Anisotropic Hardening Law of Plasticity Using an Internal Time Concept* (Deformations of Yield Surface)

Osamu WATANABE**

The present paper proposes a new law of anisotropic hardening of plasticity, using an internal time concept developed by the endochronic theory. This report describes a fundamental aspect of the present law, as well as a comparison with the previous experiment involving tension-torsion problems. The present law can accurately predict the shape change of the yield surface with only two parameters of deformation, and as a special application of the present theory, it can also express the vertex effects of the yield surface.

Key Words: Plasticity, Anisotropic Hardening Law, Constitutive Equation, Endochronic Theory, Internal Time Concept, Yield Surface

1. Introduction

Extensive experimental evidence^{(1)~(6)} has indicated that materials exhibit significant anisotropic hardening due to plastic deformations. Subsequent yield surfaces, which expand/contract and translate in stress space, are also subjected to shape distortions. The shape change of a yield surface affects the plastic behavior of solids and structures when a loading path is curved with a corner. Therefore, the practical and accurate constitutive modeling of anisotropic hardening of materials continues to be a subject of intense research interest. Inelastic constitutive relations presented so far in the literature, are based on the following theories: the polycrystalline model⁽⁷⁾ or slip theory⁽⁸⁾, based on micromechanical considerations; an expression by polynomial functions of a higher order using rational mechanics⁽⁹⁾; and plastic potential theory with anisotropic coefficients^{(10)~(14)}. Detailed surveys for anisotropic hardening may be

found in Ref.(15)~(17).

In constitutive modelings, where the vertex point is formed at a yield surface, the following theories have been applied: polycrystalline theory⁽¹⁸⁾, differential equation of deformation theory⁽¹⁹⁾, phenomenological potential theory⁽²⁰⁾, and tensor algebra based on rational mechanics⁽²¹⁾. The trends for vertex hardening theory may be surveyed in Ref.(22).

Based on the internal time theory developed by Valanis⁽²³⁾, the present authors continue to study the alternate form of the relation between differential stress and strain⁽²⁴⁾, theoretical comparison with other internal variable theories and multi-yield-surface theories⁽²⁵⁾, and a methodology for the finite element elastoplastic analysis of structures subjected to cyclic loading⁽²⁶⁾. The internal time theory^{(23)~(26)} employs the concept of the yield surface, which translates and expands in stress space, but does not deform in shape. The purpose of this work is to propose a new anisotropic hardening law of plasticity, which can describe shape deformations, using an internal time concept. The present report describes a fundamental aspect of the new hardening law, as well as a comparison with the previous experiments involving tension-

* Received 21st January, 1987. Paper No. 86-0461A.

** Institute of Engineering Mechanics, University of Tsukuba, 1-1-1 Ten-nohdai, Ni-ihari, Sakura, Ibaraki, 305, Japan

torsion problems to confirm the validity of the present theory.

2. Theory

Considerations in the present work are restricted to small strains and infinitesimal deformations. For simplicity, we use a Cartesian system of coordinates $x_i (i=1,2,3)$, with basis $e_i (i=1,2,3)$. If $A (=A_{ij}e_ie_j)$ and $B (=B_{ij}e_ie_j)$ are two second order tensors, the notations $A \cdot B = A_{ik}B_{kj}e_ie_j$, $A:B = A_{ij}B_{ij}$ and $\|A\| = (A_{ij}A_{ij})^{1/2}$ are employed.

2.1 Description of yield surface

Let $d\epsilon$, the differential strain, be decomposed into:

$$d\epsilon = d\epsilon^e + d\epsilon^p \quad (1)$$

where the superscripts (e) and (p) denote "elastic" and "plastic" components, respectively. As in the classical pressure-insensitive plasticity, we assume that the "deviatoric" and 'mean' parts of strain rate can be treated separately. We write:

$$d\epsilon = de + (d\epsilon_m)I = (de^e + de^p) + (d\epsilon_m^e + d\epsilon_m^p)I \quad (2)$$

where (de) denotes the deviatoric and $(d\epsilon_m)$ denotes the mean parts of tensor $d\epsilon$. It is noted that $d\epsilon_m = (de : I)/3$, with unit tensor $I (= \delta_{ij}e_ie_j)$. We assume that $d\epsilon_m^p = 0$, and hence $d\epsilon_m = d\epsilon_m^e$.

To describe expansion/contraction, translation and distortion of a yield surface during plastic flow, we select an intrinsic time given by the endochronic theory as a time measure in materials. The intrinsic time measure ζ , which is the so called "equivalent plastic strain length", is defined by the equation:

$$d\zeta = \|de^p\| = (de_i^p de_i^p)^{1/2} \quad (3)$$

Note that $d\zeta$ and hence ζ , like Newtonian time t , are irreversible and non-negative. The differential intrinsic time dz is defined as:

$$dz = \frac{d\zeta}{f(\zeta)} \quad (4)$$

where $f(\zeta)$ is non-negative and $f(0)=1$.

