Progress of Continuum Damage Mechanics*

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The notion of, and the progress of continuum damage mechanics along with the microscopic aspects of material damage are reviewed. Starting from a brief review of material damage and the notion of continuum damage mechanics, mechanical modelling of the damage states and the damage variables used to describe them are discussed. Then, the extension of the classical creep damage theory to the anisotropic state of creep damage and the problems of elastic-plastic damage are discussed. Besides the application of these damage theories to the damage of metals, concrete and rocks, recent works on spall damage, fatigue damage and creep-fatigue damage are reviewed. The applicability of continuum damage mechanics to the analysis of boundary value problems, and the local approach to fracture problems by means of continuum damage mechanics are also referred to.

Key Words: Continuum Damage Mechanics, Damage, Elastic Damage, Plastic Damage, Creep Damage, Fatigue Damage, Evolution Equation, Constitutive Equation, Review

1. Introduction

Material elements under external loads are often subjected to irreversible changes of microscopic internal structures, besides macroscopic deformation. Among these internal microstructural changes, the nucleation and the growth of distributed microscopic cavities and cracks not only induce the initiation of macrocracks and the final rupture, but also lead to the deterioration of materials (i.e. material damage), such as reductions in strength, rigidity and fracture toughness, or decrease in the remaining lifetime. The distributed cavities and cracks, furthermore, have significant influence on various physical phenomena, including the velocity of wave propagation in materials, heat and electric conduction, and the permeability of magnetism, radiation and fluid. Therefore, the material damage caused by distributed microscopic voids is an important topic for a wide variety of engineering problems, ranging from the fracture process of materials to the evaluation of mechanical and physical properties of the damaged materials, the remaining life predictions of material elements, the development of various nondestructive inspection methods and the interpretation of their results, etc. Thus, a new discipline, called continuum damage mechanics (CDM) or damage mechanics(10-11) has recently made significant progress in investigating, from a continuum mechanics point of view, the evolution of these distributed microscopic voids, along with their effects on the mechanical behavior of damaged materials.

The present paper is concerned with a brief review of the notion of CDM, its recent progress, with special emphasis on the mechanical modelling of the distributed cavities and cracks, and of the mathematical formulation of the mechanical behavior of materials deteriorated by the distributed cavities and cracks.

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Table 1 Classification of material damage, microscopic mechanisms and characteristic features

<table>
<thead>
<tr>
<th>Damage</th>
<th>Microscopic mechanisms, characteristic features, etc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic damage(^{(10,15,17,19)})</td>
<td>Nucleation and growth of microscopic cracks caused by the elastic deformation of the matrix. Rocks, concrete, composites, metals.</td>
</tr>
<tr>
<td>Elastic-plastic damage(^{(75,142)})</td>
<td>Nucleation and growth of microscopic voids caused by the elastic-plastic deformation of the matrix. Intersection of slips, decohesion of particles from the material matrix, cracking of particles. Metals, composite, polymers, etc.</td>
</tr>
<tr>
<td>Spall damage(^{(86,140,151)})</td>
<td>Elastic and plastic damage due to impulsive loads. Uniform distribution of microscopic cavities and cracks. Coupling between void growth and stress waves.</td>
</tr>
<tr>
<td>Fatigue damage(^{(116,174,176,184)})</td>
<td>Transgranular microscopic surface cracks caused by repeated loading. Distributed cracks for low cycle fatigue.</td>
</tr>
<tr>
<td>Creep damage(^{(65,150)})</td>
<td>Nucleation and growth of microscopic intergranular voids and cracks caused by creep. Mainly due to grain-boundary sliding and diffusion.</td>
</tr>
<tr>
<td>Creep-fatigue damage(^{(171,175,178)})</td>
<td>Damage induced by repeated loading at high temperature. Nonlinear coupling between intergranular voids and transgranular cracks.</td>
</tr>
<tr>
<td>Corrosion damage(^{(74,139)})</td>
<td>Pitting corrosion, intergranular corrosion etc. Development of microcracks under stress in corrosive environments.</td>
</tr>
<tr>
<td>Irradiation damage(^{(20,26,102)})</td>
<td>Damage caused by irradiation of neutron, particles and x-ray. Knock-on of atoms, nucleation of voids and bubbles, swelling.</td>
</tr>
</tbody>
</table>

2. Material Damage and Continuum Damage Mechanics

Among various microstructural changes of materials, the development of distributed microscopic cavities and cracks usually brings about a certain deterioration of materials. Thus, in the following, we will refer to the existence of distributed microscopic voids* (i.e., microstructural change leading to the deterioration of the mechanical properties of materials) as damage, and call the nucleation and the growth of the voids as their evolution\(^{(10,15)}\).

