Numerical Analysis of an Axisymmetric Jet Using a Streamline Coordinate System*

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The complete equations which describe a viscous axisymmetric flow are formulated using a streamline coordinate system (ϕ, ψ) , where ψ is the stream function. The independent variable ϕ is constant along the orthogonal trajectories of streamlines in a plane including a symmetric axis. The axisymmetric laminar jet leaving a Poiseuille tube for an inviscid outer fluid phase is solved numerically using parabolic differential equations which are simplified by a boundary-layer-type analysis. The prediction of the velocity distributions and shape of the free surface profile, under the influence of both gravity and surface tension, is undertaken. The results obtained by the present method are found to agree very closely with experimental results. The behavior of a laminar jet issuing from a convergent nozzle with a small convergence angle is also investigated by the same method.

Key Words: Fluid Mechanics, Numerical Analysis, Free Surface, Streamline Coordinate System, Axisymmetric Jet

1. Introduction

The analyses of a two-dimensional flow using a streamline coordinate system have been considered for a number of years⁽¹⁾. Recently, the complete equations which describe an incompressible viscous two-dimensional flow have been formulated using the streamline coordinate system⁽²⁾⁽³⁾. The two-dimensional laminar jets issuing from a channel in which there is Poiseuille flow far upstream⁽²⁾ and from a skew-symmetrical orifice, such as the restriction of spool valves widely used in hydraulic power applications⁽⁴⁾⁽⁵⁾, were solved numerically using the streamline coordinate system. Results from such studies proved that the streamline coordinate system is very useful for investigating a laminar jet which has free surfaces. Furthermore, the streamline coordinate system was applied to estimating some of the principal physical variables of fluid flow, through flow-visualization and image-processing techniques⁽⁶⁾. On the

other hand, the problems of potential axisymmetric flow from a large reservoir through a circular orifice⁽⁷⁾ and through Griffith diffusers⁽⁸⁾ were formulated by considering the streamline coordinate system which has the velocity potential and Stokes stream function as the independent variable, and the radial and axial dimensions as the dependent variables. The above numerical solutions of the axisymmetric flow field were limited to the assumption of potential flow.

In this study, the problem of an incompressible viscous axisymmetric flow is formulated using the streamline coordinate system. The axisymmetric laminar jet leaving a Poiseuille tube for an inviscid outer fluid phase is solved numerically using parabolic differential equations⁽⁹⁾ which are simplified by a boundary-layer-type analysis. The results obtained by the present method are compared with the experimental results⁽¹⁰⁾, and the method used is found to be very useful for investigating the behavior of an axisymmetric laminar jet of incompressible viscous fluid.

The axisymmetric laminar jet issuing from a circular tube with uniform cross section has long been theoretically studied by several researchers. Tillet⁽¹¹⁾

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theoretically studied the jet of liquid, open to the atmosphere, that emerges from a two-dimensional channel with the aid of the method of matched asymptotic expansions, which can be employed for the case of an axisymmetric jet leaving a circular pipe. Neglecting gravitational effects, Goren and Wronski⁽¹²⁾ considered theoretically the liquid in a laminar jet issuing from a long capillary needle into an immiscible and inviscid medium such as air. The behavior of an axisymmetric laminar jet of incompressible viscous fluid was investigated numerically by using the Galerkin formulation of the finite element method⁽¹³⁾, the finite difference method⁽¹⁴⁾ and the method developed for approximating the transverse velocity component in the incompressible boundary layer equations⁽¹⁵⁾. The cylindrical coordinate system was used for the above calculations. Duda and Vrentas⁽¹⁶⁾ developed the new coordinate system, symbolically referred to as the Protean coordinate system, whose fundamental formation is the use of the stream function as the independent variable in the radial direction of the jet. Essentially, the coordinate transformation is an extension of the method developed by von Mises⁽¹⁷⁾ for the transformation of the boundary-layer equations into a more tractable form. It is expected that the accuracy of the finite-difference solutions using the Protean coordinate system is not satisfactory in the flow fields where the radius of curvature of a streamline is small, because the Protean coordinate system is non-orthogonal.