Figure 1 shows the motion of the subsequent yield surface in deviatoric stress space. The deviatoric stresses (S and s) represent the stress states on the initial and subsequent yield surfaces, respectively. The stress point $a(S)$ on the initial surface moves to stress point $d(s)$ on the subsequent surface due to plastic strains de^p during the intrinsic time interval up to z defined in Eq.(4). If a Lagrangian description of the yield surface is used, then the point $d(s)$ on the subsequent surface, whose initial position is at S , is given at time z as:

$$s = s\left(z, \frac{de^p}{dz}[0, z], S\right) \quad (5)$$

where $de^p/dz[0, z]$ denotes the history of plastic strains. The initial yield surface can be assumed to initially expand ($S^{(0)} \rightarrow S^{(1)}$), then translate as a rigid

body ($S^{(1)} \rightarrow S^{(2)}$), and finally deform ($S^{(2)} \rightarrow S^{(3)}$).

The subsequent yield state (s) may be referred back to its initial state by:

$$S = \frac{1}{g}(s - r - q) \quad (6)$$

where g is a measure of change in size (isotropic hardening), r is a location of the yield surface center (kinematic hardening), and q is a measure of change in shape (anisotropic hardening). The endochronic theory of Valanis⁽²³⁾ defines that f in Eq.(4) is identical with g in Eq.(6). In the present work, we express f and g , separately. The quantities g , r and q can be specified as:

$$g = g(z) \quad (7a)$$

$$r = r\left(z, \frac{de^p}{dz}[0, z]\right) \quad (7b)$$

$$q = q\left(z, \frac{de^p}{dz}[0, z], S\right) \quad (7c)$$

Since it is convenient to express q as a function of s on the subsequent yield instead of S on the initial surface, substituting Eq.(6) into Eq.(7c) gives:

$$q = q\left(z, \frac{de^p}{dz}[0, z], \frac{1}{g}(s - r - q)\right) \quad (8)$$

By moving the term q , which appears in the right-hand side of Eq.(8), to the left-hand side of the same equation, we assume the following expression for q as:

$$q = q\left(z, \frac{de^p}{dz}[0, z], \frac{1}{g}(s - r)\right) \quad (9)$$

In Fig.2, the two subsequent yield surfaces $S^{(2)}$ and $S^{(3)}$ are superposed by subtracting r from s , and are normalized by g . The distortion q/g from point c on $S^{(2)}$ to point d on $S^{(3)}$ is measured from the subsequent undeformed yield surface $S^{(2)}$ as a function of $(s - r)/g$, while the isotropic hardening g and kinematic hardening r are measured from the initial yield

Fig. 1 Initial and subsequent yield surfaces

surface $S^{(0)}$.

We employ Mises surface as a criterion of yielding. For an initial Mises surface of $S^{(0)}$, we have:

$$F = \frac{1}{S_y^0} \|S\| - 1 = 0 \quad (10)$$

where S_y^0 represents the size of the initial surface. For the subsequent yield surface $S^{(3)}$, we have the following relation by introducing Eq.(6) into Eq.(10) as:

$$F = \frac{1}{S_y^0 g} \|s - r - q\| - 1 = 0 \quad (11)$$

2.2 Anisotropic hardening law

According to the experiment for aluminum alloy by Phillips and Tang⁽²⁾, it is observed that the subsequent yield surface has a region of high curvature on the forward part of the preloading, and a flattening on the rear part, and no reduction of the width in the direction orthogonal to the loading. In order to incorporate such phenomena into the theory, we can propose a new law of anisotropic hardening of plasticity, in which the following hypotheses are assumed.

(1) For monotonic proportional loading, the distortion increment dq is linearly related to the plastic strain increments de^p . For general loadings, the evolution equations of de^p are represented by the nonlinear hardening in the plastic strain rate, as developed by Chaboche and Rousselier and the present authors⁽²⁴⁾⁻⁽²⁶⁾.

(2) The magnitude of the distortion dq is proportional to the power of the stress in the direction of plastic strain rate de^p . The observed configuration of the subsequent yield surface may be represented accurately by increasing the numbers of terms in the equation.

(3) We assume no changes in the size of the yield surface in the present paper. The measure of the

isotropic hardening in Eq.(7·a) is identical with unity. Hence, $f=1$ in Eq.(4), and $dz=d\zeta$.

Based on the above assumptions, let us superpose the deviatoric plastic strain space on the deviatoric stress space. Since de^p/dz represents the cosine of the coordinate axis, as indicated in Fig. 3, the stress s^p in the direction of de^p can be given as:

$$s^p = \frac{(s-r)}{S_y^0} : \frac{de^p}{d\zeta} \quad (12)$$

The deformation $dq^{(n)}$ of the yield surface, which corresponds to a power of n for the stress s^p can be expressed for the linear hardening law as:

$$dq^{(n)} = 2\mu_0 \rho_i^{(n)}(0) (s^p)^n de^p \quad (13)$$

where μ_0 is the shear modulus and $\rho_i^{(n)}(0)$ is a material constant for anisotropic hardening. The superscript (n) in s^p means power and (n) in dq and ρ_i means a term corresponding to power n .

