Elastic Analysis of Crack Opening Displacement in a Fixed Sided Plate*

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The elastic analysis of a crack in a finite width plate is presented. Crack opening displacements are calculated for the cases of roller-fixed type and rigid-fixed type clamps, changing the position and the length of the crack. The continuously distributed dislocations model is employed for the analysis. The results show that the profile of the crack opening displacement has peaks which exceed the applied displacement in the case of long cracks.

Key Words: Elasticity, Fracture, Crack, Crack Opening Displacement, Continuously Distributed Dislocations Model

1. Introduction

Most of the elastic analyses of cracks have been intended to obtain the crack tip singularities, e. g., stress intensity factors and so on, while little attention has been paid to the crack opening displacements. The shape of the opening displacement of a Mode I crack in a plate when stressed uniformly at infinity is known to be elliptic. However, the shape of a crack opening generally varies, depending on its boundary conditions. The authors previously analyzed the elastic field of rapidly propagating cracks in fixed sided plates, and obtained interesting results about the crack opening displacements⁽¹⁾. In this work, a stationary crack with a finite length in a fixed sided plate is analyzed to obtain the crack opening displacement using the method of continuously distributed dislocations model.

2. Crack Model in a Fixed Sided Plate

The model analyzed in this paper is shown in Fig. 1. Namely, the model is such that a fixed sided plate with infinite length contains a crack with a finite length parallel to the side line of the plate. The material of the plate is assumed to be homogeneous, isotropic and linear-elastic with the shear modulus G and the Poisson's ratio ν . The crack length, the width

Fig. 1 Crack model in a fixed sided plate

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of the plate and the distance between the crack and the center line of the plate are denoted by $2a$, $2h$ and d respectively. The x -axis is taken along the center line of the plate and the y -axis is taken to be perpendicular to the x -axis, passing through the center point of the crack. The x - and y -components of the displacement are denoted by u and v respectively. As the boundary conditions of both sides of the plate, the following two types are considered.

(a) Roller-fixed type

$$\begin{aligned} v &= v_0, \tau_{xy} = 0 \text{ on } y = h (-\infty < x < \infty); \\ v &= -v_0, \tau_{xy} = 0 \text{ on } y = -h (-\infty < x < \infty). \end{aligned} \quad (1)$$

(b) Rigid-fixed type

$$\begin{aligned} v &= v_0, u = 0 \text{ on } y = h (-\infty < x < \infty); \\ v &= -v_0, u = 0 \text{ on } y = -h (-\infty < x < \infty). \end{aligned} \quad (2)$$

Then, elastic constant κ is defined as,

$$\kappa = \begin{cases} 3 - 4\nu & (\text{Plane Strain}), \\ (3 - \nu)/(1 + \nu) & (\text{Plane Stress}). \end{cases} \quad (3)$$

3. Elastic Analysis Using the Method of Continuously Distributed Dislocations Model

3.1 Method of adjusting the boundary conditions

The cases of the fixed sided plate shown in Fig. 1 can be analyzed by the use of the infinite plate as follows.

Fig. 2 Method of adjusting the boundary conditions in the case of the roller-fixed type

(a) Roller-fixed type

In this case, the infinite plate in Fig. 2 which contains periodical series of cracks at $y = 2nh + (-1)^n d$ (n : integer) ($-a < x < a$), satisfies the boundary conditions of Fig. 1 by applying the stress $\sigma_y = \sigma_y^\infty_a$ at infinity, where $\sigma_y^\infty_a$ is determined such that $\varepsilon_y \rightarrow v_0/h$ at $x \rightarrow \pm \infty$, i. e.,

$$\sigma_y^\infty_a = \frac{8}{\kappa + 1} \frac{Gv_0}{h} \quad (4)$$

Generally, in-plane crack problem (Mode I, Mode II or mixed mode) can be analyzed by replacing the crack with the zy -component dislocations array (Mode I), the zx -component dislocations array (Mode II) or both components dislocations array (mixed mode). In the present case, as the problem is of mixed mode, the cracks are replaced with the zy - and zx -components dislocations arrays. Considering the symmetry with respect to $y = h$ and $y = -h$, the density functions of these dislocation components are denoted by $f_{zy}(x)$ (independent of n) and $(-1)^n f_{zx}(x)$ (only the sign changes alternately with n). The two unknown functions $f_{zy}(x)$ and $f_{zx}(x)$ ($-a < x < a$) can be determined by the traction-free conditions on the crack surfaces, i. e.,

$$\sigma_y = 0 (-a < x < a) \quad \text{on } y = d, \quad (5)$$

$$\tau_{xy} = 0 (-a < x < a) \quad \text{on } y = d. \quad (6)$$

(b) Rigid-fixed type

Fig. 3 Method of adjusting the boundary conditions in the case of the rigid-fixed type

