Flows of Non-Newtonian Fluids*

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The present paper summarizes various problems on non-Newtonian fluid mechanics. Due to complex rheological behaviors of non-Newtonian fluids, constitutive equations have highly nonlinear character. As two approaches to obtain constitutive equations, continuum mechanics and molecular theory are described, and then a number of models are presented. Various flow problems are discussed. With regard to measuring methods, the hole pressure error and the first normal stress difference of dilute polymer solutions are stated. Moreover, steady flow in straight pipes, entrance flow through abrupt contractions, flow instabilities, Toms effect, and numerical methods are briefly described.

Key Words : Fluid Mechanics, Non-Newtonian Fluid, Viscoelastic Fluid, Constitutive Equation, Pipe Flow, Drag Reduction, Flow Instability

1. Introduction

It is well known that fluids with complex structure, such as macromolecular fluids, soap solutions, and two-phase fluids are not described by Newton's law of viscosity, which is a simple linear relation between the stress tensor and the rate-of-strain tensor. The research field elucidating complex mechanical behaviors of these fluids belongs to that of rheology. The disciplines of classical fluid mechanics, strength of materials, and plasticity have their foundations, at least in part, in rheology and in mechanics. Non-Newtonian fluid mechanics, being of greatest interest to the polymer fluid dynamicist and the polymer engineer, is also the discipline combining rheology with mechanics. However, the main difference between non-Newtonian fluid mechanics

and classical Newtonian fluid mechanics is that due to the complexity of non-Newtonian response, governing equations vary with the flow type under consideration, whereas in the classical situation, the Navier-Stokes equations are accepted unquestionably as the governing equations. Thus, the selection of an appropriate constitutive equation used for analysis of flow problems is an open and major question in this research field. A number of textbooks in non-Newtonian fluid mechanics have been published up to the present^{(1)~(12)}.

As is characteristic of an interdisciplinary field, studies on non-Newtonian fluid flows include a wide range of subjects. To review this field in depth would be very complex and difficult. Therefore, we shall review mainly the fluid dynamics problems of macromolecular fluids.

2. Constitutive Equation

In order to clarify quantitatively a given flow field, a constitutive equation relating the stress tensor to various kinematic tensors is first required. However, this equation usually gives only the relation

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between the stress tensor and the rate-of-strain tensor for a fluid element in a flow field, and so except for the case of specially simple flow, only the constitutive equation cannot determine the flow field. Consequently, we need other equations that are obtained from the conservation laws of mass, momentum and energy to solve fluid dynamic problems. The constitutive equation of incompressible Newtonian fluid is well-known :

$$\sigma = -p\delta + P, \quad P = 2\eta e \quad (1)$$

Here, σ denotes stress tensor, P is an extra stress tensor, η viscosity, p isotropic pressure, e rate-of-strain tensor, and δ stands for unit tensor. Non-Newtonian fluids have much more complex mechanical behavior than Newtonian fluids. Consequently, the constitutive equation to express such behavior may have a strongly nonlinear character. Using the complicated constitutive equation with those of change (continuity, motion, and energy), we can hardly treat many important flow fields by analytical or numerical methods. Therefore, from the engineering viewpoint, we desire a simple constitutive equation as far as it expresses correctly rheological behavior. Thus, the determination of a constitutive equation is a central problem in non-Newtonian fluid mechanics. As the source of information for obtaining this key equation, we have two approaches, namely continuum mechanics and molecular theory.

(i) Continuum mechanics : In this approach, an appropriate generalization of an empirical formula for a simple flow is usually tried for arbitrary, time-dependent flow. This generalization, however, ordinarily has a considerable amount of arbitrariness, and so some principles are required in order to restrict the arbitrariness. They include the principle of coordinate invariance and the principle of material objectivity. The principle of material objectivity provides a general approach to the formulation of a constitutive equation. It expresses that the response of a material to a given experience or history of motion is independent of any motion of the person observing the response. However, a continuum mechanics approach cannot provide the explicit functional form for constitutive equations, and also cannot give the interrelations between molecular structures and bulk properties of macromolecular fluids.

(ii) Molecular theory : It is generally very difficult to describe the internal motions of the macromolecule and its interactions with other macromolecules or with the solvent. Therefore, in this approach we make use of some kind of simplified mechanical model. However, it is evident that no one approach is capable of providing complete and totally reliable information about the constitutive equation.

We have to select a suitable constitutive equation for use in the solution of non-Newtonian fluid dynamics problems.

