Magnetic and Gravitational Natural Convection of Melted Silicon—Two-Dimensional Numerical Computations for the Rate of Heat Transfer*

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The two-dimensional natural convection of fluid under both a magnetic and a gravitational field was modeled by conservation equations. Sample computations were carried out for the fluid in a square enclosure for Rayleigh number of from 10^4 to 10^6 , for Hartman number of from 1 to 10^3 and for Prandtl number equal to 0.054, equivalent to melted silicon. The numerical computations converged successfully and the Nusselt numbers obtained were correlated to give an empirical equation for the rate of heat transfer. The steady state solutions were graphically visualized. At $Ha=10^3$ and $Ra=10^6$, the point at which the temperature profile was almost linear, the flow was almost suppressed and elongated in a regime with high wave numbers.

Key Words: Magnetic Field, Low Prandtl Number, Melted Silicon, Heat Transfer Rate, Natural Convection

Nomenclature

A : abbreviated A_z , see Eq. (25)
 $\vec{A}$: dimensionless magnetic vector potential
 A_z : dimensionless Z -component of vector potential in a magnetic field
 $\vec{B}$: magnetic induction (magnetic field)
 [Weber $\cdot$ m $^{-2}$] = [volt $\cdot$ s $\cdot$ m $^{-2}$]
 B_0 : external magnetic field [Weber $\cdot$ m $^{-2}$]
 $B_x = b_x/B_0$
 $B_y = b_y/B_0$
 b : height of an enclosure [m]
 b_x : x -component of a magnetic field [Weber $\cdot$ m $^{-2}$]
 b_y : y -component of a magnetic field [Weber $\cdot$ m $^{-2}$]
 c : specific heat [J/(kg K)]
 c_1 : a constant value for determining the grid size
 c_2 : a constant value for determining the grid size
 d_1 : a constant value for determining the grid size

d_2 : a constant value for determining the grid size
 $\vec{D}$: electric displacement [coul $\cdot$ m $^{-2}$]
 $\vec{E}$: electric field intensity [volt $\cdot$ m $^{-1}$] = [N $\cdot$ coul $^{-1}$]
 g : acceleration due to gravity [m/s 2]
 Gr : Grashof number = $g\beta(\theta_h - \theta_c)l^3/\nu^2$
 $\vec{H}$: magnetic field intensity [AT $\cdot$ m $^{-1}$]
 Ha : Hartman number = $\left(\frac{\sigma_0}{\rho\nu}\right)^{1/2} B_0 l$
 $\vec{J}$: electric current density [A $\cdot$ m $^{-2}$]
 k : thermal conductivity of the fluid [J/(m $\cdot$ s $\cdot$ K)]
 $L = l/x_0 = Ra^{1/3}$
 l : distance between vertical hot and cold walls [m]
 Nu : an average Nusselt number = $ql/[k(\theta_h - \theta_c)]$
 P : pressure [N/m 2]
 Pr : Prandtl number of the fluid = ν/α
 Pr_m : magnetic Prandtl number = ν_m/α
 q : heat flux density [J/(m 2 $\cdot$ s)]
 Ra : Rayleigh number = $Gr \cdot Pr$
 T : dimensionless temperature
 = $(\theta - \theta_0)/(\theta_h - \theta_c)$
 t : time [s]
 $\vec{u}$: velocity vector [m $\cdot$ s $^{-1}$]
 u : x -component of velocity [m/s]

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$$U = u/u_0$$

$$v : y\text{-component of velocity [m/s]}$$

$$V = v/v_0$$

$$X = x/x_0$$

$$x : \text{coordinate [m]}$$

$$Y = y/y_0$$

$$y : \text{coordinate [m]}$$

Greek letters

$$\alpha : \text{thermal diffusivity [m}^2\text{/s]}$$

$$\beta : \text{volumetric coefficient of expansion [K}^{-1}\text{]}$$

$$\epsilon : \text{dielectric constant [farad}\cdot\text{m}^{-1}\text{]} = [\text{coul}\cdot\text{volt}^{-1}\cdot\text{m}^{-1}]$$