In this case, the infinite plate in Fig. 3 with periodical cracks at $y=2nh+(-1)^n d$ (n : integer) ($-a < x < a$) satisfies the boundary conditions of Fig. 1 by distributing the point forces and applying the stress as follows. The density functions of the point forces are $p_1(x)(-\infty < x < \infty)$ at $y=(4n+1)h$ (n : integer) and $p_2(x)(-\infty < x < \infty)$ at $y=(4n-1)h$ (n : integer), and the stress $\sigma_x=\sigma_x^\infty$ and $\sigma_y=\sigma_y^\infty$ is applied at infinity such that $\varepsilon_x \rightarrow 0$ and $\varepsilon_y \rightarrow v_0/h$ at $x \rightarrow \pm\infty$, i. e.,

$$\sigma_x^\infty = \frac{3-x}{x-1} \frac{Gv_0}{h}, \quad \sigma_y^\infty = \frac{x+1}{x-1} \frac{Gv_0}{h}. \quad (7)$$

In this analysis, the cracks are replaced with the continuous dislocations arrays similarly to the previous case. The density functions of zy - and zx -components dislocations are denoted by $g_{zy}(x)$ and $(-1)^n g_{zx}(x)(-a < x < a)$. The four unknown functions $g_{zy}(x)$, $g_{zx}(x)(-a < x < a)$ and $p_1(x)$, $p_2(x)(-\infty < x < \infty)$ are to be determined by the following conditions.

$$\sigma_y = 0 (-a < x < a) \text{ on } y=d, \quad (8)$$

$$\tau_{xy} = 0 (-a < x < a) \text{ on } y=d, \quad (9)$$

$$\partial u / \partial x = 0 (-\infty < x < \infty) \text{ on } y=h \quad (10)$$

$$\partial u / \partial x = 0 (-\infty < x < \infty) \text{ on } y=-h \quad (11)$$

3.2 Elastic field of periodically situated dislocations and point forces

In order to obtain the elastic solution in the bounded plate shown in Fig. 1, the fields of periodically distributed dislocations and forces in an infinite elastic body are utilized in the present analysis.

Figure 4(a) shows the periodical set of zy -component dislocations in an infinite elastic plate. The Burgers vector is denoted by b_y and the position of

each dislocation is expressed as $y=2nh+(-1)^n d$ (n : integer) on the y -axis. The stress and distortion components: σ_y , τ_{xy} and $\partial u / \partial x$ are given as follows.

$$\begin{aligned} \sigma_y = \frac{Gb_y}{h(x+1)} & \left\{ \left[\frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h}} \right. \right. \\ & + \left. \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h}} \right\} \\ & - \frac{\pi x}{4h} \left[\frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y-d)}{2h} - 1}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \\ & \left. \left. - \frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y+d)}{2h} + 1}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \right\}, \quad (12) \\ \tau_{xy} = \frac{Gb_y}{h(x+1)} & \left[-\frac{1}{2} \left\{ \frac{\sin \frac{\pi(y-d)}{2h}}{\cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h}} \right. \right. \\ & - \left. \frac{\sin \frac{\pi(y+d)}{2h}}{\cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h}} \right\} \\ & + \frac{\pi x}{4h} \left\{ \frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y-d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \end{aligned}$$

Fig. 4 Periodically situated dislocations or point forces on the y -axis in an infinite plate

$$\left. \frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y+d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \right], \quad (13)$$

$$\begin{aligned} \frac{\partial u}{\partial x} = & \frac{b_y}{h(x+1)} \left[-\frac{3-x}{8} \left\{ \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h}} \right. \right. \\ & + \left. \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h}} \right\} \\ & + \frac{\pi x}{8h} \left[\frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y-d)}{2h} - 1}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \\ & \left. \left. - \frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y+d)}{2h} + 1}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \right] \quad (14) \end{aligned}$$