2.1 Non-Newtonian fluid models obtained from a continuum mechanics approach⁽¹⁾⁻⁽¹²⁾

Many models such as the power-law model, the Carreau model, etc. have been proposed to express non-Newtonian viscosity behavior. Here, the model denotes a mathematical model which represents the constitutive equation. The Bingham model, the Cason model, and others have been used to describe rheological behavior of plastic fluids. The current problem in rheology is the development of viscoelastic fluid models displaying complex mechanical behavior of macromolecular fluids. To date, many proposed nonlinear viscoelastic fluids have been conveniently classified as follows: (a) Nonlinearization of linear viscoelastic fluid model. (b) Simple fluid model. (c) Other viscoelastic fluid models. This classification consists of modified nonlinear elastic and nonisotropic fluid models. The models in (a), (b) and (c) are written in differential or integral form.

Here, fluids obeying the assumption that the stress in a given fluid particle depends only on the kinematic history of that particle alone are referred to as simple fluids.

Now, as stated in the preceding section, models must satisfy the principle of material objectivity, that is, constitutive equations must be objective. As reference frames, so that the objectivity of the model be automatically insured, two frames which are co-rotating and co-deforming are obtained. However, in the flow observation and the fluid mechanics treatment, we usually use the fixed reference frame. The transformation rules from the co-rotating and the co-deforming to the fixed frames are found.

2.2 Polymer fluid models based on a molecular theory approach⁽¹³⁾⁻⁽¹⁸⁾

2.2.1 Dilute polymer solutions

We idealize the polymer molecules as a collection of mass points (beads) joined together by some kind of connector (springs, rods, or dashpots). The crudest macromolecular models are the dumbbell models, namely, two beads joined by a connector. Further models exist where linear chains of beads are joined together by some kind of connector. The constitutive equations obtained from these models can be compared with the available experimental data on viscosity, complex viscosity and normal stress differences, and the qualitative agreement is quite good.

2.2.2 Concentrated polymer solutions and polymer melts

We have the network and the reptation models. The network model⁽¹⁹⁾⁻⁽²¹⁾ has been inspired by the

successful network model for rubber elasticity, and it visualizes a polymer fluid as a temporary network formed by physical entanglements. The resulting constitutive equations are slightly complex, but they should be used for solving fluid mechanics problems by numerical methods.

In the reptation model⁽²²⁾, the motion of each polymer molecule is conceived as reptation, namely, moving backwards and forwards, through a tube. The idea of the tube is introduced to take into account the constraint imposed on the polymer molecule by entanglements with neighboring molecules. The constitutive equations obtained from this model can quantitatively explain complex rheological behavior of concentrated polymer solutions and polymer melts.

3. Fluid Mechanics Problems

Even for Newtonian fluids whose mechanical behavior is expressed by the simple linear constitutive equation (Eq. (1)), there are many open questions with respect to fluid mechanics. Consequently, flow problems of non-Newtonian fluids expressed by complex nonlinear models are extremely difficult to solve by using analytical and numerical methods. Non-Newtonian fluids display anomalous flow phenomena that do not occur in Newtonian fluids. Not one of these fluids has been solved perfectly to date because of mathematical difficulty and the lack of an established model. In Newtonian fluid mechanics, for an established model, various important flow fields from an engineering viewpoint can be clarified by using analytical, numerical and experimental methods. On the contrary, in the research field of non-Newtonian fluids, a simple basic flow is examined in order to determine the values of rheological constants in models; this is the salient feature in this research field, as opposed to that of Newtonian fluid mechanics. Flow problems of non-Newtonian fluid include a wide range of complex subjects; several important problems are described in the following.

3.1 Problems in measuring methods

3.1.1 Hole pressure error

On pressure measurements in a fluid flow, use of a liquid-filled hole is very common. Broadbent et al.⁽²³⁾ have reported that such a measuring method gives rise to a systematic error for the normal stress difference in macromolecular fluids.

Many experimental and theoretical studies^{(24)~(30)} were subsequently made regarding this effect. Most of them confirmed that the value of the normal stress measured by the liquid-filled hole is smaller than that of the real one at the wall. It is considered that the low pressure reading is produced by an extra tension along a streamline which moves the fluid out of the hole.

