Dynamic Deformation Analysis of Rotating Discs*

(Influences of Circumferential Displacements on Radial Ones)

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The equation of motion in the radial direction for a rotating disc contains the product of the circumferential displacement and the angular acceleration of the disc. It is theorized that circumferential deformation has a considerable influence on radial displacement as the angular acceleration increases. This influence is discussed in this paper. Adopting the cyclically-varying rotation given by

$$\bar{\omega}(\bar{t}) = \bar{\Omega}_1 \sin(\bar{\Omega}_2 \bar{t}), \quad \bar{\Omega}_1, \bar{\Omega}_2: \text{constants};$$

the displacements are computed for various values of $\bar{\Omega}_2$. The maximum amplitudes, $\Delta \bar{u}$ and $\Delta \bar{u}'$, for the radial displacements with and without the circumferential deformation are discussed with respect to a non-dimensional quantity, $\zeta = \bar{\Omega}_2 / \bar{\Omega}_1$.

Key Words: Elasticity, Structural Analysis, Rotating Disc, Dynamic Displacement, Cyclic Rotation

1. Introduction

When steam and gas turbines start up, slow down or suddenly stop as a result of accidents such as the rupture of blades, it is necessary to perform a dynamic analysis to determine exactly the stresses in the affected discs. Dynamic stress analyses have so far been done for discs which have been subjected to the rotations in step and ramp⁽¹⁾⁻⁽³⁾, exponentially increasing⁽⁴⁾ and cyclically variable processes⁽⁵⁾ with respect to time.

The deformations in the radial and circumferential directions for rotating discs are coupled with each other in the fundamental equations. Past studies were performed to derive solutions to simplified equations of motion which decoupled the two kinds of deformations by introducing some assumptions. While the radial deformation is mainly influenced by the angular

velocity ω , the circumferential deformation depends largely on the angular acceleration $d\omega/dt$. However, as the value of $d\omega/dt$ increases as compared with ω , the original fundamental equations suggest that the influence of the circumferential deformation on the radial one should no longer be neglected.

This report presents the effects of the circumferential deformation on the radial one. To do that, the coupling effects of two components of the deformations are discussed in terms of the ratio of the angular velocity to the angular acceleration. Let the variable rotating process of the disc be a cyclic rotation; that is,

$$\omega(t) = \Omega_1 \sin(\Omega_2 t) \quad (1)$$

where Ω_1 and Ω_2 are both constants. The angular acceleration $\dot{\omega}(d\omega/dt)$ is given by

$$\dot{\omega}(t) = \Omega_1 \Omega_2 \cos(\Omega_2 t) \quad (2)$$

which is derived from Eq.(1). Let us introduce the parameter ζ

$$\zeta \equiv \frac{|\dot{\omega}|}{|\omega|} = \frac{\Omega_2}{\Omega_1} \quad (3)$$

which represents the ratio of the angular acceleration to the angular velocity. The coupling effects of the radial and circumferential deformations are discussed

* Received 13th January, 1987. Paper No.86-0021A

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in detail by changing this parameter ζ .

2. Fundamental Equations

A solid disc rotates around a point O with an angular speed $\omega(t)$, as shown in Fig.1. Let a be its outer radius. It is assumed that the disc is homogeneous, isotropic, in a plane-stressed state, subjected to axisymmetric deformation, and obeys the linear elasticity theory. Moreover, assuming that the Coriolis acceleration can be neglected, the equations of motion⁽²⁾ in the radial and circumferential directions are given by

$$\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}^2} = \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - \bar{v} \frac{d\bar{\omega}}{d\bar{t}} - \bar{r} \bar{\omega}^2 \quad (4)$$

and

$$\frac{\partial^2 \bar{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}^2} = \bar{\alpha}^2 \left[\frac{\partial^2 \bar{v}}{\partial \bar{t}^2} + \bar{r} \frac{d\bar{\omega}}{d\bar{t}} \right] \quad (5)$$

in dimensionless form. The stress components are related to the displacements as follows.