$$\xi : \text{dimensionless vorticity}$$

$$\eta : \text{dummy coordinate}$$

$$\theta : \text{temperature [K]}$$

$$\theta_c : \text{temperature of the cold wall [K]}$$

$$\theta_h : \text{temperature of the hot wall [K]}$$

$$\mu : \text{viscosity of the fluid [kg/(m s)]}$$

$$\mu_m : \text{magnetic permeability [henry}\cdot\text{m}^{-1}\text{]}$$

$$\nu : \text{kinematic viscosity [m}^2\text{/s]}$$

$$\nu_m : \text{coefficient of magnetic viscosity} \\ = 1/(\sigma_0 \mu_m) [\text{ohm}\cdot\text{m}^2\cdot\text{henry}^{-1}]$$

$$\xi : \text{dummy coordinate}$$

$$\rho : \text{density of fluid [kg/m}^3\text{]}$$

$$\rho_0 : \text{electric charge density of fluid [coul}\cdot\text{m}^{-3}\text{]}$$

$$\sigma_0 : \text{electric conductivity [ohm}^{-1}\cdot\text{m}^{-1}\text{]}$$

$$\tau : \text{dimensionless time} = t/t_0$$

$$\Phi : \text{temperature difference from conduction state}$$

$$\Phi_c : \text{conduction temperature profile}$$

$$\psi : \text{stream function [m}^2\text{/s]}$$

$$\Psi : \text{dimensionless stream function} = \psi/\alpha$$

Operator

$$D/D\tau = \partial/\partial\tau + U\partial/\partial X + V\partial/\partial Y$$

$$\nabla^2 = \partial^2/\partial X^2 + \partial^2/\partial Y^2$$

1. Introduction

Electro-conducting fluid such as liquid metal and plasma receives a Lorentz force under a magnetic field when the fluid moves and the angle of the flow direction is tilted. This is also true when the fluid convects in a gravitational field under non-uniform thermal boundary conditions. The convective rates of heat transfer and mass transfer would be affected by both the gravitational and the magnetic fields. This kind of situation is realized in the process of manufacturing crystals such as crystal silicon and Gallium Arsenide. It is important to clarify the detailed characteristics of transport phenomena in such processes so that well controlled, good quality products can be developed with improved design in the manufacturing and operational processes.

The natural convection of liquid metal has been studied by, for example, Hunt & Wilks (1). They studied the convection of a fluid at a low Prandtl

number around a semi-infinite vertical plate under a strong cross magnetic field. They reported an increase in the rate of heat transfer near a leading edge as the Prandtl number decreased.

Nagase et al.⁽²⁾ measured the rate of heat transfer for various liquid metals over a heated vertical plate and plotted the Nusselt number against $2Ha^2/Gr_x^{1/2}$. The plot of the data itself was widely scattered, reflecting the difficulty of the experiment and the complexity of the phenomena. They also reported complicated perturbations of temperature at 0.1 tesla rather than at zero.

For liquid metals in a rectangular box under an external magnetic field, Tabeling (3) studied the natural convection with a linear theory. He discussed the relationship between the flow patterns of the Benard cells and the direction of the magnetic field. Raptis & Vlahos⁽⁴⁾ studied the combined effect of the magnetic and the gravitational natural convection in porous media, which is related to the geographical phenomena. The effects of various parameters was discussed in terms of the boundary layer approximation.

Maekawa and Tanasawa⁽⁵⁾ studied the combined effects of the force of surface tension, that of the magnetic field and that of the gravity field for a horizontal fluid layer with a linear theory. They reported a simple relationship, including both the critical Marangoni and the critical Rayleigh numbers. Their work was motivated for the manufacturing of semi conducting material under a micro gravity field.

All these works have limitations, owing to either boundary layer approximation and/or linear theory. The work by Weiss⁽⁶⁾ is more closely related to the present work. He carried out a linear and a numerical analysis for the natural convection of fluid under both magnetic and gravitational fields. He studied the perturbed convection near a critical state, i. e., when Ra was about 6 500 or less. His work is, however, limited to slip boundaries which are physically unrealizable, and is further limited to a Prandtl number equal to unity, for which magnetic force is hardly effective.

In the present work, the natural convection of the fluid at $Pr=0.054$ (melted silicon) was numerically calculated under both magnetic and gravitational fields for a two-dimensional flow with rigid boundaries.