Figure 4(b) shows the periodical set of zx -component dislocations. The positions of the dislocations are given by $y = 2nh + (-1)^n d$ (n : integer) on the y -axis, and the Burgers vector is written as $(-1)^n b_x$ here. The stress and distortion components: σ_y , τ_{xy} and $\partial u/\partial x$ are given as follows.

$$\begin{aligned} \sigma_y = & \frac{Gb_x}{h(x+1)} \left[-\frac{1}{2} \left\{ \frac{\sin \frac{\pi(y-d)}{2h}}{\cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h}} \right. \right. \\ & + \left. \frac{\sin \frac{\pi(y+d)}{2h}}{\cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h}} \right\} \\ & + \frac{\pi x}{4h} \left[\frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y-d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \\ & \left. \left. + \frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y+d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \right] \quad (15) \end{aligned}$$

$$\begin{aligned} \tau_{xy} = & \frac{Gb_x}{h(x+1)} \frac{\pi x}{4h} \left[\frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y-d)}{2h} - 1}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \\ & \left. + \frac{\cosh \frac{\pi x}{2h} \cos \frac{\pi(y+d)}{2h} + 1}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \quad (16) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial x} = & \frac{b_x}{h(x+1)} \left[-\frac{x-1}{8} \left\{ \frac{\sin \frac{\pi(y-d)}{2h}}{\cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h}} \right. \right. \\ & + \left. \frac{\sin \frac{\pi(y+d)}{2h}}{\cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h}} \right\} \end{aligned}$$

$$\begin{aligned} & - \frac{\pi x}{8h} \left[\frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y-d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} - \cos \frac{\pi(y-d)}{2h} \right\}^2} \right. \\ & \left. + \frac{\sinh \frac{\pi x}{2h} \sin \frac{\pi(y+d)}{2h}}{\left\{ \cosh \frac{\pi x}{2h} + \cos \frac{\pi(y+d)}{2h} \right\}^2} \right] \quad (17) \end{aligned}$$

Figure 4(c) shows two periodical sets of x -component point forces in an infinite elastic plate. The positions of the point forces of one set are given by $y = (4n+1)h$ (n : integer) and the magnitude of these forces is denoted by P_1 , while the positions of the forces of the other set are given by $y = (4n-1)h$ (n : integer) and the magnitude of these forces is denoted by P_2 . The stress and distortion components: σ_y , τ_{xy} and $\partial u/\partial x$ are given as follows.

$$\begin{aligned} \sigma_y = & \frac{P_1}{h(x+1)} \left\{ -\frac{3-x}{8} \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h}} \right. \\ & + \frac{\pi x}{8h} \frac{\cosh \frac{\pi x}{2h} \sin \frac{\pi y}{2h} - 1}{\left(\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h} \right)^2} \left. \right\} + \frac{P_2}{h(x+1)} \left\{ -\frac{3-x}{8} \right. \\ & \times \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h}} - \frac{\pi x}{8h} \frac{\cosh \frac{\pi x}{2h} \sin \frac{\pi y}{2h} + 1}{\left(\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h} \right)^2} \left. \right\} \quad (18) \end{aligned}$$

$$\begin{aligned} \tau_{xy} = & \frac{P_1}{h(x+1)} \left\{ \frac{x-1}{8} \frac{\cos \frac{\pi y}{2h}}{\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h}} \right. \\ & + \frac{\pi x}{8h} \frac{\sinh \frac{\pi x}{2h} \cos \frac{\pi y}{2h}}{\left(\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h} \right)^2} \left. \right\} + \frac{P_2}{h(x+1)} \left\{ -\frac{x-1}{8} \right. \\ & \times \frac{\cos \frac{\pi y}{2h}}{\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h}} - \frac{\pi x}{8h} \frac{\sinh \frac{\pi x}{2h} \cos \frac{\pi y}{2h}}{\left(\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h} \right)^2} \left. \right\} \quad (19) \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial x} = & \frac{P_1}{Gh(x+1)} \left\{ -\frac{x-1}{8} \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h}} \right. \\ & - \frac{\pi x}{16h} \frac{\cosh \frac{\pi x}{2h} \sin \frac{\pi y}{2h} - 1}{\left(\cosh \frac{\pi x}{2h} - \sin \frac{\pi y}{2h} \right)^2} \left. \right\} \\ & + \frac{P_2}{Gh(x+1)} \left\{ -\frac{x-1}{8} \frac{\sinh \frac{\pi x}{2h}}{\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h}} \right. \\ & + \frac{\pi x}{16h} \frac{\cosh \frac{\pi x}{2h} \sin \frac{\pi y}{2h} + 1}{\left(\cosh \frac{\pi x}{2h} + \sin \frac{\pi y}{2h} \right)^2} \left. \right\} \quad (20) \end{aligned}$$