$$\left. \begin{aligned} \bar{\sigma}_r &= \frac{1}{1-\nu^2} \left[\frac{\partial \bar{u}}{\partial \bar{r}} + \nu \frac{\bar{u}}{\bar{r}} \right] \\ \bar{\sigma}_\theta &= \frac{1}{1-\nu^2} \left[\frac{\bar{u}}{\bar{r}} + \nu \frac{\partial \bar{u}}{\partial \bar{r}} \right] \end{aligned} \right\} \quad (6)$$

$$\bar{\tau}_{r\theta} = \partial \bar{v} / \partial \bar{r} - \bar{v} / \bar{r} \quad (7)$$

Since the outer boundary of the disc is free of stress, the boundary conditions are represented by

$$\bar{\sigma}_r = 0 \quad \text{at } \bar{r} = 1 \quad (8)$$

$$\bar{\tau}_{r\theta} = 0 \quad \text{at } \bar{r} = 1 \quad (9)$$

The initial conditions of the disc are

$$\bar{u} = \partial \bar{u} / \partial \bar{t} = 0 \quad \text{at } \bar{t} = 0 \quad (10)$$

$$\bar{v} = \partial \bar{v} / \partial \bar{t} = 0 \quad \text{at } \bar{t} = 0 \quad (11)$$

which stands for the state of rest. The non-dimensional quantities used in Eqs. (4) ~ (11) are defined by

$$\left. \begin{aligned} \bar{u} &= u/a, \bar{v} = v/a, \bar{r} = r/a \\ \bar{\sigma}_r &= \sigma_r/E, \bar{\sigma}_\theta = \sigma_\theta/E, \bar{\tau}_{r\theta} = \tau_{r\theta}/G \\ \bar{\omega} &= (a\beta)\omega, \bar{t} = t/(a\beta), \alpha = a/\beta \end{aligned} \right\} \quad (12)$$

where

$$\beta = [\rho(1-\nu^2)/E]^{1/2}, \alpha = [\rho/G]^{1/2} \quad (13)$$

3. Derivation of Solutions

Since the equation of motion, (5), includes no

Fig. 1 Rotating disc

coupling term, the circumferential displacement can be derived from it by using the boundary and initial conditions. Applying the Laplace transform to Eq. (5), and using the initial condition (11), the following equation is obtained:

$$\frac{\partial^2 \tilde{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \tilde{v}}{\partial \bar{r}} - (s^2 \bar{\alpha}^2 + \frac{1}{\bar{r}^2}) \tilde{v} = \bar{r} \bar{\alpha}^2 \tilde{\Omega} \quad (14)$$

where

$$\tilde{\Omega} = \int_0^\infty \Omega \left(= \frac{d\bar{\omega}}{d\bar{t}} \right) e^{-s\bar{t}} d\bar{t}$$

A general solution of Eq. (14) can be expressed by the modified Bessel functions of the 1st and 2nd kinds, I_1 and K_1 , respectively, as follows:

$$\tilde{v} = CI_1(s\bar{\alpha}\bar{r}) + DK_1(s\bar{\alpha}\bar{r}) - \bar{r} \bar{\alpha}^2 \tilde{\Omega}/s^2 \quad (15)$$

The boundedness of the displacement at the disc center and the boundary condition (9) demand that the solution (15) be reduced to

$$\tilde{v} = -\bar{r} \bar{\alpha}^2 \tilde{\Omega}/s^2 \quad (16)$$

By using the convolution theorem, the displacement is obtained as

$$\bar{v} = -\bar{r} \bar{\alpha}^2 \int_0^{\bar{t}} \Omega(\bar{t}-\xi) \xi d\xi \quad (17)$$

Substituting Eq. (17) into Eq. (4), we get

$$\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{\bar{u}}{\bar{r}^2} = \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \bar{r} \alpha^2 \Gamma(\bar{t}) - \bar{r} \bar{\omega}^2 \quad (18)$$

where the function $\Gamma(\bar{t})$ is defined by

$$\Gamma(\bar{t}) = \int_0^{\bar{t}} \Omega(\bar{t}) \Omega(\bar{t}-\xi) \xi d\xi \quad (19)$$