Table 1 shows the typical types of material damage encountered in engineering problems, their microscopic mechanisms and their characteristic aspects**. Every case of damage in this table has significant influence on the mechanical properties of the material; among them, are the macroscopic elastic constants, the plastic properties, the tensile strength, the rupture strain, the fracture toughness, the fatigue strength, the creep rates and creep rupture times, the velocities of elastic waves, etc. The shape and arrangement of such voids and their mechanical effects usually have marked anisotropy\(^{(10,15)}\).

When we discuss the development of the distributed voids in Table 1 and their mechanical effects, it is unfeasible to describe the details of the evolution of the individual cavities and cracks. One of the practical approaches to the problem, as originally proposed by Kachanov\(^{(11)}\), is to introduce continuum models of the damaged materials by representing the relevant damage state in terms of appropriate mechanical variables. This approach is essentially identical to the theory of the continuous distribution of dislocations\(^{(15)}\) in which the effect of dislocation arrangement are represented by a dislocation density tensor, or to the theory of plasticity where the effect of dislocation arrangement is described in terms of macroscopic hardening parameters.

For the continuum approximation of damaged materials, let us now take a body B which contains

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* Here, microscopic voids imply cavities or cracks of the size of a few crystal grains at most, or of the size of less than 1 μm\(^{(41-47)}\).

** Among these types of damage, the elastic-plastic damage is closely related to the nucleation of microscopic voids in the vicinity of macroscopic cracks, macroscopic voids and notches, or in the localized shear bands. Thus, the elastic-plastic damage is investigated intensively, not only in damage mechanics, but also in the mechanics of ductile fracture\(^{(120,13)}\).
distributed microvoids, as shown in Fig. 1, and consider an arbitrary point \( x \) in it. We assume that a volume element \( V \) exists around \( x \), which is large enough to include a sufficient number of cavities or cracks, but is sufficiently small, at the same time, so that the states of stress, strain and void distribution are sufficiently uniform. According to the notion of continuum damage mechanics, it is hypothesized that the effects of these microvoids at an arbitrary point \( x \) can be described by appropriate mechanical variables \( \Omega(x) \) or \( \omega(x) \) which we call damage variables or damage fields. Since these damage variables are macroscopic variables, which describe mechanically the state of the internal structure, they are precisely the internal state variables in thermodynamic theories of constitutive equations.

Thus, if the damage states, as shown in Table 1, can be represented by certain damage variables, we can readily develop a systematic approach to analyze the damage evolution and the mechanical behavior of damaged materials in the framework of continuum mechanics\(^{(1)}\-(11)\).

Though the present review is focused on the deterioration of the mechanical properties of materials, the notion of damage mechanics can be applied to more general phenomena. The inelastic behavior of materials with physicochemical deterioration, for example, has been discussed by Rice\(^{(14)}\) and Alfantis\(^{(17)}\).

### Table 2: Damage variables and their tensorial nature

<table>
<thead>
<tr>
<th>Damage variables</th>
<th>Reference</th>
<th>Material damage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scalar</td>
<td>Kachanov(^{(10)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Kachanov(^{(16)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Rabotnov(^{(20)})</td>
<td>Creep (anisotropic)</td>
</tr>
<tr>
<td></td>
<td>Martin-Leckie(^{(40)}), Hayhurst-Leckie(^{(41)})</td>
<td>Creep (anisotropic)</td>
</tr>
<tr>
<td></td>
<td>Davison et al.(^{(29)})</td>
<td>Spalling</td>
</tr>
<tr>
<td></td>
<td>Gurson(^{(39)})</td>
<td>Elastic-plastic</td>
</tr>
<tr>
<td>Vector</td>
<td>Kachanov(^{(16)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Hayhurst-Storakers(^{(10)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Davison-Stevens(^{(29)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Krajcinovic-Fonseka(^{(30)})</td>
<td>Spalling</td>
</tr>
<tr>
<td></td>
<td>Krajcinovic et al.(^{(33)})</td>
<td>Elastic-brittle</td>
</tr>
<tr>
<td></td>
<td></td>
<td>General, creep</td>
</tr>
<tr>
<td>Second rank tensor</td>
<td>Rabotnov(^{(20)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Vakulenko-Kachanov(^{(24)})</td>
<td>Elastic-brittle</td>
</tr>
<tr>
<td></td>
<td>Murakami-Oono(^{(39)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>M. Kachanov(^{(20)})</td>
<td>Elastic</td>
</tr>
<tr>
<td></td>
<td>Cordebois-Sidoroff(^{(33)})</td>
<td>Elastic, elastic-plastic</td>
</tr>
<tr>
<td></td>
<td>Better(^{(40)})</td>
<td>General, creep</td>
</tr>
<tr>
<td></td>
<td>Murakami(^{(29)})</td>
<td>General</td>
</tr>
<tr>
<td>Higher rank tensor (fourth rank) (a set of tensors of even rank)</td>
<td>Chaboche(^{(41)})</td>
<td>Creep</td>
</tr>
<tr>
<td></td>
<td>Leckie-Ohn(^{(46)})</td>
<td>Creep</td>
</tr>
</tbody>
</table>