In this paper, the laminar jet, whose free surface contracts rapidly and whose radius of curvature of the streamline at the free surface is small near the tube outlet, under the influence of both gravity and surface tension, is investigated when the influence of viscosity is relatively small compared with gravity. The prediction of the velocity distributions and shape of the free surface profile is compared with the experimental results.

The behavior of the laminar jet issuing from a convergent nozzle with a small convergence angle is also investigated by the same method. The results obtained are compared with the results of the laminar jet issuing from a circular tube with uniform cross section to show the effects of the convergence angle on the velocity distributions and on the shape of the free surface profile.

Nomenclature

- Fr : Froude number $= U^{*2}/(gL^*)$
 g : acceleration of gravity
 L^* : reference length
 p : dimensionless pressure $= p^*/(\rho U^{*2})$
 q : dimensionless velocity vector $= \mathbf{q}^*/U^*$

- q : dimensionless velocity $= \mathbf{q}^*/U^*$
 Re : Reynolds number $= L^*U^*/\nu$
 r : dimensionless radial coordinate $= r^*/L^*$
 U^* : reference velocity
 v_r : dimensionless radial velocity $= v_r^*/U^*$
 v_z : dimensionless axial velocity $= v_z^*/U^*$
 We : Weber number $= \rho U^{*2}L^*/\sigma^*$
 z : dimensionless axial coordinate $= z^*/L^*$
 α : convergence angle (see Fig. 2)
 ν : kinematic viscosity of fluid
 ρ : density of fluid
 σ^* : surface tension
 ϕ : new coordinate along a streamline
 ψ : dimensionless stream function $= \psi^*/(U^*L^{*2})$
 Ω : dimensionless vorticity vector $= \Omega^*/(U^*/L^*)$
 Ω : dimensionless vorticity $\Omega^*/(U^*/L^*)$

Superscript

- * : dimensional quantity (g , ν and ρ are dimensional)

2. Mathematical Formulation

2.1 Development of the streamline coordinate system

The plane including a symmetric axis of the cylindrical coordinate system is illustrated in Fig. 1. A streamline coordinate system (ϕ, ψ) is developed. The coordinate ϕ is in the streamwise direction and ψ is orthogonal to the streamlines. The coordinate ψ also denotes the stream function.

The stream function ψ is defined by Eqs. (1).

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}, \quad v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad (1)$$

where v_r and v_z are the components of dimensionless velocity in the directions r and z , respectively.

The function ϕ is defined by the following equations:

$$\frac{\partial \phi}{\partial r} = \beta v_r, \quad \frac{\partial \phi}{\partial z} = \beta v_z \quad (2)$$

where β is a factor and a function of r and z .

Fig. 1 Streamline coordinate system

Equations (1) and (2) satisfy the condition in which the ϕ -lines perpendicularly intersect the ψ -lines. Namely, the streamlines are the lines $\psi = \text{constant}$, and the orthogonal trajectories to the streamlines are the lines $\phi = \text{constant}$. In potential flows, ϕ reduces to the velocity potential and $\beta = 1$, because groups of curves perpendicularly intersecting streamlines are equipotential lines.

On the other hand, if $\phi = \phi(r, z)$ and $\psi = \psi(r, z)$, then r and z must be functions of ϕ and ψ , such that⁽⁷⁾⁽¹⁸⁾

$$\left. \begin{aligned} \frac{\partial r}{\partial \phi} &= \frac{1}{J} \frac{\partial \psi}{\partial z}, & \frac{\partial r}{\partial \psi} &= -\frac{1}{J} \frac{\partial \phi}{\partial z} \\ \frac{\partial z}{\partial \phi} &= -\frac{1}{J} \frac{\partial \psi}{\partial r}, & \frac{\partial z}{\partial \psi} &= \frac{1}{J} \frac{\partial \phi}{\partial r} \end{aligned} \right\} \quad (3)$$

in which J is the Jacobian determinant

$$J = \frac{\partial \phi}{\partial r} \frac{\partial \psi}{\partial z} - \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial r} \quad (4)$$