Applying the Laplace transform to Eq. (18), using the initial condition (10), an inhomogeneous, modified Bessel equation is obtained.

$$\frac{\partial^2 \tilde{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \tilde{u}}{\partial \bar{r}} - (s^2 + \frac{1}{\bar{r}^2}) \tilde{u} = -\bar{r} [\tilde{\omega}^2 - \bar{\alpha}^2 \tilde{\Gamma}] \quad (20)$$

where $\tilde{\Gamma}$ stands for the Laplace transform of the function $\Gamma(\bar{t})$. A general solution of Eq. (20) is given by

$$\tilde{u} = AI_1(s\bar{r}) + BK_1(s\bar{r}) + (\tilde{\omega}^2 - \bar{\alpha}^2 \tilde{\Gamma}) \bar{r}/s^2 \quad (21)$$

The boundary condition (8) and the boundedness of the displacement at the disc center determine the coefficients A and B as follows:

$$A = \frac{-2(1+\nu)(\tilde{\omega}^2 - \bar{\alpha}^2 \tilde{\Gamma})}{s^2 [sI_0(s) + 2\nu I_1(s) + sI_2(s)]}$$

$$B = 0$$

Therefore, the transformed displacement is given by

$$\tilde{u} = -\frac{2(1+\nu)(\tilde{\omega}^2 - \bar{\alpha}^2 \tilde{\Gamma}) I_1(s\bar{r})}{s^2 [sI_0(s) + 2\nu I_1(s) + sI_2(s)]} + (\tilde{\omega}^2 - \bar{\alpha}^2 \tilde{\Gamma}) \bar{r}/s^2 \quad (22)$$

Referring to the solution⁽¹⁾ obtained by the convolution and to the Cauchy integral theorem, the inversion of $\tilde{u}$ is carried out as follows:

$$\bar{u}(\bar{r}, \bar{t}) = 4(1+\nu) \sum_{n=1}^{\infty} \frac{J_1(\lambda_n \bar{r})}{\Phi(\lambda_n, \nu)} \int_0^{\bar{t}} [\bar{\omega}^2(\bar{t}-\xi) - \bar{\alpha}^2 \Gamma(\bar{t}-\xi)] \sin(\lambda_n \xi) d\xi \quad (23)$$

The stress components are represented by

$$\bar{r}_{\theta} = 0$$

$$\bar{\sigma}_r = \frac{2}{1-\nu} \sum_{n=1}^{\infty} \frac{\lambda_n J_0(\lambda_n \bar{r}) + (2\nu/\bar{r}) J_1(\lambda_n \bar{r}) - \lambda_n J_2(\lambda_n \bar{r})}{\Phi(\lambda_n, \nu)} \times \int_0^{\bar{t}} [\bar{\omega}^2(\bar{t}-\xi) - \bar{\alpha}^2 \Gamma(\bar{t}-\xi)] \sin(\lambda_n \xi) d\xi \quad (24)$$

$$\bar{\sigma}_{\theta} = \frac{2}{1-\nu} \sum_{n=1}^{\infty} \frac{\nu \lambda_n J_0(\lambda_n \bar{r}) + (2/\bar{r}) J_1(\lambda_n \bar{r}) - \nu \lambda_n J_2(\lambda_n \bar{r})}{\Phi(\lambda_n, \nu)} \times \int_0^{\bar{t}} [\bar{\omega}^2(\bar{t}-\xi) - \bar{\alpha}^2 \Gamma(\bar{t}-\xi)] \sin(\lambda_n \xi) d\xi \quad (25)$$

where the function $\Phi(\lambda_n, \nu)$ used in Eqs.(23)~(25) is defined by

$$\Phi(\lambda_n, \nu) = -\lambda_n^2(3+\nu)J_0(\lambda_n) - \lambda_n(4\nu-3\lambda_n^2/2)J_1(\lambda_n) + \lambda_n^2(3+\nu)J_2(\lambda_n) - \lambda_n^3 J_3(\lambda_n)/2 \quad (26)$$