Accordingly to Higashitani-Pritchard's analysis⁽²⁶⁾, the hole pressure error P_H is

$$P_H \approx (1/6)\Delta\sigma_{n1} \quad (2)$$

where $\Delta\sigma_{n1}$ denotes first normal stress difference. A measuring value is obtained as follows⁽²⁵⁾.

$$P_H/\Delta\sigma_{n1} = 0.15 \sim 0.18$$

The visualization of flow in the vicinity of a hole by Hou et al.⁽³⁰⁾ may be utilized for analysis of this effect.

3.1.2 Measurement of the first normal stress difference of dilute polymer solutions

Measurements of the normal stress difference in polymer melts and concentrated polymer solutions have been performed by many investigators using various measuring devices, such as cone-plate, parallel-plate, capillary, and flow birefringent instruments. Rheometry of polymer melts has been reviewed in detail by Meissner⁽³¹⁾. However, the normal stress difference of dilute polymer solutions, particularly in a range of high shear rates of laminar conditions, are extremely difficult to measure. Metzner et al.⁽³²⁾ devised a method to measure the normal stress by utilizing the swell of the jet (Barus effect) issuing from a capillary tube. Its utility, however, is limited to the case that the concentration of polymer solution is high and the normal stress is great. Shertzer and Metzner⁽³³⁾ measured the reaction force of a jet on a capillary tube and calculated the first normal stress difference. Metzner and Park⁽³⁴⁾, Patterson and Zakin⁽³⁵⁾, and Meister and Biggs⁽³⁶⁾ made similar experiments, but solutions employed in these experiments were more concentrated than ones commonly used in drag-reducing phenomena. Oliver⁽³⁷⁾ made use of the thrust of the jet impacting a vertical plate positioned at a distance downstream from the tube exit, and measured the normal stress of rather dilute polymer solutions. Mochimaru et al.⁽³⁸⁾ and Hasegawa⁽³⁹⁾ developed devices utilizing the reaction force of a jet on a capillary tube, and obtained the first normal stress differences in various dilute polymer solutions with concentrations of the order of ppm.

3.2 Steady flow in straight pipes

Green and Rivlin⁽⁴⁰⁾ have analyzed the steady flow of Reiner-Rivlin fluid through a straight pipe of elliptical cross section using perturbation method. This perturbation solution indicates the existence of circulating secondary flows in each of the four quadrants of the ellipse superposed on the rectilinear flow along the axis of the pipe. According to the new phenomenon showed in this pioneer paper, many polymer fluid dynamicists have investigated the problems of the steady flow in pipe of arbitrary cross section. Langlois and Rivlin⁽⁴¹⁾⁽⁴²⁾ have worked out the case of the Rivlin-Ericksen fluid, and obtained the analytical results that the secondary flow appears with the fourth-order, i. e.,

for a Rivlin-Ericksen fluid of grade four, but does not appear with the second-order and third-order fluids, respectively. Moreover, their analysis indicates that secondary flow does not occur when the second normal stress difference $\Delta\sigma_{n2}$ is zero, that is, $\Delta\sigma_{n2}=0$, and when $\Delta\sigma_{n2}\propto\eta(D)$ (D : shear rate, viscosity η is a function of D , that is, η shows non-Newtonian viscosity.).

Dodson et al.⁽⁴³⁾ and Townsend et al.⁽⁴⁴⁾ have carried out experimental investigations and numerical computations, based on an approximate rheological and dynamical model, for the secondary flow in a pipe of square cross section. According to Walters⁽⁴⁵⁾, the secondary flow problem is summarized as follows. (i) The secondary normal stress difference plays an important role in determining whether or not secondary flows exist. Certainly when $\Delta\sigma_{n2}$ is zero, no secondary flows are possible. The converse, however, is not true. (ii) Notwithstanding (i), the direction of the streamlines cannot be simply related to the sign of $\theta_2(\Delta\sigma_{n2}=\theta_2(D)D^2)$. (iii) The secondary flows in the cross section of a pipe are weak and have a very small effect on the mean flow rate / pressure gradient relationship. (iv) Existing experimental results for a number of elastic liquids are in agreement with the secondary flow direction shown in Fig. 1. (v) In the case of pipes of elliptic cross section, the strength of the secondary flow is greatest when the ratio of major to minor axes is approximately 1.5 : 1. (vi) Although the secondary flows are weak, they can, nevertheless, have an important effect on heat-transfer characteristics.