Substituting Eqs.(3) into the equations which are obtained by eliminating the variables v_r and v_z from Eqs.(1) and (2), respectively, gives

$$-\frac{1}{r} \frac{\partial r}{\partial \phi} = \frac{1}{\beta} \frac{\partial z}{\partial \psi} \quad (5)$$

$$\frac{1}{r} \frac{\partial z}{\partial \phi} = \frac{1}{\beta} \frac{\partial r}{\partial \psi} \quad (6)$$

Since the dimensionless velocity q is expressed as $\sqrt{v_r^2 + v_z^2}$, the substitution of Eqs.(1)~(4) into the relation ($q = \sqrt{v_r^2 + v_z^2}$) yields

$$q = \frac{1}{r \sqrt{\left(\frac{\partial z}{\partial \psi}\right)^2 + \left(\frac{\partial r}{\partial \psi}\right)^2}} \quad (7)$$

2.2 Transforming the Navier-Stokes equations and the vorticity transport equation into the new coordinate frame

The Navier-Stokes equations are written in the following vector form, for viscous flow:

$$(\mathbf{q} \cdot \nabla) \mathbf{q} = -\nabla p + \mathbf{K} + \frac{1}{Re} \nabla^2 \mathbf{q} \quad (8)$$

where $\mathbf{q}$, $\mathbf{K}$ and p are the dimensionless velocity vector ($= q\mathbf{e}_\phi$), the dimensionless body force and the dimensionless pressure, respectively. ∇ denotes a dimensionless vector differential operator which is defined by

$$\nabla = \mathbf{e}_\phi \beta q \frac{\partial}{\partial \phi} + \mathbf{e}_\psi r q \frac{\partial}{\partial \psi} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \quad (9)$$

where $\mathbf{e}_\phi$, $\mathbf{e}_\psi$ and $\mathbf{e}_\theta$ are the dimensionless unit vectors in the directions ϕ , ψ and θ , respectively. θ is the azimuthal coordinate variable.

The vorticity vector $\boldsymbol{\Omega} (= \boldsymbol{\Omega} \mathbf{e}_\theta)$ is defined by

$$\boldsymbol{\Omega} = \nabla \times \mathbf{q} = \frac{r q^2}{\beta} \frac{\partial \beta}{\partial \phi} \mathbf{e}_\theta \quad (10)$$

where $\nabla^2 \mathbf{q} = -\nabla \times \boldsymbol{\Omega}$ for incompressible flow. Substituting Eqs.(9) and (10) into Eq.(8) and rearranging by using the relation of $\mathbf{q} = q\mathbf{e}_\phi$, the equations of motion in the ϕ - and ψ - directions can be expressed in the following form:

$$\beta q^2 \frac{\partial q}{\partial \phi} = -\beta q \frac{\partial p}{\partial \phi} + K_\phi - \frac{q}{Re} \frac{\partial}{\partial \psi} (r \Omega) \quad (11)$$

$$\frac{r q^2}{\beta} \frac{\partial}{\partial \psi} (\beta q) = -r q \frac{\partial p}{\partial \psi} + K_\psi + \frac{\beta q}{Re} \frac{1}{r} \frac{\partial}{\partial \phi} (r \Omega) \quad (12)$$

where K_ϕ and K_ψ are the components of the dimensionless body force which act upon a unit mass in the directions ϕ and ψ , respectively.