3.3 Singular integral equations

(a) Roller-fixed type

From Eqs. (12), (13), (15) and (16), the stress components, σ_y and τ_{xy} are expressed as integral forms with the unknown functions $f_{zy}(x)$ and $f_{zx}(x)$ ($-a < x < a$). The boundary conditions (5) and (6) are put into the integral forms to obtain the following singular integral equations of the Cauchy type.

$$\oint_{-a}^a K_{11}(x, \xi) f_{zy}(\xi) d\xi + \oint_{-a}^a K_{12}(x, \xi) f_{zx}(\xi) d\xi + \sigma_y^\infty = 0 \quad (-a < x < a), \quad (21)$$

$$\oint_{-a}^a K_{21}(x, \xi) f_{zy}(\xi) d\xi + \oint_{-a}^a K_{22}(x, \xi) f_{zx}(\xi) d\xi = 0 \quad (-a < x < a), \quad (22)$$

where the kernel functions K_{11}, K_{12}, K_{21} and K_{22} are listed in Table 1 (which is common to the case of the rigid-fixed type).

(b) Rigid-fixed type

Similarly, from Eqs. (12) to (20), the stress and distortion components: σ_y, τ_{xy} and $\partial u/\partial x$ are expressed

in the integral forms with the unknown functions $g_{zy}(x), g_{zx}(x)$ ($-a < x < a$) and $p_1(x), p_2(x)$ ($-\infty < x < \infty$). With the conditions (8) to (11), the integral forms are reduced to the following singular integral equations of the Cauchy type.

$$\begin{aligned} & \oint_{-a}^a K_{11}(x, \xi) g_{zy}(\xi) d\xi + \oint_{-a}^a K_{12}(x, \xi) g_{zx}(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{13}(x, \xi) p_1(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{14}(x, \xi) p_2(\xi) d\xi + \sigma_y^\infty = 0 \end{aligned} \quad (-a < x < a), \quad (23)$$

$$\begin{aligned} & \oint_{-a}^a K_{21}(x, \xi) g_{zy}(\xi) d\xi + \oint_{-a}^a K_{22}(x, \xi) g_{zx}(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{23}(x, \xi) p_1(\xi) d\xi + \int_{-\infty}^{\infty} K_{24}(x, \xi) p_2(\xi) d\xi = 0 \end{aligned} \quad (-a < x < a), \quad (24)$$

$$\begin{aligned} & \oint_{-a}^a K_{31}(x, \xi) g_{zy}(\xi) d\xi + \oint_{-a}^a K_{32}(x, \xi) g_{zx}(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{33}(x, \xi) p_1(\xi) d\xi + \int_{-\infty}^{\infty} K_{34}(x, \xi) p_2(\xi) d\xi = 0 \end{aligned} \quad (-\infty < x < \infty), \quad (25)$$

