendency of width spread, and the mean gradient of the free side surface is caused by the reduction of the side parts of the material near the upper and lower rolls (Fig.(a)). Also the tendency of warping of the section along the width is expected from the shape of the contact surface (Fig.(b)).

4.3 Effect of circumferential velocities of the upper and lower rolls

The effect of roll radius of the upper and lower rolls is shown in Fig.10 ($k_{r}$ is constant). As the upper
roll radius increases, warping of the section along the thickness (Fig. 9(a)) and curvature (Fig. 9(b)) decreases. The effect of angular velocities of the upper and lower rolls is shown in Fig. 11. As upper roll angular velocities increase, warping of the section along the thickness (Fig. 10(a)) and curvature (Fig. 10(b)) decreases. A comparison of Fig. 10 with 11 shows that rolling properties agree well with each other quantitatively. This is because the rolling properties depend highly on the circumferential velocities of the upper and lower rolls. An experiment using plasticine and a plaster roll was done for Case 2-4 in Table 1. The lubricant was CaCO₃, the atmospheric temperature was 25°C, and the rolling velocity was 5 mm/s. In the experiment, the mean curvature of the upper and lower rolls at \( y = b_h/2 \) was measured, and supplemented at the dotted line in Fig. 10. The calculated results are in good agreement with the experimental ones.

### 4.4 Discussion of curling

Material will curl at the exit, when vertical forces and moments are not applied. We will discuss the condition in which straight material is obtained. When the moment toward the direction of thickness is applied at the exit, the term \(-M_{F0}\theta_b\) should be added to the functional \( \Phi \) in Eq. (1), where \( M_{F0} \) is the moment at the exit, and \( \omega_b \) is the angular velocity at the exit. We use \( M_{F0}^* \) as a variable concerning rolling properties.

Moment at the exit: \( M_{F0}^* = (M_{F0}/2r_F)/(2S_F/\sqrt{3k}) \)

(\( 2S_F \): cross section at the exit, when material curls toward the upper roll at the exit, \( M_{F0}^* > 0 \))

The effect of moment at the exit is shown in Fig. 12. From this figure, it is found that very small moment \( (M_{F0}^* = -0.02) \) is needed to obtain straight material. We can also get straight material by changing the ratio of the angular velocities of the upper and lower rolls (\( \omega_u/\omega_b = 1.005 \), in the case of Fig. 11).

### 4.5 Discussion of deformation

#### 4.5.1 Material shape

Experiments were done for Case 1 and 3 in Table
and the shape of the contact surface and cross section at the exit were measured. Calculated results and experimental ones were compared, as shown in Fig.13. Both results agree with each other well.

4.5.2 Equivalent strain rate

The distribution of equivalent strain rates in longitudinal section at Case 1 in Table 1 shown in Fig.14. This result is confirmed by a slip line field of plane strain rolling, and coincides with the experiment using plastinece.

4.5.3 Comparison with results in the previous paper

The results of this paper and those of the previous paper are compared in Fig.15. In the previous paper, the material position was fixed beforehand, and curling was not assumed to occur, so we will get rid of some velocity fields in the above discussion (V). The number of fundamental velocity fields is then 54. The width spread shown in Fig. (a) is less than the previous one where the width/height ratio of the material was small. The equivalent strain rate toward the direction of thickness at the entrance is shown in Fig. (b). It is more complex than the previous one. The reason for the difference in these two results is as follows: In the previous paper, the number of velocity fields was small and deformation could not be approximated sufficiently, especially in the region shown in Fig. (b), in which the changing rate of equivalent strain rate is large. On the other hand, in this paper the number of velocity fields is many and the above mentioned problem is solved. Hence a better analysis is given for the rolling of an angle for small width/height ratio.

5. Conclusions

A combined method of the energy method and the rigid plastic FEM is applied to an analysis of the rolling of an angle for small width/height ratio because the deformation of material is rather complex and the usual energy method cannot be applied. The results are as follows:

(1) The method of constructing the fundamental velocity field in the previous paper is applied to the analysis of rolling of an angle, and reasonable results are obtained in comparison with the experiment. Especially, in this paper, the asymmetry toward the direction of thickness such as the position of the material at the entrance, and the curling of the material at the exit are considered.

(2) Rolling properties with respect to fundamental rolling conditions are obtained, and some discussions are made from a mechanical standpoint: curling depends on the difference of circumferential velocities of the upper and lower rolls, distortion of the cross section of a material is caused by the reduction of the side parts of a material near the upper and lower rolls.

References