2. Mathematical Model

The mathematical model for natural convection of the fluid consists of the balance equations for mass, energy and momentum. For the convection in a magnetic field, the external force term includes a Lorenz force as well as a buoyant force. According to

text books on Electro-magnetics⁽⁷⁾⁽⁸⁾, the following relationships should be solved to determine the magnetic field.

$$\text{rot } \vec{E} + \partial \vec{B} / \partial t = 0 \quad (1)$$

$$\text{rot } \vec{H} - \partial D / \partial t = \vec{J} \quad (2)$$

$$\text{div } \vec{D} = \rho_0 \quad (3)$$

$$\text{div } \vec{B} = 0 \quad (4)$$

$$\vec{D} = \epsilon \vec{E} \quad (5)$$

$$\vec{B} = \mu_m \vec{H} \quad (6)$$

$$\vec{J} = \rho_0 \vec{u} + \sigma_0 (\vec{E} + \vec{u} \times \vec{B}) \quad (7)$$

Equations (1) to (4) are the Maxwell equations for an electro-magnetic field, and Eqs. (5) and (6) relate the electric field to the magnetic field. Equation (7) is Ohm's law. However, these equations can be simplified under a number of assumptions. First, the displacement current $\partial D / \partial t = 0$, because the propagation of electromagnetic waves is not considered. The fluid is assumed to be electrically neutral and the convective term $\rho_0 \vec{u}$ is neglected. These equations are finally reduced to the next equation of induction for a magnetic field $\vec{B}^{(7)}$:

$$\frac{\partial \vec{B}}{\partial t} = \text{rot}(\vec{u} \times \vec{B}) - \text{rot}\left(\frac{1}{\sigma_0} \text{rot } \frac{\vec{B}}{\mu_m}\right) \quad (8)$$

The momentum equation, including a Lorentz force in the magnetic field, is given as:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \rho \vec{g} - \vec{B} \times \text{rot}\left(\frac{\vec{B}}{\mu_m}\right) + \mu \nabla^2 \vec{u} \quad (9)$$

The energy and the mass balance equations are as follows:

$$\rho c \frac{D\theta}{Dt} = k \nabla^2 \theta \quad (10)$$

$$\nabla \cdot \vec{u} = 0 \quad (11)$$

These equations are simplified to two-dimensional ones, and a Boussinesq approximation was introduced that the physical properties other than the density in the buoyant term are constant. The two momentum equations are cross-differentiated and subtracted from each other to give a vorticity equation. After the dedimensionalization, the following equations are obtained:

Energy equation:

$$\frac{DT}{D\tau} = \nabla^2 T \quad (12)$$

Induction equations in the X and Y coordinates:

$$\frac{D\vec{B}}{D\tau} - \left(B_x \frac{\partial \vec{U}}{\partial X} + B_y \frac{\partial \vec{U}}{\partial Y} \right) = Pr_m \nabla^2 \vec{B} \quad (13), (14)$$

Maxwell equation:

$$\nabla \cdot \vec{B} = 0 \quad (15)$$

Vorticity equation:

$$\begin{aligned} \frac{D\xi}{D\tau} = & -Pr \frac{\partial T}{\partial Y} + Pr \nabla^2 \xi \\ & + Ha^2 Ra^{-2/3} Pr Pr_m \left\{ B_x \frac{\partial}{\partial X} \left(\frac{\partial B_y}{\partial X} - \frac{\partial B_x}{\partial Y} \right) \right. \\ & \left. + B_y \frac{\partial}{\partial Y} \left(\frac{\partial B_y}{\partial X} - \frac{\partial B_x}{\partial Y} \right) \right\} \end{aligned} \quad (16)$$

Definition of the vorticity:

$$\xi = -\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} = -\nabla^2 \psi \quad (17)$$

Definition of the stream function:

$$U = \frac{\partial \psi}{\partial Y} \quad (18)$$

$$V = -\frac{\partial \psi}{\partial X} \quad (19)$$

The dimensionless variables are defined as follows:

$$X = x/x_0, \quad Y = y/y_0, \quad U = u/u_0, \quad V = v/v_0, \quad \tau = t/t_0,$$

$$T = \frac{\theta - \theta_0}{\theta_h - \theta_c}, \quad B_x = b_x/B_0, \quad B_y = b_y/B_0,$$

$$x_0 = y_0 = \left[\frac{g\beta(\theta_h - \theta_c)}{\alpha\nu} \right]^{-1/3}, \quad u_0 = v_0 = \alpha/x_0,$$

$$t_0 = x_0^2/\alpha, \quad Ha = \left(\frac{\sigma_0}{\rho\nu} \right)^{1/2} B_0 l, \quad \theta_0 = (\theta_h + \theta_c)/2,$$