3.3 Entrance flow through abrupt contractions

3.3.1 Entrance flow of inelastic non-Newtonian fluids

(i) Creeping flow: For very low Reynolds numbers (less than 0.1) the sharp-edged circular entrance flow may be considered as creeping or inertialess. This entrance flow problem for Newtonian fluids has been solved using a variational method by Weissberg⁽⁴⁶⁾.

The corresponding problem for power-law fluids has been treated by Tomita⁽⁴⁷⁾, and Boles and Bogue⁽⁴⁸⁾. Later, Boger et al.⁽⁴⁹⁾ computed the end correction of entrance flow for power-law fluids using the finite-element techniques developed by Tanner et al.⁽⁵⁰⁾, and Vrentas et al.⁽⁵¹⁾ computed that for shear-thinning fluids (Carreau and Ellis models) by finite-difference methods. Also, the same problem for power-law and Carreau models was numerically solved by Kim-E et al.⁽⁵²⁾ using finite-element methods. All the authors who used numerical methods considered the case in a circular 4 : 1 contraction.

(ii) Flow of inlet region: As a fluid enters a tube at high Reynolds number (several hundred or more), it forms a boundary layer on the wall which increases in thickness as the fluid flows down the tube. In the process of forming the boundary layer, an additional pressure loss occurs in the inlet length, which is the distance from the tube entrance having a flat velocity profile to the plane having a fully developed velocity profile. In this case flow upstream of the tube entrance is not considered. The problem of the developing boundary layer, which is called the hydrodynamic entrance region problem, has been treated by many researchers for Newtonian fluids. Analytical treatments using the boundary layer approximations were first applied for the entrance flow of inelastic non-Newtonian fluid (power-law fluid) by Bogue⁽⁵³⁾, Tomita⁽⁵⁴⁾, and Collins and Schowalter⁽⁵⁵⁾. Recently, Tachibana et al.⁽⁵⁶⁾ carried out the theoretical analysis using finite-difference methods and experimental measurements of pressure drop for the steady flow of the inelastic power-law fluid in the inlet region of rectangular ducts. According to Boger⁽⁵⁷⁾, due to the revolutionary progress in digital computation, the entrance flow problem is now solved for Newtonian and shear-thinning inelastic non-Newtonian fluids without the restrictions imposed by the assumptions of either a flat entry velocity profile (boundary layer

Fig. 1 Schematic diagram of secondary flow in a pipe of square cross section (from reference (45))

theory) or a fully developed entry velocity profile (creeping flow).

3.3.2 Entrance flow of viscoelastic fluids

Understanding entrance flows of viscoelastic fluids is of importance in the fundamental flow-property measurement with a capillary rheometer and in the extrusion of polymer melts and solutions. Therefore, many experimental and analytical treatments (usually numerical) are made on the entrance flow problem. An experimental method determining the end correction for viscoelastic fluids is given by Bagley⁽⁵⁸⁾. Flow visualization studies are being carried out by many researchers^{(59)~(62)}, and the existence of a large recirculating vortex in the corner of an abrupt circular entry flow at low Reynolds number is shown. Many researchers consider that fluid elasticity is responsible for the vortex growth. However, it is confirmed by White and Baird⁽⁶³⁾ that the presence and characteristics of the vortex strongly depend on the elongational flow properties. Hasegawa and Iwaida⁽⁶⁴⁾ carried out experiments on the elongational flow of dilute polymer solutions through small apertures. Numerical solutions have largely been confined to creeping flows of constant-viscosity elastic fluids, represented by constitutive equations such as the convected Maxwell and Oldroyd B models⁽⁶⁵⁾⁽⁶⁶⁾. The hydrodynamic entrance region problem for viscoelastic fluids was first analyzed by Metzner et al.⁽⁶⁷⁾, and Kobayashi and Tomita⁽⁶⁸⁾ who also measured pressure drops and compared these results with the analytical results. A particularly detailed review of the entrance flow problem was given by Boger⁽⁶⁷⁾.