For a nonlinear hardening law, the deformation increment dq can be given during plastic flow $d\zeta$ as:

$$dq^{(n)} = \sum_i dq_i^{(n)} \quad (14)$$

$$dq_i^{(n)} = 2\mu_0 \rho_{i1}^{(n)} (s^p)^n de^p - \alpha_i^{(n)} q_i^{(n)} d\zeta \quad (15)$$

where $\rho_{i1}^{(n)}$ and $\alpha_i^{(n)}$ are material constants. Equation (14) can be converted into the following integral form:

$$q^{(n)} = 2\mu_0 \int_0^\zeta \rho_i^{(n)} (\zeta - \zeta') (s^p)^n \frac{de^p}{d\zeta'} d\zeta' \quad (16)$$

where

$$\rho_i^{(n)}(\zeta) = \sum_i \rho_{i1}^{(n)} \exp(-\alpha_i^{(n)} \zeta) \quad (17)$$

Total deformation dq can be obtained by superposing each term as:

$$q = \sum_{n=1}^N q^{(n)} \quad (18)$$

We now express $q^{(n)}$ in Eq.(16) for the cases of $n=0, 1$ and 2 as:

Fig. 2 Deformation of yield surface

Fig. 3 New anisotropic hardening law

(1) Case of $n=0$: From Eq.(16), we have

$$\mathbf{q}^{(0)} = 2\mu_0 \int_0^\zeta \rho_l^{(0)}(\zeta - \zeta') \frac{d\mathbf{e}^p}{d\zeta'} d\zeta' \quad (19)$$

This expression coincides with the kinematic hardening $\mathbf{r}$ given in Ref.(24)~(26).

(2) Case of $n=1$: By introducing Eq.(12) into Eq.(16), we have

$$\mathbf{q}^{(1)} = 2\mu_0 \int_0^\zeta \rho_l^{(1)}(\zeta - \zeta') \left\{ \frac{(\mathbf{s} - \mathbf{r})}{S_y^0} : \frac{d\mathbf{e}^p}{d\zeta'} \right\} \frac{d\mathbf{e}^p}{d\zeta'} d\zeta' \quad (20)$$

In component form, Eq.(20) may be written as:

$$q_{ij}^{(1)} = R_{ijkl}^{(1)} \left(\frac{s_{kl} - r_{kl}}{S_y^0} \right) \quad (21)$$

where

$$R_{ijkl}^{(1)} = 2\mu_0 \int_0^\zeta \rho_l^{(1)}(\zeta - \zeta') \frac{de_{ij}^p}{d\zeta'} \frac{de_{kl}^p}{d\zeta'} d\zeta' \quad (22)$$

It is not necessary to include the term $(s_{kl} - r_{kl})/S_y^0$ of Eq.(21) in the calculus of integration, since we employ total Lagrangian mapping based on the subsequent yield surface $S^{(2)}$.

(3) Case of $n=2$: Similarly, from Eq.(16), we have

$$\mathbf{q}^{(2)} = 2\mu_0 \int_0^\zeta \rho_l^{(2)}(\zeta - \zeta') \left\{ \frac{(\mathbf{s} - \mathbf{r})}{S_y^0} : \frac{d\mathbf{e}^p}{d\zeta'} \right\}^2 \frac{d\mathbf{e}^p}{d\zeta'} d\zeta' \quad (23)$$

In component form, Eq.(23) may be written as:

$$q_{ij}^{(2)} = R_{ijklmn}^{(2)} \left(\frac{s_{kl} - r_{kl}}{S_y^0} \right) \left(\frac{s_{mn} - r_{mn}}{S_y^0} \right) \quad (24)$$

where

$$R_{ijklmn}^{(2)} = 2\mu_0 \int_0^\zeta \rho_l^{(2)}(\zeta - \zeta') \frac{de_{ij}^p}{d\zeta'} \frac{de_{kl}^p}{d\zeta'} \frac{de_{mn}^p}{d\zeta'} d\zeta' \quad (25)$$

3. Tension-torsion Problem

The theory described in Chapter 2 has been applied to the prediction of the subsequent yield surface reported by Phillips and Tang⁽²⁾ in tension-torsion problems. We define the tension-torsion response through the relation: $\sigma_{11} \neq 0, \sigma_{12} \neq 0$ otherwise $\sigma_{ij} = 0$. For plastic strains, we can assume that:

$$d\epsilon_{11}^p = d\epsilon_{11}^p \neq 0, \quad d\epsilon_{12}^p = d\epsilon_{12}^p \neq 0 \quad (26 \cdot a)$$

$$d\epsilon_{22}^p = d\epsilon_{33}^p = -\frac{1}{2} d\epsilon_{11}^p, \quad d\epsilon_{23}^p = d\epsilon_{31}^p = 0 \quad (26 \cdot b)$$