$$Pr_m = \frac{\nu_m}{\alpha}, \quad Pr = \frac{\nu}{\alpha}$$

$$L = l/x_0 = Ra^{1/3}, \quad Ra = g\beta(\theta_h - \theta_c)l^3/(\alpha\nu)$$

In a primary state of computation, the magnetic field B_x and B_y could have been solved successfully, but they failed to satisfy another requirement, Eq. (15). This is because B_x and B_y were not solved to satisfy Eq. (15). Weiss⁽⁶⁾ employed a vector potential, $\vec{A}$, in the magnetic field, and we followed his approach. It is defined by the next equation

$$\vec{B} = \nabla X \vec{A} \quad (20)$$

The magnetic vector potential satisfies the next equation

$$\frac{D\vec{A}}{D\tau} = Pr_m \nabla^2 \vec{A} \quad (21)$$

The induction equations for a magnetic field, Eqs. (13) and (14), can be derived by taking derivatives of Eq. (21) either by Y or X , respectively. For a two-dimensional field, $B_z = 0$ and $\partial/\partial Z = 0$ hold, and then we get:

$$B_x = \frac{\partial A_z}{\partial Y} \quad (22)$$

and

$$B_y = -\frac{\partial A_z}{\partial X} \quad (23)$$

The Maxwell relationship for the magnetic field is satisfied automatically as follows:

$$\nabla \cdot \vec{B} = \frac{\partial B_x}{\partial X} + \frac{\partial B_y}{\partial Y} = \frac{\partial^2 A_z}{\partial X \partial Y} - \frac{\partial^2 A_z}{\partial Y \partial X} = 0 \quad (24)$$

The Z -component of the magnetic vector potential is abbreviated to A and the next equation to be solved is:

$$\frac{\partial A}{\partial \tau} + U \frac{\partial A}{\partial X} + V \frac{\partial A}{\partial Y} = Pr_m \left(\frac{\partial^2 A}{\partial X^2} + \frac{\partial^2 A}{\partial Y^2} \right) \quad (25)$$

The solution scheme is as follows:

- 1) Temperature from Eq. (12).
- 2) The vorticity and the stream function from Eqs. (16) and (17).
- 3) The velocity components from Eqs. (18) and (19).

- 4) The magnetic vector potential from Eq. (25).
- 5) Two components of the magnetic field from Eqs. (22) and (23).

The finite difference approximation and the computational scheme are the same as the one reported by Ozoe et al.⁽¹⁰⁾ and would not be repeated herein. At the beginning, the flow field is solved for a non-magnetic condition. This provides an initial condition for the computation of the magnetic field. The transient computation was continued for a convergence.

For a fluid at a low Prandtl number, the thermal conductivity is large and the temperature becomes a linear profile which results in inaccurate values of the temperature gradients. When $Pr=0.054$, the variable

for temperature was changed from T to Φ as follows, where Φ_c represents a linear temperature profile as defined in Eq. (27) :

$$T = \Phi_c + \Phi \quad (26)$$

$$\Phi_c = 0.5 - (Y/Ra^{1/3}) \quad (27)$$

The equations (12) and (16) become as follows :

$$\frac{D\Phi}{D\tau} - \frac{V}{Ra^{1/3}} = \nabla^2 \Phi \quad (28)$$

$$\begin{aligned} \frac{D\xi}{D\tau} = & -Pr \frac{\partial \Phi}{\partial Y} + \frac{Pr}{Ra^{1/3}} + Pr \nabla^2 \xi \\ & + Ha^2 Ra^{-2/3} Pr Pr_m \left\{ B_x \frac{\partial}{\partial X} \left(\frac{\partial B_y}{\partial X} - \frac{\partial B_x}{\partial Y} \right) \right. \\ & \left. + B_y \frac{\partial}{\partial Y} \left(\frac{\partial B_x}{\partial X} - \frac{\partial B_y}{\partial Y} \right) \right\} \end{aligned} \quad (29)$$

The boundary condition for Φ becomes as follows :

$$\Phi = 0 \text{ at } Y=0 \text{ and } Ra^{1/3}$$

Eqs. (28) and (29) were employed for $Pr=0.054$.