3. **Mechanical Description of the Damage States and Damage Variables**

Two problems arise when we describe the states of material damage in terms of the damage variables mentioned above. The first problem is as to what kind of mathematical property (tensorial nature) we should specify for these variables in order to represent such damage states, while the second is in what way the magnitude of these variables should be quantified. Table 2 shows the examples of the damage variables proposed so far. Though most of the damage variables in Table 2 have been defined on the basis of microscopic arrangements of the distributed voids, their magnitudes are usually quantified with reference to some macroscopic phenomena affected by the distributed voids.

The scalar damage variables in Table 2 are adequate for the isotropic damage, i.e., for the damage characterized by the random distribution of voids or by the homogeneous distribution of spherical cavities; they are not applicable to the damage with marked directionality in void geometries or to the damage with spherical cavities with significant directionality in their spacial distribution. However, in the case of a small cavity density, the global deformation is almost isotropic, and hence the scalar damage variables are frequently employed for the constitutive equations of damaged materials\(^{(1)}\)-(11)\),\(^{(35)}\)\(^{(36)}\),\(^{(37)}\), even if the void arrangement is anisotropic.

It would be straightforward to describe the effect of distributed plane cracks by a vector damage variable perpendicular to the plane of the cracks\(^{(18)}\)-(20). However, the vector damage variable cannot represent the set of plane-cracks of different orientations by means of a simple addition of the corresponding vectors.

On the other hand, despite the complexity of their mathematical structure, tensors of second and higher rank can describe the state of distribution of the microscopic voids and their mechanical effects more accurately\(^{(24)}\)-(34). Among others, the second rank symmetric tensor has been employed in a number of anisotropic damage theories\(^{(24)}\)-(37) because of its mathematical simplicity. However, it should be noted that they cannot describe the damage states with a more complicated symmetry than orthotropy\(^{(31)}\).
4. Continuum Damage Mechanics Theories for Material Damage

4.1 Creep Damage

The notion of continuum damage mechanics discussed above was first proposed by Kachanov\(^{(10)}\) when he developed a mathematical theory of evaluating brittle creep rupture times of tensile bars. If we assume that the net area reduction due to cavity growth is the principal mechanism of the material damage, the state of damage can be represented by a scalar damage variable $\phi$ such that $\phi = 1$ and $\phi = 0$ specify the undamaged initial state and the final rupture state, respectively\(^{(4)}\). Then, by taking the maximum tensile stress $\sigma_\text{max}$ as the principal factor governing the progression of the damage, Kachanov formulated the evolution equation of the damage variable $\phi$ as follows:

$$\dot{\phi} = B(\sigma_\text{max}(\phi))$$ \hspace{1cm} (1)

where $\phi$ denotes the differentiation with respect to time $t$, and $B$ and $\nu$ are material constants.

Though Kachanov did not discuss in any detail the physical meaning of $\phi$, it can be interpreted as being the ratio between the load-carrying net area $A$ of a given section and that of the corresponding apparent area $A_0$:

$$\phi = A/A_0 \quad (0 < \phi < 1)$$ \hspace{1cm} (2)

Instead of the variable $\phi$, Rabotnov\(^{(35)(36)}\) thereafter introduced an alternative internal state variable

$$\omega = 1 - \phi \quad (0 < \omega < 1)$$ \hspace{1cm} (3)

which can be interpreted as being the reduction of the load-carrying net area as a result of the void formation. The quantity in Eq. (1)

$$\bar{\sigma} = \dot{\omega} = \sigma/(1-\omega)$$ \hspace{1cm} (4)

implies a stress magnified by the net area reduction, and is called the net stress or the effective stress.