The quantities $\mathbf{s}$, $\mathbf{r}$, $\mathbf{q}$, and $\mathbf{q}^{(n)}$ can be given by

$$s_{22} = s_{33} = -\frac{1}{2} s_{11}, \quad r_{22} = r_{33} = -\frac{1}{2} r_{11} \quad (27 \cdot a)$$

$$q_{22} = q_{33} = -\frac{1}{2} q_{11}, \quad q_{22}^{(n)} = q_{33}^{(n)} = -\frac{1}{2} q_{11}^{(n)} \quad (27 \cdot b)$$

Accordingly, we normalize $\mathbf{s}, \mathbf{r}, \mathbf{q}$ and $d\mathbf{e}^p$ as follows:

$$\bar{s}_{11} = \sqrt{\frac{3}{2}} s_{11}, \quad \bar{r}_{11} = \sqrt{\frac{3}{2}} r_{11} \quad (28 \cdot a)$$

$$\bar{s}_{12} = \sqrt{2} s_{12}, \quad \bar{r}_{12} = \sqrt{2} r_{12} \quad (28 \cdot b)$$

$$\bar{q}_{11} = \sqrt{\frac{3}{2}} q_{11}, \quad d\bar{e}_{11}^p = \sqrt{\frac{3}{2}} de_{11}^p \quad (28 \cdot c)$$

$$\bar{q}_{12} = \sqrt{2} q_{12}, \quad d\bar{e}_{12}^p = \sqrt{2} de_{12}^p \quad (28 \cdot d)$$

where the superscript ' $\sim$ ' denotes the newly defined

values in Eq.(28). Substituting Eq.(28) into Eqs.(3), (12) and (11) gives:

$$d\zeta^2 = d\bar{e}_{11}^p + d\bar{e}_{12}^p \quad (29 \cdot a)$$

$$s^p = \left(\frac{\bar{s}_{11} - \bar{r}_{11}}{S_y^0} \right) \frac{d\bar{e}_{11}^p}{d\zeta} + \left(\frac{\bar{s}_{12} - \bar{r}_{12}}{S_y^0} \right) \frac{d\bar{e}_{12}^p}{d\zeta} \quad (29 \cdot b)$$

$$F = \left\{ \left(\frac{\bar{s}_{11} - \bar{r}_{11} - \bar{q}_{11}}{S_y^0} \right)^2 + \left(\frac{\bar{s}_{12} - \bar{r}_{12} - \bar{q}_{12}}{S_y^0} \right)^2 \right\}^{1/2} - 1 \quad (29 \cdot c)$$

The rigid body translation $\bar{r}_{11}$, $\bar{r}_{12}$ and the distortions $\bar{q}_{11}$, $\bar{q}_{12}$ can be given from Eqs.(19) and (16) as:

$$\bar{r}_{11} = 2\mu_0 \int_0^\zeta \rho_l^{(0)}(\zeta - \zeta') \frac{d\bar{e}_{11}^p}{d\zeta'} d\zeta' \quad (30 \cdot a)$$

$$\bar{r}_{12} = 2\mu_0 \int_0^\zeta \rho_l^{(0)}(\zeta - \zeta') \frac{d\bar{e}_{12}^p}{d\zeta'} d\zeta' \quad (30 \cdot b)$$

$$\bar{q}_{11}^{(n)} = 2\mu_0 \int_0^\zeta \rho_l^{(n)}(\zeta - \zeta') (s^p)^n \frac{d\bar{e}_{11}^p}{d\zeta'} d\zeta' \quad (31 \cdot a)$$

$$\bar{q}_{12}^{(n)} = 2\mu_0 \int_0^\zeta \rho_l^{(n)}(\zeta - \zeta') (s^p)^n \frac{d\bar{e}_{12}^p}{d\zeta'} d\zeta' \quad (31 \cdot b)$$

3.1 Proportional loading

First of all, we discuss the monotonic radial loading in stress space ($\bar{s}_{11} - \bar{s}_{12}$). We define the orientation of a loading with respect to coordinate systems in stress space by the angles ϕ_{11} and ϕ_{12} , as shown in Fig. 4, to obtain the following relations:

$$\frac{d\bar{e}_{11}^p}{d\zeta} = \cos \phi_{11} = \text{const.} \quad (32 \cdot a)$$

$$\frac{d\bar{e}_{12}^p}{d\zeta} = \cos \phi_{12} = \text{const.} \quad (32 \cdot b)$$