3. The System Studied and the Boundary Conditions

Figure 1 shows the system considered in this report. The fluid in a square enclosure is heated from one vertical side wall at θ_h and is cooled from an opposing wall at θ_c , and is insulated along the horizontal

Table 1 Coordinates of grid points employed for the computation.

no.	X_i	Y_i
1	0.0000	0.0000
2	0.0206	0.0073
3	0.0437	0.0167
4	0.0696	0.0288
5	0.0986	0.0444
6	0.1311	0.0644
7	0.1675	0.0902
8	0.2084	0.1234
9	0.2542	0.1661
10	0.3056	0.2211
11	0.3631	0.2918
12	0.4277	0.3829
13	0.5000	0.5000
14	0.5723	0.6171
15	0.6369	0.7012
16	0.6944	0.7789
17	0.7458	0.8339
18	0.7916	0.8766
19	0.8325	0.9098
20	0.8689	0.9356
21	0.9014	0.9556
22	0.9305	0.9712
23	0.9563	0.9833
24	0.9794	0.9927
25	1.0000	1.0000

Fig. 1 Thermal boundary conditions of the system. The fluid is heated and cooled isothermally from the opposing vertical walls and is insulated along the top and bottom walls.

Fig. 2 Computed grid size. The coordinates of the grid are listed in Table 1.

tal boundaries. These boundaries are made of rigid walls. Figure 2 shows unequal 24 grid divisions both in the X - and in the Y -directions. Table 1 lists the coordinates of these grid points which can be computed by the following equations:

$$\begin{aligned} X &= (e^{\xi/c_2} - 1)/c_1 & \text{at } 0 \leq \xi \leq L/2 \\ &= (1 + c_1 - e^{\xi/c_2})/c_1 & \text{at } L/2 \leq \xi \leq L \end{aligned} \quad (30)$$

$$\begin{aligned} Y &= (e^{\eta/d_2} - 1)/d_1 & \text{at } 0 \leq \eta \leq L/2 \\ &= (1 + d_1 - e^{\eta/d_2})/d_1 & \text{at } L/2 \leq \eta \leq L \end{aligned} \quad (31)$$

where

$$c_1 = 5.8765, \quad c_2 = 0.36476$$

$$d_1 = 39.283, \quad d_2 = 0.16516$$

The boundary conditions are given as follows.

1) Temperature

$$T = 0.5 \quad \text{at } Y = 0$$

$$T = -0.5 \quad \text{at } Y = L$$

$$\partial T / \partial X = 0 \quad \text{at } X = 0 \text{ and } L$$

2) Velocity

$$U = V = 0 \quad \text{at } X = 0 \text{ and } L$$

$$\text{at } Y = 0 \text{ and } L$$

3) Stream function

$$\psi = 0 \quad \text{at } X = 0 \text{ and } L$$

$$\text{at } Y = 0 \text{ and } L$$

4) Vorticity

$$\zeta = \partial V / \partial X \quad \text{at } X = 0 \text{ and } L$$

$$\zeta = -\partial U / \partial Y \quad \text{at } Y = 0 \text{ and } L$$

5) Magnetic vector potential

$$A = 0 \quad \text{at } Y = 0$$

$$A = Ra^{1/3} \quad \text{at } Y = L$$

$$A = Y \quad \text{at } X = 0 \text{ and } L$$

6) Magnetic field

$$B_x = 1 \quad \text{and } \partial B_y / \partial X = 0 \quad \text{at } X = 0 \text{ and } L$$

$$B_y = 0 \quad \text{and } \partial B_x / \partial Y = 0 \quad \text{at } Y = 0 \text{ and } L$$

The boundary conditions for the magnetic field were determined as follows. $B_x = 1$ was assigned for the uniform external magnetic field. Since $B_x = \partial A / \partial Y$, $A = Y$ holds at $X = 0$ and L . $A = 0$, when $Y = 0$, was arbitrarily set as a constant. These gave $A = L$ when $Y = L$. $B_y = -\partial A / \partial X$ becomes zero owing to the

$$B_x = 1, \quad A = Y$$

Fig. 3 Magnetic boundary conditions for dimensionless values. The uniform magnetic field is given along the top and bottom surfaces.

condition $A = \text{constant}$ at $Y = 0$ and L . When $Y = 0$, $\partial^2 A / \partial X^2 = 0$, and then $\partial^2 A / \partial^2 Y = \partial B_x / \partial Y = 0$, since when $Y = 0$, $U = V = 0$ and $\partial A / \partial \tau = 0$ at a steady state in Eq. (25). However, B_x at $Y = 0$ and L , and B_y at $X = 0$ and L , may be computed with Eq. (22) by the extrapolation from inside the fluid regime to establish accurate transient results. These are shown in Fig. 3.