Taking the curl of each term of Eq.(8) to eliminate the pressure term, the vorticity transport equation, which is written in the following dimensionless form, is obtained.

$$-\boldsymbol{\Omega} \cdot \nabla \mathbf{q} + \mathbf{q} \cdot \nabla \boldsymbol{\Omega} = \nabla \times \mathbf{K} + \frac{1}{Re} \nabla^2 \boldsymbol{\Omega} \quad (13)$$

Taking the $\mathbf{e}_\theta$ component out of Eq.(13), we finally obtain

$$\begin{aligned} \beta q^2 \frac{\partial \Omega}{\partial \phi} - \frac{\beta q^2 \Omega}{r} \frac{\partial r}{\partial \phi} &= \beta q \frac{\partial K_\psi}{\partial \phi} - r q \frac{\partial K_\phi}{\partial \psi} \\ &\quad - \frac{\beta}{r} K_\psi \frac{\partial}{\partial \phi} (r q) + \frac{r}{\beta} K_\phi \frac{\partial}{\partial \psi} (\beta q) \\ &\quad + \frac{1}{Re} \left[\beta q^2 \frac{\partial \beta}{\partial \phi} \frac{\partial \Omega}{\partial \phi} + \beta^2 q^2 \frac{\partial^2 \Omega}{\partial \phi^2} + \left(2 r q^2 \frac{\partial r}{\partial \phi} \right. \right. \\ &\quad \left. \left. - \frac{r^2 q^2}{\beta} \frac{\partial \beta}{\partial \phi} \right) \frac{\partial \Omega}{\partial \psi} + r^2 q^2 \frac{\partial^2 \Omega}{\partial \psi^2} - \frac{\Omega}{r^2} \right] \end{aligned} \quad (14)$$

Under the influence of gravity as the body force, the combination of Eqs.(1)~(4) leads to Eqs.(15):

$$K_\phi = \frac{r q}{Fr} \frac{\partial r}{\partial \psi}, \quad K_\psi = \frac{r q}{Fr} \frac{\partial z}{\partial \phi} \quad (15)$$

where Fr is the Froude number, defined as:

$$Fr = \frac{U^{*2}}{L^* g} \quad (16)$$

There are two ways to choose some dependent variables for solving the equations mentioned in this chapter. One way is that Eqs.(5)~(7), (11) and (12) can be solved by choosing r , z , p , q and β as the dependent variable after eliminating Ω by substituting Eq.(10) into Eqs.(11) and (12). The other is that Eqs.(5)~(7), (10) and (14) can be solved by choosing r , z , q , β and Ω as the dependent variable.

3. Application of the Governing Equations to the Axisymmetric Jet

The governing equations mentioned in Chapter 2 are applied to the axisymmetric jet as shown in Fig. 2.

3.1 Numerical calculation

The dependent variables r , z , q , β and Ω are chosen in order to solve Eqs.(5)~(7), (10) and (14). The numerical marching integration from an upstream station to a downstream one is employed to obtain finite-difference solutions to simplified forms of the partial differential equations (vorticity transport equation, etc.), converting the elliptic partial differential equations to parabolic equations and, thus, formulating an initial-value problem.

The quantities at a point P are denoted by subscripts i and j in Fig. 2. The index i increases with ϕ ,

Fig. 2 Axisymmetric jet

and j with ϕ , such that $\phi = (i-1)\Delta\phi$ and $\psi = (j-1)\Delta\psi$, where $\Delta\phi$ and $\Delta\psi$ are the spacings of the finite difference mesh. The distance $\overline{AB}$ is chosen as a reference length L^* and the average velocity in the exit section $\overline{AB}$ as a reference velocity U^* . Furthermore, the values of the quantities at each point are denoted by the subscripts shown in Fig. 2.