In Kachanov's theory outlined above, the damage state is assumed to have no influence on the creep rate, and cannot describe the tertiary creep observed in creep tests under constant stress. Thus, Rabotnov\(^{(30)}\) extended Kachanov's theory to incorporate this effect, and assumed the following constitutive and evolution equations by using the damage variable $\omega$ of Eq. (3):

$$\dot{\varepsilon} = A F(\sigma, \omega), \quad \dot{\omega} = G(\sigma, \omega)$$ \hspace{1cm} (5-a)

The explicit forms of Eq. (5-a) were given, for example, as follows\(^{(30)}\):

$$\dot{\varepsilon} = A_0 \sigma'(1-\omega)^\alpha, \quad \dot{\omega} = B \sigma'(1-\omega)^\nu$$ \hspace{1cm} (5-b)

where $A$, $B$, $m$, $n$, $\nu$ and $\mu$ are material constants.

The Kachanov-Rabotnov theory mentioned above is a prototype of the damage theories developed thereafter, and has provided an important basis for continuum damage mechanics. Assuming isotropy of the materials, along with that of material damage, Leckie and Hayhurst\(^{(37)(38)}\), for example, generalized Eq. (5) to multiaxial states of stress. By using the scalar\(^{(29)-(41)}\) and the vector damage variables\(^{(10)(19)}\), Eqs. (5-a) and (5-b) were also extended to multiaxial anisotropic creep damage.

As mentioned already, scalar and vector damage variables cannot describe the anisotropic material damage properly. Therefore, by assuming that the principal effect of the material damage consists of the net area reduction due to the three-dimensional distribution of microvoids, Murakami and Ohno\(^{(28)}\) described anisotropic damage states by a second rank symmetric damage tensor

$$\mathbf{\Omega} = \sum \Omega_i \mathbf{n}_i \otimes \mathbf{n}_i$$ \hspace{1cm} (6)

where $\Omega_i$ and $\mathbf{n}_i$ are the principal values and the corresponding principal directions respectively of the damage tensor $\mathbf{\Omega}$. According to Eq. (6), $\mathbf{\Omega}$ can be interpreted as the void area density in the plane perpendicular to the principal direction $\mathbf{n}_i$ of the damage. By examining the damage effects in planes other than the principal planes, Murakami and Ohno further showed that the effects of the Cauchy stress $\mathbf{\sigma}$ acting on a body is magnified as a result of the net area reduction due to void formation, and given by the following net stress tensor\(^{(28)(38)}\):

$$\bar{\mathbf{\sigma}} = (1/2)(I - \mathbf{\Omega})^{-1} \mathbf{\sigma} + \mathbf{\sigma}(I - \mathbf{\Omega})^{-1}$$ \hspace{1cm} (7)

where $I$ is a unit tensor of rank two. The damage tensor $\mathbf{\Omega}$ and the net stress tensor $\mathbf{\sigma}$ of Eqs. (6) and (7) are the three-dimensional versions of $\omega$ and $\bar{\sigma}$ of Eqs. (3) and (4) in the Kachanov-Rabotnov theory.

Recently, Murakami\(^{(28)}\) derived the results of Eqs. (6) and (7) in a more consistent way, by directly extending the notion of classical Kachanov-Rabotnov theory and by introducing a fictitious deformation from the current damaged configuration to the corresponding fictitious undamaged configuration.

The rate of damage growth in metals at each instant depends on the current state of material damage, on that of stress, as well as on the past history of the inelastic deformation of the material. If the effect of stress on the damage growth is given by the net stress $\bar{\mathbf{\sigma}}$ of Eq. (7), the evolution equation of damage tensor $\mathbf{\Omega}$ can be specified as follows\(^{(28)}\):

$$\mathbf{\dot{\Omega}} = \mathbf{G}(\mathbf{\sigma}, \mathbf{\Phi}, \kappa), \quad \mathbf{\Phi} = (I - \mathbf{\Omega})^{-1} \hspace{1cm} (8)$$

where $\kappa$ is the strain-hardening parameter of the material.