Here, λ_n is the n -th positive root satisfying the characteristic equation.

$$\lambda_n J_0(\lambda_n) + 2\nu J_1(\lambda_n) - \lambda_n J_2(\lambda_n) = 0, \quad n = 1, 2, \dots, \infty \quad (27)$$

4. Cyclic Rotating Process

This report adopts the cyclically-varying rotating process given by Eq.(1). Employing the non-dimensional quantities, Eq.(1) is rewritten by

$$\bar{\omega}(\bar{t}) = \bar{\Omega}_1 \sin(\bar{\Omega}_2 \bar{t}) \quad (28)$$

Substituting Eq.(28) into Eqs.(23) and (25), the convolution integral which is included in the stress and displacements, are calculated as follows:

$$C = \int_0^{\bar{t}} [\bar{\omega}^2(\bar{t}-\xi) - \bar{\alpha}^2 \Gamma(\bar{t}-\xi)] \sin(\lambda_n \xi) d\xi$$

$$= \frac{\bar{\Omega}_1^2}{2} \left\{ \frac{1 - \cos(\lambda_n \bar{t})}{\lambda_n} + \frac{\cos(\lambda_n \bar{t}) - \cos(2\bar{\Omega}_2 \bar{t})}{2} \right.$$

$$\times \left[\frac{1}{\lambda_n + 2\bar{\Omega}_2} + \frac{1}{\lambda_n - 2\bar{\Omega}_2} \right] \left. - \bar{\alpha}^2 \frac{\bar{\Omega}_1^2}{2} \left\{ -\frac{1 - \cos(\lambda_n \bar{t})}{\lambda_n} \right. \right.$$

$$- [\cos(\lambda_n \bar{t}) - \cos(\bar{\Omega}_2 \bar{t})] \left[\frac{1}{\lambda_n + \bar{\Omega}_2} + \frac{1}{\lambda_n - \bar{\Omega}_2} \right] \left. + \frac{\cos(\lambda_n \bar{t}) - \cos(2\bar{\Omega}_2 \bar{t})}{2} \left[\frac{1}{\lambda_n + 2\bar{\Omega}_2} + \frac{1}{\lambda_n - 2\bar{\Omega}_2} \right] \right\} \quad (29)$$

The second term in Eq.(29) corresponds to the influence of the circumferential deformation, while, the characteristic values derived from Eq.(27) give the non-dimensional natural frequencies of the in-plane vibrations for the rotating solid discs⁽⁶⁾⁽⁷⁾. For the case in which the circumferential deformations are neglected, the radial displacement $\bar{u}$ and the radial and circumferential stresses diverge when the circular frequency of the cyclic variable rotating process given by Eq.(1) attains one-half of the natural frequency of the in-plane vibration; that is

$$\bar{\Omega}_2 = \lambda_n/2 \quad (30)$$

Therefore, $\lambda_n/2$ becomes the resonance circular frequency. However, for the case which takes into consideration the influence of the circumferential deformation, the resonance frequency takes the value

$$\bar{\Omega}_2 = \lambda_n \quad (31)$$

in addition to ones given by Eq.(30).

5. Computational Results

Numerical computations are carried out for a disc with an outer radius of $a=0.15$ m. The disc material is SNCM8, whose properties are $E=205.95$ GPa (2.1×10^6 kgf/cm²), $\rho=7.702 \times 10^3$ kg/m³ (0.786×10^{-5} kgf·s²/cm⁴) and $\nu=0.278$. The amplitude of the cyclic rotating process in Eq.(28) is given by $\bar{\Omega}_1=0.29$ 2×10^{-2} (104.7 rad/s, which corresponds to $N=1000$ rpm). The time-responses of displacement and stress are computed for the various values of $\bar{\Omega}_2$.