4. Computed Results

4.1 The natural convection of the fluid at a low Prandtl number without a magnetic field

The Prandtl number of the electro-conducting fluid is generally small. The thermal conductivity of a low-Prandtl number fluid is usually large, and therefore the convection rate of such a fluid becomes large, even with a small difference in temperature. The computation of such strong natural convection becomes difficult. Bertin & Ozoe⁽⁹⁾ recently computed the natural convection of a fluid at low Prandtl number with the variables shown by Eqs. (26) and (27). Ozoe et al.⁽¹⁰⁾ has reported steady state convection at $Ra = 10^6$ and $Pr = 9.17$. In this work, the computation was started from their solution, and proceeded to those at $Pr = 1$ and 0.1 successively. The transient responses of the central stream function ψ_c are shown in Fig. 4. For $Pr = 0.054$, Eqs. (28) and (29) were

Fig. 4 Transient responses of the central stream function after a step change of the Prandtl number at $Ra = 10^6$ for a non-magnetic field.

Table 2 The effect of the Prandtl number on the Nusselt number and the central stream function at $Ra = 10^6$ without a magnetic force.

Pr	Nu	Nu*	ψ_c	ψ_c / Pr
9.17	9.38	10.859	21.745	2.371
1	9.36	9.309	17.385	17.385
0.1	7.00	6.697	12.169	121.69
0.054	6.08	5.960	10.506	194.55

Nu^* = Correlation equation by Churchill⁽¹¹⁾.

employed to solve for temperature, where temperature is a difference from that at a conduction state. The steady state solution was successfully obtained for these conditions. The contour maps of the temperature and the stream function are shown in Figs. 5(a) to 5(d). The effect of the Prandtl number is apparent at $Pr=0.054$. The central values of the stream function are listed in Table 2. When the Prandtl number decreases, the central value of the stream function divided by the kinetic viscosity, ψ_c/Pr increases very much. This suggests strong convection for a low value of the Prandtl number.

On the other hand, the average Nusselt number decreases from 9.38 to 6.09. Churchill⁽¹¹⁾ obtained an empirical equation, for an average Nusselt number, of natural convection in a rectangular enclosure for all ranges of the Prandtl number. The average Nusselt numbers computed by his equation are also listed for each value of the Prandtl number and the agreement is satisfactory, reconfirming these results. At $Pr=0.054$, Nu by this work and Nu^* by the Churchill correlation differ 2%. The accuracy of the average

Nusselt number computed in this work is expected to be at the same order of magnitude as this.

4.2 Natural convection of the fluid at $Pr=0.054$ under a magnetic field of various strengths

The system parameters are the Rayleigh number Ra , the Prandtl number Pr , the Hartman number Ha and the magnetic Prandtl number Pr_m . For the sake of simplicity, $Pr=0.054$ and $Pr_m=1$ were fixed. The computations were carried out for $Ra=10^6$ and Ha equal to 1, 10, 10^2 and 10^3 , respectively. Figure 6 shows the transient responses of the central value of the stream function. The response for $Ha=10^3$ is shown in Fig. 7. This final response differs significantly from the previous ones, on the point that the central value of the stream function oscillates between +50 and -90 or less. This means that the direction of the convection oscillates alternately, i.e., clockwise and anti-clockwise. This oscillation dampens slowly almost to zero after a long time. The average Nusselt number, the central stream function and the other properties are listed in Table 3 for these combinations of the parameters. The average Nusselt number decreases

Fig. 5 Computed isothermal lines and contour maps of the stream function at $Ra=10^6$.

with an increase of the Hartman number, which has the effect of suppressing the convection. The average Nusselt number increases slightly at $Ha=10$, at which the computation was started from that at $Ha=1$. Although the convergence may not be perfect, the

Curve 1 is for a steady state at $Ha=1$.
Curve 2 is for a steady state at $Ha=10$.
Curve 3 is for a steady state at $Ha=100$.

Fig. 6 Transient responses of the central stream function after a step change of the Hartman number at $Ra=10^6$, $Pr=0.054$ and $Pr_m=1$.

computation was discontinued to save computing time.

According to the linear stability theory for the horizontal fluid layer⁽¹²⁾⁽¹³⁾, convection is suppressed under the following conditions:

$$Ra < \pi^2 Ha^2 \quad (32)$$

At $Ra=10^6$, it becomes $318 < Ha$. In Table 3, the average Nusselt number decreases to almost unity between $Ha=100$ and 10^3 , which is in accordance with the above condition.