Unlike the damage growth, the deformation of damaged materials depends not only on the net area reduction caused by void formation, but also on their three-dimensional arrangement. Thus, the net stress tensor $\mathbf{\sigma}$ of Eq. (7) can no longer be applied for this purpose. In view of the fact that the deformation characteristics of materials are in general specified by a fourth rank tensor, Murakami and Imaizumi\(^{(42)}\)
defined the net stress tensor for deformation $\tilde{\sigma}$ by use of a fourth rank tensor $\Gamma(\Omega)$;
\[
\tilde{\sigma} = (1/2)[ \Gamma \colon \sigma + (\Gamma \colon \sigma)^T ]
\]  
(9)

Thus, the constitutive equation for creep can be expressed as follows:
\[
\dot{\varepsilon} = G(\tilde{\sigma}, \Omega, x)
\]  
(10)

The validity of the notion of Eqs. (6) ~ (10) has been examined in detail by performing model tests on perforated specimens\(^{[46]}\), and by creep damage tests on thin-walled copper tubes\(^{[47]}\). Equations (6) ~ (10) have recently been applied to analyse the coupling of creep damage and plastic damage\(^{[46]}\), and to the creep crack growth analysis under non-proportional loading\(^{[29]}\).

A second rank damage tensor, similar to Eq. (6), has also been proposed by M. Kachanov\(^{[29]}\) and Betten\(^{[30]}\). By taking the number and the volume of voids as the measure of material damage, Leckie and Onat\(^{[34]}\), on the other hand, described the damage states by a set of tensors of even ranks. The details of these theories will be omitted, because they have been reviewed elsewhere\(^{[9]}\).

Furthermore, Lemaître and Chaboche\(^{[35]~-[40]}\) characterized the damage states by the changes of elastic constants of the material and developed a comprehensive theory for material damage, in general\(^{[31]~-[40]}\). For one-dimensional problems, if we employ the notion of the net stress of Eq. (4) and define the net stress $\tilde{\sigma}$ in a damaged material under stress $\sigma$ and elastic strain $\varepsilon^e$ as a stress which would induce the identical strain $\varepsilon^e$ in a corresponding undamaged material, we have the relations
\[
\varepsilon^e = \frac{\sigma}{E} = \frac{\tilde{\sigma}}{\tilde{E}} = \tilde{\sigma} / (1 - \omega)
\]  
(11)

\[
\omega = 1 - \frac{E}{\tilde{E}}
\]  
(12)

where $E$ and $\tilde{E}$ are elastic constants of the damaged and undamaged materials. Equations (11) and (12) imply that damage states can be identified by the value of $\tilde{E}$.

If we apply this idea to multiaxial states of stress, the damage variable $D$ would lead to an eighth rank tensor which transforms the fourth rank elasticity tensor $E$ of the undamaged material to the corresponding elasticity tensor $\tilde{E}(D)$ of the damaged material\(^{[35]}\). However, in order to avoid the mathematical complexity of eighth rank damage tensors, Chaboche\(^{[35]}\) postulated a fourth rank damage tensor $D$ which defines the following transformation from $E$ to $\tilde{E}(D)$:
\[
\tilde{E}(D) = (I - D)^{\frac{1}{4}} E
\]  
(13)

where $I$ and $(:)$ stand for a unit tensor of rank four and the contraction of rank two, respectively. Thus, the damage tensor and the net stress tensor are given by analogous relations to Eq. (12):
\[
D = 1 - \tilde{E} / E
\]  
(14)
\[
\tilde{\sigma} = (I - D)^{\frac{1}{4}} \sigma
\]  
(15)

Krajcinovic developed a thermodynamical theory\(^{[71]}\) for brittle and ductile damage, including creep damage\(^{[72]}\), for damaged materials containing a number of plane-microcracks, by employing a vector damage variable
\[
\omega = \omega_0 \mathbf{N}
\]  
(16a)
\[
\dot{\omega} = \dot{\omega}_0 \mathbf{N} + \omega_0 \dot{\mathbf{N}}
\]  
(16b)

where $\mathbf{N}$ and $\omega_0$ stand for the unit vector perpendicular to the crack plane and the crack area density, respectively. The details of his theory will be presented later. Burke et al.,\(^{[49]}\) on the other hand, represented the damage states of materials by means of cavity volume fractions, and discussed a thermodynamical theory of the three-dimensional constitutive equation of creep damage.

### 4.2 Elastic damage and elastic-plastic damage

The monotonic increase of stress often induces distributed microscopic cavities in metals, polymers, composites, concrete and rocks. The material damage caused by the progression of elastic or elastic-plastic deformation due to stress increase is called elastic damage or elastic-plastic damage, and has been discussed from the damage mechanics point of view in many papers.

For uniaxial states of stress, Hult and Broberg\(^{[74],[94]}\) extended the Kachanov-Rabotnov the-
theory of the constitutive and the evolution equations of
general elastic-plastic damage.