From Eqs.(30) and (31), we can find that the orientation of plastic strain rate $d\mathbf{e}^p$ coincides with those of kinematic hardening $d\mathbf{r}$ and anisotropic hardening $d\mathbf{q}$, and therefore that of stress increments $d\mathbf{s}$. Substituting Eq.(29.b) into Eq.(31) and employing the binomial theorem gives the following expression for $\mathbf{q}^{(n)}$ as a function of $(\mathbf{s} - \mathbf{r})$:

$$\bar{q}_{11}^{(n)} = \sum_{k=0}^n a_{11}^{(n,k)} \left(\frac{\bar{s}_{11}}{S_y^0} \right)^{n-k} \left(\frac{\bar{s}_{12}}{S_y^0} \right)^k \quad (33 \cdot a)$$

Fig. 4 Proportional loading problem

$$\bar{q}_{12}^{(n)} = \sum_{k=0}^n a_{12}^{(n,k)} \left(\frac{\hat{s}}{S_y^0} \right)^{n-k} \left(\frac{\hat{s}_{12}}{S_y^0} \right)^k \quad (33 \cdot b)$$

where

$$\hat{s}_{11} = \bar{s}_{11} - \bar{r}_{11} \quad (34 \cdot a)$$

$$\hat{s}_{12} = \bar{s}_{12} - \bar{r}_{12} \quad (34 \cdot b)$$

$$a_{11}^{(n,k)} = b^{(n)} {}_n C_k (\cos \phi_{11})^{n-k+1} (\cos \phi_{12})^k \quad (34 \cdot c)$$

$$a_{12}^{(n,k)} = b^{(n)} {}_n C_k (\cos \phi_{11})^{n-k} (\cos \phi_{12})^{k+1} \quad (34 \cdot d)$$

$$b^{(n)} = 2\mu_0 \int_0^{\xi} \rho^{(n)}(\xi - \xi') d\xi' \quad (34 \cdot e)$$

Let us consider the case where $\phi_{11}=0$ and $\phi_{12}=\pi/2$, that is, a uniaxial tension stress state. The yield condition F is given by :

$$F = \left[\left\{ \frac{\hat{s}_{11}}{S_y^0} - \sum_{n=1}^N \frac{b^{(n)}}{S_y^0} \left(\frac{\hat{s}_{11}}{S_y^0} \right)^n \right\}^2 + \left(\frac{\hat{s}_{12}}{S_y^0} \right)^2 \right]^{1/2} - 1 \quad (35)$$

Figure 5 shows the shape distortion of the subsequent yield surface, when the parameters $b^{(1)}$ and $b^{(2)}$ are independently taken into account. The parameters $b^{(1)}$

Fig. 5 Effect of deformation parameters $b^{(1)}$ and $b^{(2)}$

and $b^{(2)}$ give the symmetry and nonsymmetry deformations, respectively, with respect to the axis orthogonal to the loading direction. Both modes $b^{(1)}$ and $b^{(2)}$ exist in real material, and further, higher modes of deformation than $n=3$ may also exist.

We will compare the experimental result labelled as "S-5" by Phillips and Tang⁽²⁾. The four sequences of yield surfaces were measured in tension loading. Figure 6 shows the comparison between the first subsequent yield surface and the theoretical one by two- and three-term approximation. Fairly good agreement is obtained in three-term approximation; two-term approximation is accurate enough to describe a shape change in practical use.

Figure 7 shows the comparison between the theoretical results and the sequences of the subsequent

Fig. 6 Two and three-term approximations of deformations

Fig. 7 Comparison with experiment by Phillips and Tang in proportional loading

yield surface in the experiment. The distortion of the yield surface in the calculation were approximated by only two parameters, $b^{(1)}$ and $b^{(2)}$. The present results may be seen to agree excellently with experimental data, with small differences occurring near the location of yield surface center $\bar{s}_{11} = \bar{r}_{11}$.

3.2 Determination of material constant

When the subsequent yield surface s are expressed as a function of the intrinsic time ζ , and hence the equivalent plastic strain length for uniaxial tension loading in $\bar{s}_{11} - \bar{s}_{12}$ space' we can determine the material constants given in Eqs.(34•e) and (30•a) as follows.

(1) In the normalized deviatoric stress space ($\hat{s}_{11}/S_y^0, \hat{s}_{12}/S_y^0$), we approximate distortions of the yield surface by $b^{(1)}$ and $b^{(2)}$, and obtain the relations for $b^{(1)}(\zeta)$ and $b^{(2)}(\zeta)$ at each loading event.

(2) Next, we similarly obtain a relation of translation r_{11} against the intrinsic time ζ . We approximate the shape change first, since some errors may be included in determining a location $\bar{r}_{11}$, and best-shape approximation may not be obtained after the determination of rigid body translation.

(3) Coefficients $b^{(1)}, b^{(2)}$ and $\bar{r}_{11}$ in Eqs.(34•e) and (30•a) can be integrated as:

$$b^{(n)} = 2\mu_0 \sum_i \frac{\rho_i^{(n)}}{\alpha_i^{(n)}} \{1 - \exp(-\alpha_i^{(n)} \zeta)\} \quad (36\cdot a)$$

$$\bar{r}_{11} = 2\mu_0 \sum_i \frac{\rho_i^{(0)}}{\alpha_i^{(0)}} \{1 - \exp(-\alpha_i^{(0)} \zeta)\} \quad (36\cdot b)$$

Using the procedure given in Ref.(28), we obtain the material constants ρ_{1i} and $\alpha_i^{(n)}$ in the order of the smaller $\alpha_i^{(n)}$ from $b^{(n)} - \zeta$ and $\bar{r}_{11} - \zeta$ curves.