3.4 Flow instabilities

In many polymer processing operations, the rate of production is limited by the onset of flow instabilities. When polymer melts are extruded from a reservoir through a capillary or a slit, there exists a critical shear rate, below which the extruded polymer surface is smooth, but at and above which the extruded polymer surface is irregularly distorted. This phenomenon has come to be called melt fracture, although the mechanism might best be described as flow instability in many cases. We call this phenomenon the primary instable flow. A detailed review on this flow instability is given by Petrie and Denn⁽⁶⁹⁾. When shear rates exceed, by a factor of about ten, the critical shear rates of the primary instable flow, the extruded polymer solution does not show the Barus effect at the exit of the capillary even though its diameter is approximately equal to the inner diameter of the capillary. The extruded polymer solution begins to swell at some distance from the exit of the capillary and, roughly speaking, the distance changes periodically. After swelling, its flow becomes irregular

and is referred to as the primary instable flow. Making use of dilute polymer solutions, Brenschede and Klein⁽⁷⁰⁾ and others have studied such flow behavior, but they have shown only the relation of shear rates to the form of the extruded polymer solution. Tomita et al.⁽⁷¹⁾ have measured the distances and pressures in a capillary, which both change continuously with time, and have clarified the correlation between these. We call such a phenomenon the secondary instable flow. Figure 2 shows three such flow patterns, that is, (a) stable flow, (b) primary instable flow, and (c) secondary instable flow, respectively. Here $\Gamma = 4Q/(\pi R^3)$ (Q : flow rate, R : inner radius) is the apparent shear rate. It is considered that the principal causes of the occurrence of the secondary instable flow are the oscillatory flows in the vicinity of the reservoir exit and the tube inlet.

From flow visualization studies⁽⁷²⁾⁽⁷³⁾, Tordella differentiated melt fracture phenomena into two types: at the die entry and on the die wall. The latter is called slip-stick melt fracture. Doi and Edwards⁽²²⁾ have shown mathematically using their constitutive equation that the flow curve having a maximum followed by a dip are obtained, and the maximum represents an instability point. By theoretical calculations using a constitutive equation obtained from modifying the Doi-Edwards molecular theory and also by experiments, Lin⁽⁷⁴⁾⁽⁷⁵⁾ has clarified that the instability point corresponds to the onset of slip-stick melt fracture, which is a consequence of the existence of the dip on the flow curve. Tomita and Shimbo⁽⁷⁶⁾ analyzed the primary instable flow by a equation of oscillation, and the analytical results obtained for the onset of instable flow were in good agreement with experimental val-

Fig. 2 Behavior of extruded polymer solution from a capillary (from reference (71))

ues. The equation of oscillation is obtained from both the pressure balance in the reservoir-capillary system and the momentum balance of the jet between the exit of the capillary and some downstream cross section where the velocity of the jet is uniform. Through the examination of this equation, it is known that the instable flow is a kind of self-excited oscillation.

In extruding viscoelastic fluids from a duct, the pressure near the duct exit decreases with increasing flow rate when the flow rate is higher than a certain value. Shimbo et al.⁽⁷⁷⁾ determined the velocity profiles near the exit of two-dimensional ducts for an aqueous solution of polymer by tracer method. With the use of observed velocity profiles and the Denn model, they proved the abnormal behavior of pressure near the duct exit.

3.5 Toms effect

Dilute solutions of linear, random-coiling macromolecules in turbulent pipe flow display a significant reduction of frictional resistance. This anomalous phenomenon is termed the Toms effect, after Toms who first recognized it about forty years ago. The Toms effect has been widely studied, partly because it is of practical importance, and partly because it may contribute to the better understanding of turbulent structure. There are many reviews, explanatory articles^{(78)~(80)} and bibliographies^{(81)~(82)} on the Toms effect. Mean velocity profiles, turbulence intensities, etc. are usually measured using a method whereby probes are inserted in the flow, but this method is plagued by errors for flows of dilute polymer solutions. Because a laser-Doppler anemometer^{(83)~(84)} need not come into contact with the fluid in question, it has recently become popular for measuring flow quantities of polymer solutions. Turbulent flow structures in the vicinity of walls are also investigated by flow visualization using laser-holography⁽⁸⁵⁾, etc. Although the cause and the mechanism of the Toms effect are being examined by many researchers, we still lack established theory on this effect. Flows in convergent, divergent^{(86)~(87)} and curved pipes of dilute polymer solutions⁽⁸⁸⁾ are being studied theoretically and experimentally in connection with the Toms effect. On the Toms effect of external flows, many problems such as boundary layer flow along a flat plate at zero incidence, that with pressure gradient past any shape body^{(89)~(90)}, flow near a rotating disk^{(91)~(94)} and so on are also being investigated.