The minimum eigenvalue is

$$\lambda_1 = 2.045 \quad (32)$$

which is obtained from Eq.(27).

Figure 2 shows the time-response of the radial displacement at the outer boundary for $\bar{\Omega}_2=0.5$ ($\zeta=171.3$). In this figure, the solid line indicates the displacement given by Eq.(23) and the dotted line gives the displacement which neglects the influence of circumferential displacement; that is,

$$\bar{u}' \equiv \{\text{set } \bar{\alpha}=0 \text{ in Eq.(23)}\} \quad (33)$$

Fig. 2 Time-response of radial displacement for $\bar{\Omega}_2=0.5$

Fig. 3 Time-response of radial displacement for $\bar{\Omega}_2=0.8$

The double-dotted line indicates the displacement derived from the quasi-static theory. The displacement without the influence of the circumferential deformation changes with a cycle similar to the quasi-static one. However, the displacement $\bar{u}$ with the influence of circumferential deformation varies considerably from $\bar{u}'$. The maximum amplitude of $\bar{u}'$, without consideration of the circumferential deformation, is $\Delta\bar{u}'=0.218\times 10^{-5}$; and the maximum amplitude with it becomes $\Delta\bar{u}=0.751\times 10^{-5}$, which is 3.445 times larger than the former value. The period of $\bar{u}$ is given by 0.345×10^{-3} s.

Figure 3 shows the time-response of the radial displacement at the outer boundary for $\bar{\Omega}_2=0.8$ ($\zeta=274.1$). Here, $\bar{u}'$ behaves like a "beat oscillation", with a period of 0.435×10^{-3} s, which approximately agrees with that of $\bar{u}$. Though $\bar{u}'$ changes for both the positive and negative sides, the displacement with the influence of circumferential deformation varies almost entirely in the positive side. The maximum amplitude of $\bar{u}'$ is $\Delta\bar{u}'=0.414\times 10^{-5}$, in which case $\Delta\bar{u}$ becomes 0.929×10^{-5} , which is about 1.24 times larger than $\Delta\bar{u}'$. As compared with the case shown in Fig.2, the time-response shown in Fig.3 is subjected to less influence by the circumferential deformation.

The time-response for $\bar{\Omega}_2=0.9$ ($\zeta=308.3$) is shown in Fig.4. Equations (23) and (29) indicate that the circumferential displacement $\bar{v}$ varies with twice the period of the radial displacement $\bar{u}'$ without the influence of circumferential displacement. Therefore, there are two cases in which $\bar{u}'$ and $\bar{v}$ change in phase and out of phase, respectively. Then, the amplitude of $\bar{u}$ becomes alternately large and small. The envelope of its amplitude also behaves as a "beat oscillation" whose period is 0.775×10^{-3} s, which agrees with that of $\bar{u}'$. The maximum amplitude of $\bar{u}'$ and $\bar{u}$ are $\Delta\bar{u}'=0.1321\times 10^{-4}$ and $\Delta\bar{u}=0.205\times 10^{-4}$, respectively. Since the ratio of the two values becomes 0.859, this

Fig. 4 Time-response of radial displacement for $\bar{\Omega}_2=0.9$

means that the effect of circumferential deformation on $\bar{u}$ decreases.

Figure 5 shows the same response for $\bar{\Omega}_2=0.95$ ($\zeta=325.4$) as does Fig.4. The value of $\bar{\Omega}_2$ comes closer to the lowest natural frequency, 1.017, of the in-plane vibration than the previous case. Therefore, the phase of $\bar{u}$ is delayed by 180° , as compared with that of $\bar{u}'$. The peak value of $\bar{u}$ appears at the trough of $\bar{u}'$. The period of "the beat oscillations" of both $\bar{u}$ and $\bar{u}'$ is 1.29×10^{-3} s, while the maximum amplitudes of $\bar{u}'$ and $\bar{u}$ are 0.238×10^{-4} and 0.205×10^{-4} , respectively, the ratio of which becomes approximately 0.86.