Let us first assume that the internal states of the
material related to the material damage and to the
dislocation arrangement can be represented by a
vector damage variable \( \omega \) of Eq.(16) and a vector
strain-hardening parameter \( a \). Then, if we express the
Helmholtz free energy of the damaged material by
\( \phi(e^* , a, \omega, T) \), the Clausius-Duhem inequality furnishes
\[
\sigma_{ij} = \rho \frac{\partial \phi}{\partial \varepsilon^*_{ij}}, \quad A_i = \rho \frac{\partial \phi}{\partial a_i}, \quad R_i = \rho \frac{\partial \phi}{\partial \omega_i}
\]
where \( T \), \( \rho \), and \( \sigma \) are temperature, density and
stress, while \( A \) and \( R \) are the thermodynamic forces
conjugates to \( a \) and \( \omega \). Let us further define the ther-
modynamical flux vector \( J \) and the corresponding
conjugate thermodynamical force vector \( X \) as follows:
\[
J = (\varepsilon^*_{ij}, \dot{a}_i, \dot{\omega}_i, \dot{q}_i)^T
\]
\[
X = \{ A_{ij} - A_{ij} - R_i - \frac{1}{\gamma} \text{grad} T \}
\]
Then, the Clausius-Duhem inequality can be expressed in
the form of the entropy production inequality
\[
\rho \dot{D} = X^T \dot{X} \geq 0
\]
where \( \dot{q}_i \) is the heat flux vector.

When the functional relation of the ther-
modynamical flux \( J \) on the thermodynamical force \( X \)
satisfies certain appropriate conditions, we can postu-
late the existence of the flow potential \( F(X) \) related to the
entropy production rate \( \dot{D} \) of Eq.(21) \( \text{[16][50][57]} \);
the flux vector \( J \) can be given as follows:
\[
J_n = \frac{\partial F}{\partial x_n}
\]
or
\[
\varepsilon^*_{ij} = -\frac{\partial F}{\partial a_{ij}}, \quad \dot{a}_i = -\frac{\partial F}{\partial A_i}, \quad \omega_i = -\frac{\partial F}{\partial \omega_i}
\]
Thus, once the explicit forms of the Helmholtz free
energy and the flow potential \( F \) are specified, the
elastic-plastic constitutive equation and the evolution
equation of material damage can be easily obtained by
Eqs.(19) and (22). Analogous derivations can be also
performed for the tensor damage variable \( Q \) and the
tensor strain-hardening parameter \( a_{ij} \).

As an application of the above theory, Krajcinovic\( \text{[25]} \) discussed the damage process of an
elastic-brittle material caused by the development of
plane-cracks by using the free energy and the flow
potential, of the form
\[
\phi = \phi(e^* , \omega, T), \quad F = L_m R_a R_a - R_b^2
\]
and showed that this theory can describe not only the
development of cleavage cracks perpendicular to the
maximum principal direction, but also the initiation of
plane-cracks in a shear direction and their rotation
due to shearing stress.
This theory has been applied also to the case of creep damage\(^{26,27}\). In this case, Krajinovic employed a tensorial kinematic hardening parameter \(a_s\), together with a scalar isotropic hardening parameter \(p\) for creep hardening, and postulated the free energy and the flow potential of the form

\[
\phi = \phi^{(v)}(\varepsilon^*, a_s, \beta, T) + \phi^{(c)}(a, p, \xi, \eta, T)
\]

\[
F = F_0 (\sigma, A, P; \omega) + F_1 (R; \omega)
\]  
(24)

where \(A\) and \(P\) are the thermodynamic forces conjugate to \(a\) and \(p\).

Anisotropic theories for elastic-plastic damage based on a second rank damage tensor have been developed in the papers of Vakulenko and M. Kachanov\(^{24-26}\) and of Dragon et al.\(^{27-55}\). The details of these theories will be omitted, because they have been presented elsewhere\(^{26,19}\).

The above papers of Krajinovic, Vakulenko and M. Kachanov, and Dragon et al. are mainly concerned with the brittle damage of rock-like materials. The significant material damage of metals, on the other hand, is usually induced by large elastic-plastic deformations. Thus, there are only a limited number of studies on the elastic-plastic damage of metals under multiaxial loadings, mainly because of the experimental difficulty of realizing sufficiently uniform and large deformations, together with the complicated coupling between the stiffness reduction caused by the cavity formation and the strain hardening due to plastic deformation.