4.3 Correlation for the rate of heat transfer

Further computations were carried out at $Ra=10^5$ and $Ha=0, 33.333$ and 100 , and $Ra=10^4$ and $Ha=$

Fig. 7 A transient response of the central stream function for a steady state at $Ha=1000$, $Ra=10^6$, $Pr=0.054$ and $Pr_m=1$.

Table 3 The effect of the Hartman number on the central stream function and the average Nusselt number at $Ra=10^6$, $Pr=0.054$ and $Pr_m=1$.

Ha	Nu	ψ_c	(Bx)max	(Bx)min	(By)max	(By)min	Umax
1	6.078	10.499	6.53	0.0574	0.955	-4.51	1.233
10	6.094	10.473	6.52	0.0607	0.955	-4.55	1.233
100	4.622	6.552	4.17	0.287	1.188	-4.31	0.879
1000	1.004	0.206	1.05	0.957	0.0355	-0.108	0.0369

Table 4 Computed results to correlate the average Nusselt number at $Pr_m=1$ and $Pr=0.054$.

Ra	Ha	Nu	$[Nu-1]/[Nu(Ha=0)-1]$	$Ha/Ra^{1/3}$	Ha^2/Ra
10^4	0	1.850	1	0	0
	100	1.004	0.0047	4.6416	1
10^5	0	3.304	1	0	0
	33.333	2.907	0.828	0.7181	0.0111
	100	1.262	0.114	2.154	10^{-1}
10^6	0	6.084	1	0	0
	1	6.078	0.999	0.01	10^{-6}
	10	6.094	1.002	0.1	10^{-4}
	10^2	4.622	0.712	1	10^{-2}
	10^3	1.004	0.001	10	1

100. These results are all listed in Table 4. The average Nusselt numbers are plotted against $\log Ha$ in Fig. 8. The separate curves are the results of different values of the Rayleigh number. Since these curves appear to be similar in their shapes, we can expect to construct a single correlation curve. The Nusselt numbers, reconstructed in a following ratio,

$$\frac{Nu-1}{Nu(Ha=0)-1}$$

are plotted against $\log(Ha^2/Ra)$ in Fig. 9, following Eq. (32). The critical Hartman number from Eq. (32) agrees well with the point extrapolated to $Nu=1$. This appears to support the computed result, in spite of the difference in the boundary conditions. However, the Nusselt numbers for two different systems, which give almost similar values for $\log(Ha^2/Ra)=-2$, do not agree with each other. Then, the Nusselt numbers are plotted against $\log(Ha/Ra^{1/3})$ in Fig. 10. $Ha/Ra^{1/3}$ is a natural combination employed in the vorticity equation (16). It gave a much better single smooth line. This suggests that the Lorentz force term in the momentum equation appears to be the most effective term in comparison with other terms. The combination of the dimensionless parameter Ha^2/Ra was

Fig. 8 Tentative plot of the average Nusselt numbers for both the Hartman number and the Rayleigh number.

Fig. 9 General plot for the average Nusselt numbers against a dimensionless group Ha^2/Ra . The critical value $Ha^2/Ra=1/\pi^2$ is indicated on the abscissa.

obtained from the stability analysis which would be applicable at a critical state. The combination $Ha^2/Ra^{2/3}$ appears to give a better consistent correlation in the convection domain, and this latter one was employed to give a correlation equation for the rate of heat transfer as described next.

Churchill & Usagi⁽¹⁴⁾ reported the correlation scheme to fit the data for those in Fig. 10. Following their method, the next equation was obtained.

$$\frac{Nu-1}{Nu(Ha=0)-1} = 1 - \frac{1}{\left[1 + \left(\frac{2.2Ra^{1/3}}{Ha}\right)^{9.012}\right]^{1/5.62}} \quad (33)$$

where $Nu(Ha=0)$ is the average Nusselt number for a non-magnetic field in a square enclosure. It is given by Churchill⁽¹¹⁾ as follows, as a function of the Rayleigh number, the Prandtl number and the aspect ratio b/l .