Table 1 List of the kernel functions of the singular integral equations

$K_{11} = \frac{G}{h(x+1)} \left[\left\{ \coth \frac{\pi(x-\xi)}{4h} + \frac{\sinh[\pi(x-\xi)/(2h)]}{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]} \right\} - \frac{\pi(x-\xi)}{4h} \left[\frac{1}{2} \operatorname{cosech}^2 \frac{\pi(x-\xi)}{4h} - \frac{\cosh[\pi(x-\xi)/(2h)] \cos[\pi d/h] + 1}{\{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]\}^2} \right] \right]$
$K_{12} = \frac{G}{h(x+1)} \left[-\frac{1}{2} \frac{\sin[\pi d/h]}{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]} + \frac{\pi(x-\xi)}{4h} \frac{\sinh[\pi(x-\xi)/(2h)] \sin[\pi d/h]}{\{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]\}^2} \right]$
$K_{13} = \frac{1}{h(x+1)} \left[-\frac{3-x}{8} \frac{\sinh[\pi(x-\xi)/(2h)]}{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]} + \frac{\pi(x-\xi)}{8h} \frac{\cosh[\pi(x-\xi)/(2h)] \sin[\pi d/(2h)] - 1}{\{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]\}^2} \right]$
$K_{14} = \frac{1}{h(x+1)} \left[-\frac{3-x}{8} \frac{\sinh[\pi(x-\xi)/(2h)]}{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]} - \frac{\pi(x-\xi)}{8h} \frac{\cosh[\pi(x-\xi)/(2h)] \sin[\pi d/(2h)] + 1}{\{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]\}^2} \right]$
$K_{21} = \frac{G}{h(x+1)} \left[\frac{1}{2} \frac{\sin[\pi d/h]}{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]} - \frac{\pi(x-\xi)}{4h} \frac{\sinh[\pi(x-\xi)/(2h)] \sin[\pi d/h]}{\{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]\}^2} \right]$
$K_{22} = \frac{G}{h(x+1)} \left[\frac{\pi(x-\xi)}{4h} \left[\frac{1}{2} \operatorname{cosech}^2 \frac{\pi(x-\xi)}{4h} + \frac{\cosh[\pi(x-\xi)/(2h)] \cos[\pi d/h] + 1}{\{\cosh[\pi(x-\xi)/(2h)] + \cos[\pi d/h]\}^2} \right] \right]$
$K_{23} = \frac{1}{h(x+1)} \left[\frac{x-1}{8} \frac{\cos[\pi d/(2h)]}{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]} + \frac{\pi(x-\xi)}{8h} \frac{\sinh[\pi(x-\xi)/(2h)] \cos[\pi d/(2h)]}{\{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]\}^2} \right]$
$K_{24} = \frac{1}{h(x+1)} \left[-\frac{x-1}{8} \frac{\cos[\pi d/(2h)]}{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]} - \frac{\pi(x-\xi)}{8h} \frac{\sinh[\pi(x-\xi)/(2h)] \cos[\pi d/(2h)]}{\{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]\}^2} \right]$
$K_{31} = \frac{1}{h(x+1)} \left[-\frac{3-x}{4} \frac{\sinh[\pi(x-\xi)/(2h)]}{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]} + \frac{\pi(x-\xi)}{4h} \frac{\cosh[\pi(x-\xi)/(2h)] \sin[\pi d/(2h)] - 1}{\{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]\}^2} \right]$
$K_{32} = \frac{1}{h(x+1)} \left[-\frac{x-1}{4} \frac{\cos[\pi d/(2h)]}{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]} - \frac{\pi(x-\xi)}{4h} \frac{\sinh[\pi(x-\xi)/(2h)] \cos[\pi d/(2h)]}{\{\cosh[\pi(x-\xi)/(2h)] - \sin[\pi d/(2h)]\}^2} \right]$
$K_{33} = \frac{1}{Gh(x+1)} \left[-\frac{x-1}{8} \coth \frac{\pi(x-\xi)}{4h} - \frac{\pi(x-\xi)}{32h} \operatorname{cosech}^2 \frac{\pi(x-\xi)}{4h} \right]$
$K_{34} = \frac{1}{Gh(x+1)} \left[-\frac{x-1}{8} \tanh \frac{\pi(x-\xi)}{4h} + \frac{\pi(x-\xi)}{32h} \operatorname{sech}^2 \frac{\pi(x-\xi)}{4h} \right]$
$K_{41} = \frac{1}{h(x+1)} \left[-\frac{3-x}{4} \frac{\sinh[\pi(x-\xi)/(2h)]}{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]} - \frac{\pi(x-\xi)}{4h} \frac{\cosh[\pi(x-\xi)/(2h)] \sin[\pi d/(2h)] + 1}{\{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]\}^2} \right]$
$K_{42} = \frac{1}{h(x+1)} \left[\frac{x-1}{4} \frac{\cos[\pi d/(2h)]}{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]} + \frac{\pi(x-\xi)}{4h} \frac{\sinh[\pi(x-\xi)/(2h)] \cos[\pi d/(2h)]}{\{\cosh[\pi(x-\xi)/(2h)] + \sin[\pi d/(2h)]\}^2} \right]$
$K_{43} = \frac{1}{Gh(x+1)} \left[-\frac{x-1}{8} \tanh \frac{\pi(x-\xi)}{4h} + \frac{\pi(x-\xi)}{32h} \operatorname{sech}^2 \frac{\pi(x-\xi)}{4h} \right]$
$K_{44} = \frac{1}{Gh(x+1)} \left[-\frac{x-1}{8} \coth \frac{\pi(x-\xi)}{4h} - \frac{\pi(x-\xi)}{32h} \operatorname{cosech}^2 \frac{\pi(x-\xi)}{4h} \right]$

$$\begin{aligned} & \int_{-a}^a K_{41}(x, \xi) g_{zy}(\xi) d\xi + \int_{-a}^a K_{42}(x, \xi) g_{zx}(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{43}(x, \xi) p_1(\xi) d\xi \\ & + \int_{-\infty}^{\infty} K_{44}(x, \xi) p_2(\xi) d\xi = 0 \end{aligned} \quad (-\infty < x < \infty), \quad (26)$$

where the kernel functions K_{11} , K_{12} , K_{13} etc. are listed in Table 1.