The effect of elastic-plastic damage on the change of the elastic moduli has been investigated, mostly in order to identify the damage state of the material\(^{23,24-27,47}\). Cordebois and Sidoroff\(^{31,32}\) discussed this problem by means of the second rank symmetric damage tensor \(\Omega\) and the effective stress \(\bar{\sigma}\) of Eq. (15). Since the state of damage induced by uniaxial stress is transverse isotropic, the damage tensor \(\Omega\) has the components

\[
\Omega = \begin{bmatrix}
\Omega_1 & 0 & 0 \\
0 & \Omega_2 & 0 \\
0 & 0 & \Omega_3 \\
\end{bmatrix}
\]

(25)

where suffix 1 refers to the direction of tensile stress. By expressing the elastic constitutive equations of a damaged material in terms of the effective stress \(\bar{\sigma}\), Young's modulus \(E\), the Poisson ratio \(\nu\) and the modulus of rigidity \(G\) of the damaged material can be obtained as follows:

\[
\begin{align*}
\bar{E} &= E(1 - \Omega_2)^2 \\
\bar{\nu} &= \nu(1 - \Omega_2)(1 - \Omega_3) \\
\bar{G} &= G_E(1 - \Omega_2)/(1 + \nu)
\end{align*}
\]

(26)

where \(E\), \(\nu\) and \(G\) are the corresponding material constants of the undamaged material. Equation (26) enables us to identify the damage tensor \(\Omega\) by measuring the elastic constants of the damaged material, \(E\) and \(\bar{\nu}\) and \(\bar{G}\). Cordebois et al.\(^{31,32}\) discussed the validity of the above theory on the basis of the results of the tensile tests of an aluminum alloy and copper.

With regards to the evolution equation and the constitutive equation of the elastic-plastic damage, Cordebois and Sidoroff\(^{31,32}\) developed a thermodynamical theory similar to that of Krajinovic, mentioned above. Then, Cordebois et al. introduced an internal state variable \(\beta\) for damage which plays a similar role to the strain-hardening parameter \(a\) for plastic deformation, and postulated, instead of the Helmholtz free energy \(\phi\), the Gibbs thermodynamical potential

\[
\phi = \phi(\sigma, a, \Omega, \beta)
\]

(27)

where \(\Omega\) is the second rank symmetric damage tensor mentioned already. For the plastic strain \(\dot{\varepsilon}\) and the damage rate \(\dot{\beta}\), they postulated a flow potential of

\[
F = F_0 (\sigma, A; \Omega) + F_1 (R, B)
\]

(28)

Their theory corresponds to that of Krajinovic, if we choose the thermodynamical flux \(J\) and the conjugate force \(X\) as follows:

\[
J = (\dot{\varepsilon}^*, a, \dot{\Omega}, \dot{\beta})^T,
\]

\[
X = (a_s, -A, -R_s, -B)
\]

(29)

The components of \(J\) can be obtained by Eq. (22). The symbols \(A\), \(B\) and \(R_s\), on the other hand, are thermodynamical force conjugate to \(a\), \(\beta\) and \(\Omega_s\), and are furnished by \(\phi\) of Eq. (27) and similar relations to Eq. (19). They showed explicit forms of Eqs. (27) and (28), and discussed the method used to identify \(A(\alpha)\), \(B(\beta)\) and \(R(\Omega)\), as well as the limitations of their theory. However, a comparison of this theory with the corresponding experimental results remains open.

An anisotropic damage theory for coupling between the elastic-plastic damage and the subsequent creep damage has been developed by using the damage variable \(\Omega\) of Eq. (6)\(^{140}\). The variable \(\Omega\) was also applied to describe the inelastic damage in polymeric materials\(^{38}\).

4.3 Spall damage

Besides the above elastic-plastic damage due to quasistatic loadings, a considerable number of theoretical and experimental studies have been performed on the material damage induced by intensive stress waves. This type of damage is called spall damage. Davison, Stevens and others, for example, developed thermodynamical theories of spall damage by use of a vector\(^{28}\) and a scalar damage variable\(^{59}\). Seaman et al.\(^{60-62}\), on the other hand, proposed a mathematical model to describe the change in number, size and density of voids and cracks brough about by the ductile and the brittle damage under dynamic loading.

4.4 Fatigue damage, creep-fatigue damage

Hitherto, fracture life predictions for fatigue
damage and creep-fatigue damage have been performed mainly by conventional phenomenological theories, such as the Coffin-Manson law, the Paris law and the linear cumulative damage law\textsuperscript{(66)(67)}. However, in these theories, there are no systematic ways to elaborate them further or to generalize them to more complicated problems. Continuum damage mechanics provides a more systematic approach to these problems.