3.6 Numerical methods

In recent years, a large number of numerical methods have been developed for solving various flow problems of Newtonian fluid. Constitutive equations of non-Newtonian fluids have generally a nonlinear character; hence, flows in complex geometries are

very difficult to analyze theoretically. Consequently, most flow problems in non-Newtonian fluid mechanics demand numerical solutions. In the numerical computation of complex flows involving viscoelastic fluids, the implicit form and the nonlinear character of the constitutive equations give rise to a fundamental difficulty not encountered in Newtonian fluid mechanics. Various flow problems are numerically analyzed by means of finite-difference^{(95)~(997)} and finite-element methods^{(98)~(100)}. In the foreword to the proceedings of the fourth workshop on numerical methods in viscoelastic flow, Crochet⁽¹⁰¹⁾ stated the following: 「…… One should now reject the usual adage to the effect that “all viscoelastic calculations diverge at a Deborah number of order 1”. Secondly, it is obvious that the field has now entered an age of maturity where the generation of appropriate numerical methods does not simply consist of mimicking Newtonian algorithms; rather, the numericists together with the theoreticians are addressing the specific characters of viscoelastic flow. ……」. The textbook on the numerical simulation of non-Newtonian flow by Crochet et al.⁽¹⁰²⁾ has been published.

4. Concluding Remarks

Studies on the constitutive equation and various flows of the macromolecular fluid have been reviewed. Up to the present, many constitutive equations have been proposed, but it is believed by most investigators that none provides an all-purpose solution for every flow problem. In order to solve flow problems of non-Newtonian fluids, a constitutive equation which is both simple and which can describe correctly complex rheological behavior of a fluid is strongly demanded. As the constitutive equation is considered to have nonlinear character, however, flow problems are usually solved by means of numerical method. Consequently, an excellent numerical method must be developed. It is hoped that the present review will be a useful guide to the readers who are interested in non-Newtonian fluid mechanics.

References

- (1) Lodge, A. S., *Elastic Liquids*, (1964), p. 389, Academic Press.
- (2) Fredrickson, A. G., *Principles and Applications of Rheology*, (1964), p. 326, Prentice-Hall Inc.
- (3) Yamamoto, M., *Deformation of a Body*, (in Japanese), (1972), p. 373, Seibundo Sinkoshia.
- (4) Astarita, G. and Marrucci, G., *Principles of Non-Newtonian Fluid Mechanics*, (1974), p. 289, McGraw-Hill.
- (5) Hutton, J. F., Pearson, J. R. A. and Walters, K., *Theoretical Rheology*, (1975), p. 377, Applied