Let us suppose that the ϕ -direction is a predominant direction of flow, and the upstream conditions can be used to determine the downstream flow properties (i.e. $\partial/\partial\phi \ll \partial/\partial\psi$). Then, the first and third terms, and the first and second terms in the bracket, on the right-hand side of Eq.(14) can be eliminated after substituting Eqs.(5),(6) and (15) into Eq.(14). Thus, the differential equation(14) can be expressed by using forward difference formulas in the ϕ -direction and central difference formulas in the ψ -direction, in the following finite-difference form:

$$\Omega_p = A_E \Omega_E + A_S \Omega_S + A_T \Omega_T + A_C \quad (17)$$

where

$$\left. \begin{aligned} A_E &= \frac{r_P}{r_E} - \frac{r_E \Delta\phi}{\beta_E^2 \Delta\phi^2 R_e} \left(2r_E \beta_E + \frac{\beta_E \Delta\phi^2}{r_E^3 q_E^2} \right) \\ A_S &= \frac{r_E \Delta\phi}{4\beta_E^2 \Delta\phi^2 R_e} \left[4r_E \beta_E + 2\beta_E (r_S \right. \\ &\quad \left. - r_T) - r_E (\beta_S - \beta_T) \right] \\ A_T &= \frac{r_E \Delta\phi}{4\beta_E^2 \Delta\phi^2 R_e} \left[4r_E \beta_E - 2\beta_E (r_S \right. \\ &\quad \left. - r_T) + r_E (\beta_S - \beta_T) \right] \\ A_C &= \frac{r_E \Delta\phi}{4\beta_E q_E F_r \Delta\phi^2} \left[-(q_S r_S - q_T r_T) (r_S - r_T) \right. \end{aligned} \right\}$$

$$\left. \begin{aligned} &-4q_E r_E (r_S - 2r_E + r_T) + \frac{r_E}{\beta_E} (r_S - r_T) \\ &\times (\beta_S q_S - \beta_T q_T) \end{aligned} \right] \quad (18)$$

On the basis of the same assumption, elimination of the variable z from Eqs.(5) and (6) gives the following finite-difference equation:

$$r_P^2 = \frac{1}{2} \left[r_K^2 + r_L^2 - \frac{1}{4\beta_E} (\beta_S - \beta_T) (r_K^2 - r_L^2) \right] \quad (19)$$

Combination of Eqs.(5) and (7) leads to Eq.(20):

$$r_{in}^2 = r_{i,n-1}^2 + 2\Delta\psi \sqrt{\frac{1}{q_{i-1,n}^2} - \beta_{i-1,n}^2} \left(\frac{r_{in} - r_{i-1,n}}{\Delta\phi} \right)^2 \quad (20)$$

In Eqs.(19) and (20), the values of the variables q and β at the point (i, j) are approximated by those at the point $(i-1, j)$, after approximating the partial derivatives by backward difference formulas in the ϕ -direction and central difference formulas in the ψ -direction. This is done so as to enable the equations to be solved by numerical marching integration.

Furthermore, we obtain the following:

from Eq.(6),

$$z_P = z_E + \frac{\Delta\phi}{4\beta_E \Delta\psi} (r_S^2 - r_T^2) \quad (21)$$

from Eq.(7),

$$q_P = 1 / \sqrt{r_P^2 \left(\frac{z_R - z_L}{2\Delta\psi} \right)^2 + \frac{1}{4} \left(\frac{r_R^2 - r_L^2}{2\Delta\psi} \right)^2} \quad (22)$$

and from Eq.(10),

$$\beta_P = \beta_L + 2\Delta\psi \frac{\beta_L (\Omega_P + \Omega_L)}{(r_P + r_L)(q_P^2 + q_L^2)} \quad (23)$$

The boundary conditions in the exit section are determined as follows.

Fully developed Poiseuille flow is given at the exit $\overline{AB}$ in the case of the jet issuing from a circular tube ($\alpha=0$ in Fig. 2). On the other hand, for the jet issuing from a convergent nozzle ($\alpha>0$), a Stokes solution⁽¹⁹⁾, which corresponds to purely radial flow with the apex o' , is given at the exit section $\overline{AB}$. Ackerberg⁽²⁰⁾ considered the steady, axisymmetric, converging motion of a viscous incompressible fluid inside an infinite right circular cone. He reported that there is no purely radial flow in the region of a large convergence angle. Consequently, considering his investigation, the value of the convergence angle α is chosen in the region of a small value.