In principle, the fatigue damage process under a uniaxial state of stress can be analysed by integrating Eq. (17) along individual load cycles\textsuperscript{(60)}. However, it will be impractical to perform such integrations over the whole process of each load cycle. Thus, Chaboche and Lemaitre\textsuperscript{(39)-(46)} integrated Eq. (18\*b) over one cycle and developed an evolution equation of fatigue damage $\omega$ with cycle number $N$ as an independent variable. They applied the resulting equation to simulate the experimental results of fatigue of some metals under various variable load conditions\textsuperscript{(39)-(46)(64)}. Thereafter, Savalle and Cailletaud\textsuperscript{(65)} elaborated the theory of Chaboche et al\textsuperscript{(39)-(45)} by introducing different damage variables for the nucleation and growth of surface cracks.

As in the case of elastic-plastic damage, the above fatigue damage theories can be extended to the three-dimensional case by postulating the isotropy of damage. Recently, Marquis et al.\textsuperscript{(68)} applied such damage law to an FEM analysis of the fatigue damage process of some structural elements of complicated geometries, and compared the results with those due to the conventional methods of life prediction, together with the corresponding experiments.

The lifetime evaluation of components under repeated loads at elevated temperatures necessitates a consideration of the creep damage mentioned in Section 4.2, besides the above fatigue damage. Continuum damage mechanics has also been applied to the problems of creep-fatigue damage, and has succeeded in predicting the significant non-linear interaction in these phenomena\textsuperscript{(35)(63)(65)(68)(69)-(70)}. However, most of these papers are restricted to the uniaxial states of stress, and describe the creep and the fatigue damage in terms of a single damage variable. The evolution equation is formulated in a similar way to the creep damage and fatigue damage mentioned above. Savalle and Cailletaud\textsuperscript{(65)}, on the contrary, introduced two separate damage variables for the fatigue process, in addition to a damage variable for creep, and analysed coupled creep and fatigue damage under various combinations of these two types of damage. For further details of these problems, a number of reviews\textsuperscript{(2)(3)(60)(67)} are available and will be useful for the interested readers.

4.5 Application of damage mechanics to boundary value problems

Continuum damage mechanics is a new branch of continuum mechanics. Thus, the constitutive and the evolution of material damage can be applied to various boundary value problems by means of the standard methods of inelastic analyses. With regards to these problems, refer to the excellent monographs of Kachanov\textsuperscript{(11)(18)}.

Another important application of damage mechanics is concerned with the process of the initiation and growth of macroscopic cracks. Macroscopic cracks are usually initiated by the nucleation, growth and coalescence of microscopic voids and cracks, and develop by the coalescence of microscopic cavities in front of crack tips. Microscopic voids and cracks in the vicinity of the crack tip have an essential influence on their propagation. Since conventional fracture mechanics postulates a discrete crack in an intact material, and correlates the crack growth rate to global fracture parameters, it sometimes has limited applicability to the problems of damage and fracture. Continuum damage mechanics, on the other hand, provides a systematic approach to these crack problems by analyzing the process of the initiation and growth of macroscopic cracks caused by the progression of the local damage. This local approach of fracture\textsuperscript{(71)(77)} is also applicable to the problems of crack bifurcation under non-proportional loading\textsuperscript{(26)} and to the three-dimensional crack problems.

Recently, significant progress has been made in the local approach of fracture\textsuperscript{(71)(77)}, and will constitute one of the important fields of damage mechanics.

5. Conclusions

The deterioration of materials as a result of the nucleation and growth of distributed microscopic cracks and cavities is observed frequently in various fields of engineering. Thus, continuum damage mechanics, which provides a systematic approach to analyze these phenomena in the framework of continuum mechanics, has a wide scope of applicability.

The present review is concerned with a brief outline of the notion and the applicability of continuum damage mechanics; i.e., modelling of distributed microscopic cracks and voids, evolution equations of damage and constitutive equations of damaged materials, together with their application to some boundary value problems. The micromechanics and the statistical approaches to damage problems, however, were omitted from this review. Further extensive reviews of continuum damage mechanics can be found in the papers of Hult\textsuperscript{(2)}, Lemaitre\textsuperscript{(10)}, Chaboche\textsuperscript{(24)-(27)}, Krajinovic\textsuperscript{(109)} and Murakami\textsuperscript{(109)}.
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