3.4 Crack opening displacements

If the dislocation density functions $f_{zy}(x)$, $f_{zx}(x)$ or $g_{zy}(x)$, $g_{zx}(x)$ are obtained by solving the singular integral equations shown in the previous section, the crack opening displacements $v_c(x)$ (y -component of the relative displacement between the upper and the lower surfaces of the crack) and $u_c(x)$ (x -component) are derived by the following integrals.

(a) Roller-fixed type

$$v_c(x) = \int_x^a f_{zy}(\xi) d\xi, \quad u_c(x) = \int_x^a f_{zx}(\xi) d\xi. \quad (27)$$

(b) Rigid-fixed type

$$v_c(x) = \int_x^a g_{zy}(\xi) d\xi, \quad u_c(x) = \int_x^a g_{zx}(\xi) d\xi. \quad (28)$$

4. Results of Numerical Calculation

The dislocation density functions are derived as

the numerical solutions of the singular integral equations shown in section 3.3. Figures 5 and 6 show the crack opening displacements $v_c(x)$ obtained by substituting the dislocation density functions into Eq. (27) or (28). Figure 5 is the case of the roller-fixed type, and is therefore independent of the value of x . Figure 6 is the case of the rigid-fixed type with $x=2.2$. In both cases, it is generally shown that if the crack length is the same, the larger the distance between the crack and the center line of the plate becomes (the larger the value of d is), the smaller the crack opening becomes. Furthermore, when the ratio of the crack length and the width of the plate a/h extends beyond a critical value, $v_c(x)$ near $x=0$ (the center of the crack) exceeds the applied displacement $2v_0$ a little; and if the value of a/h becomes larger, the maximum values of $v_c(x)$ appear at two positions symmetrical to $x=0$, though these effects are too small to be shown in Fig. 5 or 6. Figure 7 shows the magnified values of $v_c(x)$ near the maximum points in the case of $d/h=0$.

In the case of $d \neq 0$, $u_c(x)$ is non-zero; however, its absolute value is much smaller than that of $v_c(x)$ and therefore, $u_c(x)$ scarcely affects the crack opening shape.

Fig. 5

Fig. 6

Fig. 7

Stress intensity factors K_I and K_{II} are represented by using the dislocation density functions as follows.

(a) Roller-fixed type

$$K_I = \frac{2\sqrt{2\pi}G}{\chi+1} \lim_{x \rightarrow a-} \sqrt{a-x} f_{zy}(x),$$

$$K_{II} = \frac{2\sqrt{2\pi}G}{\chi+1} \lim_{x \rightarrow a-} \sqrt{a-x} f_{zx}(x). \quad (29)$$

(b) Rigid-fixed type

$$K_I = \frac{2\sqrt{2\pi}G}{\chi+1} \lim_{x \rightarrow a-} \sqrt{a-x} g_{zy}(x),$$

$$K_{II} = \frac{2\sqrt{2\pi}G}{\chi+1} \lim_{x \rightarrow a-} \sqrt{a-x} g_{zx}(x). \quad (30)$$

Comparing the values of stress intensity factors obtained by these equations with the results of Isida (which are given in Table 4.5 at p. 149 of Ref. (2) for $d/h=0$, and $\chi=27/13$ in the case of the rigid-fixed type), it was found that both values agreed perfectly in four figures in most cases. Therefore, it is judged that this calculation has sufficient accuracy.

5. Conclusions

In this work, crack opening displacements in fixed sided plates were analyzed by using the method of continuously distributed dislocations model for various lengths and positions of the crack. As a result, it was found that when the ratio of the crack length and the width of the plate becomes large, crack opening displacement has peaks which exceed the applied

displacement.

References

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- (2) Isida, M., Elastic Analyses and Stress Intensity Factors of Cracks, (in Japanese), (1976), p. 149, Baifu-kan.