The surface normal stress σ_n is given by the following equation⁽¹³⁾:

$$\sigma_n = -\frac{\lambda}{W_e} \quad (24)$$

where W_e and λ are Weber number and the mean curvature⁽¹⁵⁾⁽²¹⁾ of the free surface. W_e and λ are defined by

$$W_e = \frac{\rho U^{*2} L^*}{\sigma^*} \quad (25)$$

$$\lambda = \left[\frac{1}{r} - \frac{r}{\beta} \frac{\partial}{\partial \psi} (\beta q) \right]_{\psi=\psi_f} \quad (26)$$

where ψ_f is the value of ψ at the free surface and σ^* is the surface tension.

On the other hand, the normal stress σ_n is expressed as

$$\sigma_n = -p - \frac{2}{Re} \frac{q\beta}{r} \frac{\partial}{\partial \phi} (rq) \quad (27)$$

On the basis of the assumption $\partial/\partial \phi \ll \partial/\partial \psi$, eliminating the viscous term of Eq.(27), the substitution of Eqs. (15), (24) and (27) into Eq.(11) yields

$$\frac{\partial q^2}{\partial \phi} = 2 \left[-\frac{1}{We} \frac{\partial \lambda}{\partial \phi} + \frac{1}{Fr} \frac{r}{\beta} \frac{\partial r}{\partial \psi} - \frac{1}{Re} \frac{1}{\beta} \frac{\partial}{\partial \psi} (r\Omega) \right] \quad (28)$$

Furthermore, the shear stress τ is given by

$$\tau = \frac{rq}{Re} \left[\frac{\partial q}{\partial \psi} + \frac{1}{\beta} \frac{\partial}{\partial \psi} (\beta q) \right] \quad (29)$$

Substitution of Eq.(10) into Eq.(29) yields

$$\Omega = -r \frac{\partial q^2}{\partial \psi} \quad (30)$$

if $\tau=0$ is used as the boundary condition at the free surface.

The boundary conditions at the symmetric axis ($\psi=0$) are determined as follows:

$r_{i1}=0$, $z_{i1}=z_{i2}$, $\beta_{i1}=0.005$, $\Omega_{i1}=0$ and q_{i1} is calculated by the finite-difference equation obtained by Eq. (7).

The sequence of calculation steps is the following:

1. The initial values of r , z , q , β and Ω in the exit section (1,j) are given.
2. r_{2j} satisfying Eqs.(19) and (20) may be determined through a successive-over-relaxation (SOR) scheme. This iteration is continued until the value of $\max[(r_{2j}^{(k)} - r_{2j}^{(k-1)})/r_{2j}^{(k)}]$ is less than 10^{-4} , where k is an

iteration level.

3. The inner values of Ω_{2j} , z_{2j} , q_{2j} and β_{2j} are obtained by Eqs.(17),(21),(22) and (23).

4. The values of z_{2n} , β_{2n} , q_{2n} and Ω_{2n} at the free surface are calculated by the finite-difference equations obtained by Eqs.(6),(23),(28) and (30), respectively, and the values at the axisymmetric axis (2,1) are given by the corresponding boundary conditions.

5. After setting $i=i+1$, steps 2,3 and 4 are then repeated until the end of the flow domain is reached. On the last sweep, the converged distribution of velocities, vorticities, etc. are printed out as required. In the present calculation, the number of mesh points in the ψ -direction n and the spacings of the finite difference mesh are determined as follows:

$$n=41, \Delta\phi=2.5 \times 10^{-5}, \Delta\psi=3.125 \times 10^{-3}.$$

3.2 Results and discussion

In Fig. 3, the free surface profiles calculated by the present method are illustrated by the continuous lines. The corresponding experimental results⁽¹⁰⁾ are indicated by the broken lines. We find that our numerical solutions are in very close agreement with the experimental results, with respect to the shape of the free surface. There is a small difference between the calculated results and the corresponding experimental results near the tube outlet in Case 4, because the radius of curvature of the streamline is small. In Cases 1, 2, 3 and 4, the calculated contraction of jets is greater than the experimental one near the tube outlet. Duda and Vrentas⁽¹⁶⁾ showed the similar tendencies in their investigation. The experimental contraction of jets, however, is greater than the calculated one in the downstream region. The difference between the calculated and experimental contraction in the downstream region, which seems to be caused by the accumulation of integral error, increases gradually as the distance z increases, but it is very small.