Figure 8 shows the calculated results $b^{(1)}/S_y^0, b^{(2)}/S_y^0$, and $\bar{r}_{11}/S_y^0$ as a function of ζ , since the same curve can be adapted to fit three different experimental data, we can obtain the following material constants by two expansion terms.

$$\rho^{(n)}(\zeta) = \rho^{(n)} \exp(-\alpha^{(n)} \zeta) + \rho_2^{(n)} \quad (37)$$

where the coefficients, regardless of power number n , are:

Fig. 8 Coefficients r_{11} , $b^{(1)}$ and $b^{(2)}$ against intrinsic time ζ

$$\rho_{11}^{(n)} = 1630 \left(\frac{S_y^0}{2\mu_0} \right), \quad \alpha_1^{(n)} = 710 \quad (38\cdot a)$$

$$\rho_{12}^{(n)} = 27 \left(\frac{S_y^0}{2\mu_0} \right) \quad (38\cdot b)$$

Note that the sign of ρ_{1i} for $n=1$ is negative. The theoretical curve given in Eq.(37) is also plotted in Fig.8. The number in Fig.8 is the order of the sequence of the measured yield surfaces.

3.3 Formation of vertex point on yield surface

When the number of expansion terms N is equal to two, we have the following expression for a yield surface in the case of $\hat{s}_{12}/S_y^0 = 0$:

$$F = \left| \frac{\hat{s}_{11}}{S_y^0} - \frac{b^{(1)}}{S_y^0} \left(\frac{\hat{s}_{11}}{S_y^0} \right) - \frac{b^{(2)}}{S_y^0} \left(\frac{\hat{s}_{11}}{S_y^0} \right)^2 \right| - 1 = 0 \quad (39)$$

In order to have real roots $\hat{s}_{11}/S_y^0$ of Eq.(35), a discriminant D of a polynomial function of second order in F should be positive or zero as:

$$D = \left(\frac{b^{(1)}}{S_y^0} - 1 \right)^2 \pm 4 \frac{b^{(2)}}{S_y^0} \geq 0 \quad (40)$$

Hence, we have the following relation between $b^{(1)}$ and $b^{(2)}$:

$$-\left(\frac{b^{(1)}}{S_y^0} - 1 \right)^2 \leq 4 \frac{b^{(2)}}{S_y^0} \leq \left(\frac{b^{(1)}}{S_y^0} - 1 \right)^2 \quad (41)$$

Figure 9 shows the deformed yield surfaces for both $D=0$ and $b^{(2)}=0$ in the case of $b^{(1)}/S_y^0 = -0.2$ and -2.0 . The present theory can represent the vertex point on the yield surface as a special combination of parameters $b^{(1)}$ and $b^{(2)}$, and hence $D=0$, as shown in Fig.9. The size of the yield surface along the $\hat{s}_{11}$ -axis attains a maximum value at $D=0$ and a minimum one at $b^{(2)}=0$.

Figure 10 shows the size of the surface $\Delta \hat{s}_{11}$ as a function of deformation parameter $b^{(1)}/S_y^0$ for the cases of $D=0$ and $b^{(2)}=0$. The size of yield surfaces in other cases exists between the two curves in Fig.10. The magnitude of the yield surface, which is equal to 2 in the initial stage, generally decreases during plastic flow, with exceptions only for $b^{(1)}/S_y^0 = 0.5$.

3.4 Orthogonal load path

Let us consider a non-proportional loading path with a corner as shown in Fig.11, which consists of the

Fig. 9 Formation of vertex point on yield surface

Fig. 10 Size of yield surface for deformation parameter $b^{(1)}$

Fig. 11 Orthogonal load path problem

first and second loading paths depicted as I and II, respectively. The direction of the second loading II is orthogonal to that of the first loading I in stress space, and its extension passes through the center of the yield surface at the final stage of the first loading I. In the region of the linear hardening law in plastic strain, the direction of the stress path coincides with that of plastic strain. For the loadings I and II, we define the intrinsic time increments $\Delta\zeta_I$ and $\Delta\zeta_{II}$ and the directions with respect to the $\bar{s}_{11}$ -axis by ϕ_{11}^I and ϕ_{11}^{II} , and those with respect to the $\bar{s}_{12}$ -axis by ϕ_{12}^I and ϕ_{12}^{II} , respectively.