- Science Pub.
- (6) Tomita, Y., Rheology, (in Japanese), (1975), p. 444, Coronashia.
 - (7) Bird, R. B., Armstrong, R. C. and Hassager, O., Dynamics of Polymeric Liquids, Fluid Mechanics, Vol. 1 (1977), p. 470, John Wiley & Sons.
 - (8) Harris, J., Rheology and Non-Newtonian Flow, (1977), p. 388, Longman.
 - (9) Middleman, S., Fundamentals of Polymer Processing, (1977), p. 525, McGraw-Hill.
 - (10) Schowalter, W. R., Mechanics of Non-Newtonian Fluids, (1978), p. 300, Pergamon Press.
 - (11) Zahorski, S., Mechanics of Viscoelastic Fluids, (1982), p. 321, Martinus Nijhoff.
 - (12) Boehme, G., Non-Newtonian Fluid Mechanics, (1987), p. 351, North-Holland.
 - (13) Takahashi, M. and Onogi, S., J. Soc. Rheol., (in Japanese), Vol. 2, No. 3 (1974), p. 95 : Vol. 3, No. 1 (1975), p. 1.
 - (14) Osaki, K., J. Soc. Rheol., (in Japanese), Vol. 5, No. 4 (1977), p. 169.
 - (15) Bird, R. B., Hassager, O., Armstrong, R. C. and Curtiss, C. F., Dynamics of Polymeric Liquids, Vol. 2, Kinetic Theory, (1977), p. 257, John Wiley & Sons.
 - (16) Takano, Y., J. Soc. Rheol., (in Japanese), Vol. 2, No. 3 (1974), p. 87.
 - (17) Osaki, K., J. Soc. Rheol., (in Japanese), Vol. 9, No. 4 (1981), p. 139.
 - (18) Williams, M. C., AIChE J., Vol. 21 (1975), p. 1.
 - (19) Lodge, A. S., Rheol. Acta, Vol. 7 (1968), p. 379.
 - (20) Yamamoto, M., J. Phys. Soc. Jpn., Vol. 11 (1956), p. 413 : Vol. 12 (1957), p. 1148 : Vol. 13 (1958), p. 1200.
 - (21) Phan-Thien, N. and Tanner, R. I., J. Non-Newtonian Fluid Mech., Vol. 2 (1977), p. 353.
 - (22) Doi, M. and Edwards, S. F., J. Chem. Soc. Faraday Trans. II, Vol. 74 (1978), p. 1789, p. 1802, p. 1818 : Vol. 75 (1979), p. 38.
 - (23) Broadbent, J. M., Kaye, A., Lodge, A. S. and Vale, D. G., Nature, Vol. 217 (1968), p. 55.
 - (24) Tanner, R. I. and Pipkin, A. G., Trans. Soc. Rheol., Vol. 13, No. 4 (1969), p. 471.
 - (25) Pritchard, W. G., Rheol. Acta, Vol. 9, No. 2 (1970), p. 200.
 - (26) Higashitani, K. and Pritchard, W. G., Trans. Soc. Rheol., Vol. 16, No. 4 (1972), p. 687.
 - (27) Tomita, Y. and Hasegawa, T., Appl. Polym. Symp., Vol. 20 (1972), p. 83.
 - (28) Higashitani, K. and Lodge, A. S., Trans. Soc. Rheol., Vol. 19, No. 2 (1975), p. 307.
 - (29) Jackson, N. R. and Finlayson, B. A., J. Non-Newtonian Fluid Mech., Vol. 10 (1982), p. 71.
 - (30) Hou, T-H., Tong, T. T. and De Vargas, L., Rheol. Acta, Vol. 16, No. 5 (1977), p. 544.
 - (31) Meissner, J., Ann. Rev. Fluid Mech., Vol. 17 (1985), p. 45, Annual Reviews Inc.
 - (32) Metzner, A. B., Houghton, W. T., Sailor, R. A. and White, J. L., Trans. Soc. Rheol., Vol. 5 (1961), p. 133.
 - (33) Shertzer, C. R. and Metzner, A. B., (ed. Providence, R. I.), Proc. 4th Int. Congr. Rheol., (1963), p. 603.
 - (34) Metzner, A. B. and Park, M. G., J. Fluid Mech., Vol. 20, No. 2 (1964), p. 291.
 - (35) Patterson, G. K. and Zakin, J. L., AIChE J., Vol. 14, No. 3 (1968), p. 434.
 - (36) Meister, B. J. and Biggs, R. D., AIChE J., Vol. 15, No. 5 (1969), p. 634.
 - (37) Oliver, D. R., Canad. J. Chem. Engng., Vol. 44 (1966), p. 100.
 - (38) Mochimaru, Y., Tomita, Y. and Hasegawa, T., Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 42, No. 354 (1976), p. 564.
 - (39) Hasegawa, T., Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 44, No. 381 (1978), p. 1606.
 - (40) Green, A. E. and Rivlin, R. E., Q. Appl. Math., Vol. 14 (1956), p. 299.
 - (41) Langlois, W. E. and Rivlin, R. S., Tech. Rep. 3, Army Ordnance R & D Project, TB 2-0001, Brown Univ. (1959).
 - (42) Langlois, W. E. and Rivlin, R. S., Rend di Mat., Vol. 22 (1963), p. 169.
 - (43) Dodson, A. G., Townsend, P. and Walters, K., Comp. Fluids, Vol. 2 (1974), p. 317.
 - (44) Townsend, P., Walters, K. and Waterhouse, W. W., J. Non-Newtonian Fluid Mech., Vol. 1 (1976), p. 107.
 - (45) Walters, K., J. Non-Newtonian Fluid Mech., Vol. 5 (1979), p. 113.
 - (46) Weissberg, H. L., Phys. Fluids, Vol. 5 (1962), p. 1033.
 - (47) Tomita, Y., Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 25, No. 157 (1959), p. 938.
 - (48) Boles, R. L. and Bogue, D. C., 27th Annual Tech. Conf. Technical Papers, Soc. Plastics Eng., (1968-5).
 - (49) Boger, D. V., Gupta, R. and Tanner, R. I., J. Non-Newtonian Fluid Mech., Vol. 4 (1978), p. 239.
 - (50) Tanner, R. I., Nickell, R. E. and Bilger, R. W., Comput. Math. Appl. Mech. Eng., Vol. 6 (1975), p. 155.
 - (51) Vrentas, J. S., Duda, J. L. and Hong, S.-A., J. Rheol., Vol. 26, No. 4 (1982), p. 349.
 - (52) Kim-E, M. E., Brown, R. A. and Armstrong, R. C., J. Non-Newtonian Fluid Mech., Vol. 13 (1983), p. 341.
 - (53) Bogue, D. C., Ind. Eng. Chem., Vol. 51 (1959), p. 874.
 - (54) Tomita, Y., Chem. Eng. Jpn., (in Japanese), Vol. 23, No. 8 (1959), p. 525.
 - (55) Collins, M. and Schowalter, W. R., AIChE J., Vol. 9, No. 1 (1963), p. 804.
 - (56) Tachibana, M., Kawabata, N. and Genno, H., J. Rheol., Vol. 30, No. 3 (1986), p. 517.
 - (57) Boger, D. V., Ann. Rev. Fluid Mech., Vol. 19 (1987), p. 157.
 - (58) Bagley, E. B., J. Appl. Phys., Vol. 28, No. 5 (1957), p. 624.
 - (59) Giesekus, H., Rheol. Acta, Vol. 7, No. 27 (1968), p.