The axial velocity distributions at cross sections of jets are shown in Figs. 4 and 5. The axial velocities increase over a cross section of a jet until a uniform velocity distribution is attained. The results are similar to the analytical results of the two-dimensional laminar jet⁽²²⁾. The axial velocity distributions calculated by the present method are found to agree very closely with experimental results⁽¹⁰⁾ measured by a laser Doppler velocimeter.

We find that our numerical solutions are in very close agreement with the experimental results, with respect to the shape of the free surfaces and the velocity distributions, as described above. Consequently, the present method proved to be very useful for investigating the behavior of an axisymmetric jet of incompressible viscous fluid.

The calculated results of the jet issuing from a

Fig. 3 Free surface profiles of numerical results and experimental ones

Fig. 4 Axial velocity distribution of a jet issuing from a circular tube ($Re=2162$)

Fig. 5 Axial velocity distribution of a jet issuing from a circular tube ($Re=1836$)

Fig. 6 Free surface profiles of jets issuing from a circular tube and a convergent nozzle

convergent nozzle are as follows. The shapes of the free surfaces of jets issuing from a circular tube with uniform cross section and a convergent nozzle are shown in Fig. 6. For the jet issuing from a convergent nozzle, the jet contraction is greater than that of the jet issuing from a circular tube for $Re=744$ and 2218 .

The typical behavior of the velocity at the surface of jets is shown as a function of dimensionless jet length in Fig. 7. For the jet issuing from a convergent nozzle the velocities are slightly greater than those of the jet issuing from a circular tube over a free surface.

The axial velocity distributions over cross sec-

Fig. 7 Dependence of surface velocities on jet length

tions of jets are shown in Fig. 8 for $Re=2218$, $Fr=0.1205$ and $We=4.14$. For the jet issuing from a convergent nozzle, the velocities are slightly greater than those of the jet issuing from a circular tube over a cross section, and the difference in axial velocities between the two types of the jets is remarkable near the free surface.

4. Conclusions

Assuming an incompressible and axisymmetric flow, the complete equations which describe a viscous flow have been formulated using the streamline coordinate system (ϕ, ψ) , where ϕ is the coordinate of the flow direction and ψ is the stream function. The

Fig. 8 Axial velocity distributions of jets issuing from a circular tube and a convergent nozzle

axisymmetric laminar jet leaving a Poiseuille tube for an inviscid outer fluid phase, under the influence of both gravity and surface tension, has been solved numerically using parabolic differential equations which are simplified by a boundary-layer-type analysis. Comparing the shape of the free surface and the velocity distribution obtained by the present method with the experimental results, the present method proved to be very useful for investigating the behavior of an axisymmetric laminar jet of incompressible viscous fluid. Furthermore, the behavior of a laminar jet issuing from a convergent nozzle with a small convergence angle was also investigated by the same method. Comparing the results with those of the jet issuing from a circular tube, a remarkable behavior of the jet issuing from a convergent nozzle is described. On the basis of the results obtained in the present investigation, the following conclusions can be drawn.

(1) The present method using the parabolic partial differential equations converted from the elliptic partial differential equations formulated using the streamline coordinate system is very useful for investigating an axisymmetric laminar jet of incompressible viscous fluid with a free surface.

(2) For the jet issuing from a convergent nozzle, the jet contraction is greater than that of the jet issuing from a circular tube with uniform cross section.

(3) For the jet issuing from a convergent nozzle, the axial velocities are slightly greater than those of the jet issuing from a circular tube over a cross section, and the difference in axial velocities between the two types of the jets is remarkable near the free surface.

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