In the region where the linear hardening law of

Fig. 12 Deformation of yield surface under orthogonal load path

Fig. 13 Comparison with the experiment by Phillips and Tang in orthogonal load path

Eq.(13) holds, or deformation parameters $b^{(1)}$ and $b^{(2)}$ are linearly related to the intrinsic time in nonlinear hardening law, we have the following:

$$b_l^{(n)} = 2\mu_0 \rho_l^{(n)}(0) \Delta\zeta_l \quad (l = I, II) \quad (42)$$

We can obtain $a_{11}^{(n,k)}$ and $a_{12}^{(n,k)}$ of Eq.(34) by superposing the values at each loading step as:

$$a_{11}^{(n,k)} = \sum_{l=1}^{II} b_l^{(n)} {}_n C_k (\cos \phi_{11}^l)^{n-k+1} (\cos \phi_{12}^l)^k \quad (43 \cdot a)$$

$$a_{12}^{(n,k)} = \sum_{l=1}^{II} b_l^{(n)} {}_n C_k (\cos \phi_{11}^l)^{n-k} (\cos \phi_{12}^l)^{k+1} \quad (43 \cdot b)$$

Figure 12 shows the theoretical results of the first and second subsequent yield surfaces with $b^{(1)}/S_y^0 = -b^{(2)}/S_y^0 = -0.5$ for both Case A, where the direction ϕ_{11} changes from 0 to 90 degrees, and Case B, where the direction ϕ_{11} changes from 45 to 135 degrees. The obtained configurations of the yield surface showed

congruence, as anticipated.

Figure 13 shows the comparison between the theoretical prediction with the results of the experiment designated as 'S-7' by Phillips and Tang⁽²⁾. The values of parameters $b^{(1)}$, $b^{(2)}$ and $\bar{\tau}_{11}$ used in the calculation are also plotted in Fig. 7. The origin of the intrinsic time ζ is located at the point where the loading direction changes. We can observe from Fig. 7 that the intrinsic time is so small that we can regard these points within the region of the linear hardening law. The values of $b^{(1)}$, $b^{(2)}$ and $\bar{\tau}_{11}$ correlate well with those in the specimen 'S-5' made of the same material.

3.5 Reversibility of deformations

Let us discuss the reversibility of deformations of the yield surface, when the loading direction turns over in the region of the linear hardening law in Eq. (13). We define the loading reversal through the relation:

$$de_{11}^p = -de_1^p \quad (44)$$

The stress s^p in the direction of the plastic strain rate de^p may also be reversed from Eq. (12) as:

$$s_{11}^p = -s^p \quad (45)$$

We can obtain the following deformation increment $dq^{(n)}$ from Eq. (13) during the same intrinsic time interval $d\zeta$:

$$dq_{11}^{(n)} = \begin{cases} dq^{(n)} & (n \text{ is odd}) \\ -dq^{(n)} & (n \text{ is even}) \end{cases} \quad (46)$$

From Eq. (46), we can predict that deformation for $n=1$ may develop further, while that for $n=2$ may recover to the original shape. Note that the same tendency may be qualitatively found in the nonlinear hardening law.

4. Conclusions

The present paper proposes a new anisotropic hardening law of plasticity to model the distortion of the yield surface. The intrinsic time in the endochronic theory is selected as the coordinate to express the history of plastic flow. The gross modeling of the overall distortion is obtained by only two parameters of deformation based on the assumptions that the deformations are proportional to the stress in the loading direction. The validity of the present theory is confirmed by good agreement with the experiment involving tension-torsion problems using aluminum alloy. The theory presented herein results in constitutive relations that are simple in form; the material constants are few in number and can be easily determined as shown in this paper. It is, therefore, hoped that the present theory may become one of the useful constitutive equations for a practical estimation of plastic material behavior.