Figure 6 shows the time-response of the radial displacement for $\bar{\Omega}_2=1.0$ ($\zeta=325.4$), which is very close to $\lambda_1/2$. For this case, $\bar{u}$ approximately obeys $1/|\lambda_1-2\bar{\Omega}_2|$. The maximum amplitudes of $\bar{u}'$ and $\bar{u}$ are 0.94×10^{-4} and 0.672×10^{-4} , respectively. The ratio of those values is about 0.715, which is lower than for the previous case.

Fig. 5 Time-response of radial displacement for $\bar{\Omega}_2=0.95$

Fig. 6 Time-response of radial displacement for $\bar{\Omega}_2=1.0$

6. Discussion on the Influences of Circumferential Deformation

Substituting Eq.(28) into Eq.(17), the circumferential displacement is obtained as follows:

$$\begin{aligned}\bar{v} &= -\bar{r}\bar{a}^2(\bar{\Omega}_1/\bar{\Omega}_2)[1-\cos(\bar{\Omega}_2\bar{t})] \\ &= -\bar{r}\bar{a}^2[1-\cos(\bar{\Omega}_2\bar{t})]/\zeta\end{aligned}\quad (34)$$

Therefore, the amplitude of $\bar{v}$ decreases as the frequency of the cyclic rotating process increases. This characteristic is included in the second term Eq. (4), which influences the radial displacement.

In the range of $\bar{\Omega}_2=0.5\sim 1.0$, Figs.2~6 showed that the ratio of the maximum amplitudes $\Delta\bar{u}$ and $\Delta\bar{u}'$ decreases with the increase of $\bar{\Omega}_2$. That is, the circumferential deformation reduces the radial displacement by taking into consideration the coupling effects. Figure 7 shows the relation between the non-dimensional circular frequency $\bar{\Omega}_2$ of the cyclic rotating process and the maximum amplitudes of $\Delta\bar{u}$ and $\Delta\bar{u}'$, in logarithmic scale. Here, λ_1 corresponds to the lowest natural frequency of the in-plane vibration. The displacement including the term $1/|\lambda_1-2\bar{\Omega}_2|$ blows up as $\bar{\Omega}_2$ approaches $\lambda_1/2$. The amplitude $\Delta\bar{u}$ increases slightly up to $\bar{\Omega}_2=0.8$, but beyond this value, it increases considerably, while, the maximum amplitude of the displacement without the influence of circumferential deformation grows gradually up to $\bar{\Omega}_2=0.8$. However, it increases rapidly above that value. In other words,

Fig. 7 Influence of circular frequency of cyclic rotation on maximum amplitude of displacement

$$\begin{aligned}\Delta\bar{u} &> \Delta\bar{u}' && \text{for } \bar{\Omega}_2 \leq 0.9 \\ \Delta\bar{u} &< \Delta\bar{u}' && \text{for } \bar{\Omega}_2 > 0.9\end{aligned}\quad (35)$$

The circumferential deformation helps the growth of the radial displacement in the range of small circular frequencies of cyclic rotation. However, as the frequency increases, this trend decreases, and the circumferential deformation influences the decreases of the radial deformation above $\bar{\Omega}_2=0.9$.

In the following, it is discussed why the circumferential deformation constrains the increase of the radial displacement in the high frequency region of the cyclic rotation by using the circumferential displacement $\bar{v}$ given by Eq.(34).

For the case of $\bar{\Omega}_2=0.8$ and 0.95 , Figs.8 and 9 show the variations of the circumferential displacement $\bar{v}$ and the radial ones, $\bar{u}$ and $\bar{u}'$, over the period in which the radial displacement $\bar{u}'$ attains its maximum. It is shown that the order of the circumferential displacement is 10^3 times that of $\bar{u}'$. This fact indicates that the circumferential displacement becomes the most substantial component in the radial displacement $\bar{u}$. The phase of $\bar{u}'$ is opposite to that of $\bar{v}$ around the

Fig. 8 Time-response of radial and circumferential displacement for $\bar{\Omega}_2=0.8$

Fig. 9 Time-response of radial and circumferential displacement for $\bar{\Omega}_2=0.95$

Fig. 10 Variation of $\Delta\bar{u}/\Delta\bar{u}'$ with respect to ζ

time of the maximum amplitude of $\bar{v}$ for $\bar{\Omega}_2=0.8$, as shown in Fig.8. This behavior has the effect in which $\bar{u}'$ causes the decrease of $\bar{v}$.