$$\begin{aligned} Nu(Ha=0) &= \overline{Nu}_l \\ \overline{Nu}_l^3 &= \left\{ \left[1 + \left(\frac{Ra_l}{4472} \right) \frac{l}{b} \right]^{-8} + \left[\frac{Ra_l(l/b)f(Pr)}{57\phi(Ra_b, b/l)} \right]^{-2} \right\}^{-3/8} \\ &+ \frac{Ra_l f(Pr)}{8000} \end{aligned} \quad (34)$$

where

$$f(Pr) = [1 + (0.5/Pr)^{9/16}]^{-16/9} \quad (35)$$

$$\phi(Ra_b, b/l) = \{1 + 0.231[Ra_b(b/l)^4]^{-1/4}\}^2 \quad (36)$$

At least for a square channel, i.e., $b=l$ and for $Pr_m=1$, the combination of these correlation equations gives an average Nusselt number for a system in which the Prandtl number is equal to 0.054, the Ra is less than 10^6 and the Ha is less than 10^3 . However, these are expected to be applicable for all ranges of the Prandtl number, the Hartman number and the Rayleigh number, as far as the laminar convection prevails.

4.4 Visualization of the flow modes

The computed results can be described with the contour maps of various flow characteristics, temperatures and the magnetic fields. The velocity vectors,

Fig. 10 A more consistent plot for the average Nusselt numbers against a dimensionless group $Ha/Ra^{1/3}$. The correlation curve is given by Eq. (33).

Fig. 11 Computed contour maps at a steady state for $Ha=1$, $Ra=10^6$, $Pr=0.054$ and $Pr_m=1$.

Fig. 12 Computed contour maps at a steady state for $Ha=10$. Other conditions and graphs are the same as those of Fig. 11.

isotherms, iso-stream functions, the magnetic vector potential and the magnetic field at a steady state are graphically shown for $Ha=1, 10, 100$ and 10^3 and at $Ra=10^6$ in Figs. 11-14, respectively. Among these graphs, the stream function at $Ha=10^3$ is quite different from

the others. It contains thin roll cells between the hot and the cold plates. These roll cells were produced one by one in the alternate change in the direction of flow, as seen in Fig. 7, for the central value of the stream function. At the beginning, the flow was in a normal

Fig. 13 Computed contour maps at a steady state for $Ha = 100$. Other conditions and graphs are the same as those of Fig. 11.

Fig. 14 Computed contour maps at a steady state for $Ha = 1000$. Other conditions and graphs are the same as those of Fig. 11.

direction under a gravity field, i. e., clockwise. When a strong magnetic force is applied the direction of flow was inverted gradually (anticlockwise) and its strength peaked with $\psi_c = -90$. Then the flow turns to dampen, and again convects clockwise with $\psi_c = 60$ at its highest strength. After five or six oscillations like these, small vortices appeared along the boundaries and they propagated into the whole regime with further oscillations after this. Many roll cells were propagated in this way. These overshootings of the flow would be resulted because of the inertial force of the fluid. Since this computation was carried out with a finite difference method, much smaller roll cells than grid sizes can not be computed and numbers of roll cells may depend on the grid sizes employed.

Transverse roll cells have been known to appear between two narrow vertical plates at a high Rayleigh number. However, the appearance of such shallow roll cells between a square enclosure has apparently not been computed. Apparently, the only relevance of this result is the picture reported by Lehnert & Little⁽¹⁵⁾ which is listed by Chandrasekhar⁽¹¹⁾. The caption for that picture says "the cells are seen to be elongated in the field direction and to extend across the whole vessel". This description is exactly applicable to those cells reported herein. Chandrasekhar⁽¹¹⁾ and Maekawa & Tanasawa⁽⁵⁾ reported that for a horizontal fluid layer heated from below, the wave number increases slightly with an increase of the Hartman number for the range of $Ha^2 = 100$ or less, for Benard convection. Their results suggest the appearance of many roll cells under a strong magnetic field. However, there appears to be no papers which have shown a possible growing process of the many roll cells, as shown herein. At this Hartman number, the temperature and the magnetic vector potential are almost linear at a steady state.

5. Conclusions

The natural convection of a fluid at $Pr = 0.054$ was successfully computed for $Ha = 0, 1, 10, 10^2$ and 10^3 and at $Ra = 10^6$. Additional computations were for $Ha = 100$ at $Ra = 10^4$, and $Ha = 0, 33.333$ and 100 at $Ra = 10^5$. These converged solutions gave average Nusselt numbers for which a correlation equation was derived as a function of the Hartman number, the Rayleigh number and the Prandtl number at $Pr_m = 1$.

The convection modes were visualized for these conditions. The multiple elongated roll cells appeared in a square regime at $Ha = 10^3$ and $Ra = 10^6$ at an almost conductive state. This has apparently not been computed so far; nor has including its transient growing process.

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