References

- (1) Naghdi, P. M., Essenberg, F. and Koff, W., "An Experiment Study of Initial and Subsequent Yield Surfaces in Plasticity," *Trans. ASME, J. Appl. Mech.*, Vol. 25, No. (1957), p. 201.
- (2) Phillips, A. and Tang, J.-L., "The Effect of Loading Path on the Yield Surface at Elevated Temperatures," *Int. J. Solids Struct.*, Vol. 8 (1972), p. 463.
- (3) Shiratori, E., Ikegami, K., Kaneko, K. and Takada, J., "Effect of Cyclic Preloading on Subsequent Yield Surfaces," *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese) Vol. 42, No. 363 (1976-11), p. 3340.
- (4) Liu, K. C. and Greenstreet, W. L. (ed. Stricklin, J. A. and Saczalski, K. J.), "Experimental Studies to Examine Elastic-Plastic Behaviors of Metal Alloys used in Nuclear Structures," *Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects*, ASME AMD-Vol. 20 (1976-12), p. 35.
- (5) Hecker, S. S. (ed. Stricklin, J. A. and Saczalski, K. J.), "Experimental Studies of Yield Phenomena in Biaxially Loaded Metals," *Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects*, ASME AMD, Vol. 20 (1976-12), p. 1.
- (6) Ikegami, K. (ed. Boehler, J. P.), "Experimental Plasticity on the Anisotropy of Metals," *Proc. Euromech Colloquim 115*, (1979-6), p. 201.
- (7) Tokuda, M. and Kratochvil, J., "Prediction of Subsequent Yield Surface by a Simple Mechanical Model of Polycrystal," *Arch. Mech.*, Vol. 36, No. 5-6 (1984), p. 661.
- (8) Dluzewski, P. H., "Slip Theory and Inelastic Deformations; Relations between the Theory and the Experimental Results," *Arch. Mech.*, Vol. 36, No. 2 (1984), p. 173.
- (9) Kreibitz, R., "Some Considerations on the Yield Condition in the Theory of Plasticity," *Acta Mech.*, Vol. 55 (1985), p. 41.
- (10) Baltov, A. and Sawczuk, A., "A Rule of Anisotropic Hardening," *Acta Mech.*, Vol. 1 (1965), p. 81.
- (11) Phillips, A. and Weng, G.J., "An Analytical Study of an Experimentally Verified Hardening Law," *Trans. ASME, J. Appl. Mech.*, Vol. 42, No. 2 (1975-6), p. 375.
- (12) Kaneko, K., Ikegami, K. and Shiratori, E., "The Yield Condition and Flow Rule of a Metal for the Various Pre-strain Paths," *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 41, No. 350 (1975-10), p. 2793.
- (13) Yoshida, F., Murakami, T., Kaneko, K., Ikegami, K. and Shiratori, E., "Plastic Behavior in the Combined Loading along Straight Stress Paths with a Bend," *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 43, No. 368 (1977-4), p. 1231.
- (14) Eisenberg, M. A. and Yen, C.-F., "The Anisotropic

- Deformation of Yield Surfaces," Trans. ASME, J. Eng. Mat. Tech., Vol. 106, No. 4 (1984-10), p. 355.
- (15) Gotoh, M., "Trend and Future of Theory of Plasticity," J. Japan Soc. Tech. Plast., (in Japanese), Vol. 18, No. 200 (1977), p. 664.
 - (16) Shindo, A., "A Historical Perspective of the Studies on the Theory of Plasticity and the Constitutive Equations," J. Soc. Mat. Sci. Jpn., (in Japanese), Vol. 27, No. 296 (1978-5), p. 401.
 - (17) Ohashi, Y., "Inelastic Deformation and Constitutive Law of Metallic Materials subjected to Complex Loading," Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 48, No. 428, A(1982-4), p. 389.
 - (18) Hutchinson, J. W., "Elastic-Plastic Behaviour of Polycrystalline Metals and Composites," Proc. Roy. Soc. Lond., Vol. 319, A(1970), p. 247.
 - (19) Budiansky, B., "A Reassessment of Deformation Theories of Plasticity," Trans. ASME, J. Appl. Mech., Vol. 26, No. 2 (1959-6), p. 247.
 - (20) Christoffersen, J. and Hutchinson, J. W., "A Class of Phenomenological Corner Theories of Plasticity," J. Mech. Phys. Solids, Vol. 27 (1979), p. 465.
 - (21) Gotoh, M., "A Class of Plastic Constitutive Equations with Vertex Effect-I. General Theory," Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 47, No. 424, A(1981-12), p. 1389.
 - (22) Gotoh, M., "Review of Plastic Constitutive Equations with Vertex Effect," J. Jpn. Soc. Tech. Plast., (in Japanese), Vol. 24, No. 267 (1983-4), p. 313.
 - (23) Valanis, K. C., "Fundamental Consequence of a New Intrinsic Time Measure-Plasticity as a Limit of the Endochronic Theory," Arch. Mech., Vol. 32, No. 2 (1980), p. 171.
 - (24) Watanabe, O. and Atluri, S.N., "Constitutive Modeling of Cyclic Plasticity and Creep, Using an Internal Time Concept," Int. J. Plast., Vol. 2 (1986), p. 107.
 - (25) Watanabe, O. and Atluri, S.N., "Internal Time, General Internal Variable, and Multi-Yield-Surface Theories of Plasticity and Creep: A Unification of Concepts," Int. J. Plast., Vol. 2 (1986), p. 37.
 - (26) Watanabe, O. and Atluri, S.N., "A New Endochronic Approach to Computational Elastoplasticity: Example of a Cyclically Loaded Cracked Plate," Trans. ASME, J. Appl. Mech., Vol. 52, No. 4 (1985-12), p. 857.
 - (27) Chaboche, J. L. and Rousselier, G., "On the Plastic and Viscoplastic Constitutive Equations-Part I: Rules Developed with Internal Variable Concept," Trans. ASME, J. Press. Vessel Tech., Vol. 105, No. 2 (1983-5), p. 153.
 - (28) Watanabe, O., "A New Integral Constitutive Equation and Determination of Material Constants for the Endochronic Theory of Cyclic Plasticity," Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 52, No. 481, A(1986-9), p. 2285.