The time difference at the peaks of $\bar{v}$ and $\bar{u}'$ is 0.014×10^{-3} s, while for $\bar{\Omega}_2=0.9$, it becomes 0.006×10^{-3} s, which is 50% smaller than the previous value. This result indicates that the radial displacement is depressed by the circumferential deformation. As the circular frequency of the cyclic rotation $\bar{\Omega}_2$ approaches $\lambda_1/2$, the phase of $\bar{v}$ is delayed by 180° , as compared with the of $\bar{u}'$. Due to this phenomenon, $\bar{u}'$ depresses the maximum amplitude of $\bar{v}$ and $\Delta\bar{u}'$ becomes larger than $\Delta\bar{u}$ beyond $\bar{\Omega}_2=0.9$.

It is shown, in Figs.8 and 9, that the circumferential displacement v oscillates with twice the period of the radial displacement. This behavior is predicted from the fact that Eq.(29) includes the term $\cos(2\bar{\Omega}_2\bar{t})$ and $\bar{v}$, given by Eq.(34), contains the term $\cos(\bar{\Omega}_2\bar{t})$.

For the parameter ζ defined by Eq.(3), the ratio of $\Delta\bar{u}$ to $\Delta\bar{u}'$ is shown in Fig.10. The value of $\Delta\bar{u}/\Delta\bar{u}'$ is 1.0 for $\zeta=310$ ($\bar{\Omega}_2=0.905$) and above this value for $\Delta\bar{u}'$, the maximum amplitude of the radial displacement without the influence of circumferential deformation becomes larger than $\Delta\bar{u}$.

7. Conclusions

The equation of motion in the radial direction contains a product of the circumferential displacement and the angular acceleration; this product acts as the external forcing term. It is expected that the circumferential displacement conspicuously influences

the radial displacement with increasing angular acceleration. This report discussed the mutual interactions between the circumferential and radial displacements with respect to the angular acceleration. Adopting the cyclic rotating process and computing the time-responses of the displacements for various values of the circular frequency of that process, the following conclusions were obtained.

(1) The radial displacement $\bar{u}$, with consideration of the circumferential displacement, derived from the simple sinusoidal oscillations. This explains why the circumferential displacement oscillates with twice the period of the radial displacement.

(2) The displacement $\bar{u}$ varies almost entirely on the positive side.

(3) The displacement u varies cyclically, and shows a "beat oscillation", whose period gets longer with increasing $\bar{\Omega}_2$, and approaches that of $\bar{u}'$.

(4) For the non-dimensional parameter ζ given by Eq.(3), the maximum amplitudes, $\Delta\bar{u}$ and $\Delta\bar{u}'$, appeared in a period of the beat oscillation, and are given by the following relations.

$$\Delta\bar{u} > \Delta\bar{u}' \quad \text{for } \zeta < 310$$

$$\Delta\bar{u} < \Delta\bar{u}' \quad \text{for } \zeta > 310$$

In other words, below $\zeta=310$, the circumferential deformation pulls up the radial displacement, and the situation becomes the opposite above $\zeta=310$.

The ratio of $\Delta\bar{u}$ to $\Delta\bar{u}'$ has the value 3.5 for $\zeta=170$, and decreases with the increase of ζ . Moreover, it attains a value less than 1.0 beyond $\zeta=310$.

(5) The previous result was anticipated from the following fact. The phase of the circumferential displacement is delayed 180° , as compared with that of $\bar{u}'$. Therefore, both the displacements cancelled each other out.

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