- 127.
- (60) Ramamurthy, A. V., *Trans. Soc. Rheol.*, Vol. 18, No. 3 (1974), p. 431.
- (61) Ballenger, T. F. and White, J. L., *J. Appl. Polym. Sci.*, Vol. 15, No. 8 (1971), p. 1949.
- (62) White, J. L. and Kondo, A., *J. Non-Newtonian Fluid Mech.*, Vol. 3 (1977), p. 41.
- (63) White, S. A. and Baird, D. G., *J. Non-Newtonian Fluid Mech.*, Vol. 20 (1986), p. 93.
- (64) Hasegawa, T. and Iwaida, T., *J. Non-Newtonian Fluid Mech.*, Vol. 15 (1984), p. 257, p. 279.
- (65) Mendelson, M. A., Yeh, P. W., Brown, R. A. and Armstrong, R. C., *J. Non-Newtonian Fluid Mech.*, Vol. 10 (1982), p. 31.
- (66) Keunings, R. and Crochet, M. J., *J. Non-Newtonian Fluid Mech.*, Vol. 14 (1984), p. 279.
- (67) Metzner, A. B. and White, J. L., *AIChE J.*, Vol. 11, No. 6 (1965), p. 989.
- (68) Kobayashi, T. and Tomita, Y., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 36, No. 287 (1970), p. 1109.
- (69) Petrie, C. J. S. and Denn, M. M., *AIChE J.*, Vol. 22, No. 2 (1976), p. 209.
- (70) Brenschede, E. and Klein, J., *Rheol. Acta*, Vol. 9 (1970), p. 130.
- (71) Tomita, Y., Simbo, T. and Ishibashi, Y., *J. Non-Newtonian Fluid Mech.*, Vol. 5 (1979), p. 497.
- (72) Tordella, J. P., *J. Appl. Polym. Sci.*, Vol. 1 (1963), p. 215.
- (73) Tordella, J. P., *Rheology*, Chap. 2, Vol. 5 (1969), Academic Press.
- (74) Lin, Y-H., *J. Rheol.*, Vol. 29, No. 6 (1985), p. 605.
- (75) Lin, Y-H., *J. Non-Newtonian Fluid Mech.*, Vol. 23 (1987), p. 163.
- (76) Tomita, Y. and Simbo, T., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 53, No. 487 (1987), p. 858.
- (77) Simbo, T., Tomita, Y. and Yokoyama, T., *J. Soc. Rheol. Jpn.*, (in Japanese), Vol. 11, No. 3 (1983), p. 142.
- (78) Lumley, J. L., *Annual. Rev. Fluid Mech.*, Vol. 1 (1969), p. 367.
- (79) Hoyt, J. W., *Trans. ASME, Ser. D*, Vol. 94, No. 2 (1972), p. 258.
- (80) Virk, P. S., *AIChE J.*, Vol. 21, No. 4 (1978), p. 625.
- (81) White, A. and Hemmings, J. A. G., *Drag Reduction by Additives, Review and Bibliography*, BHRA Fluid Engineering (1976).
- (82) *Drag Reducing Fluids, A Bibliography with Abstracts, Search Period Covered (1969-7)-(1978)*, National Technical Information Service, NTIS/PS-78/0730.
- (83) Reischman, M. M. and Tiederman, W. G., *J. Fluid Mech.*, Vol. 70, No. 2 (1975), p. 369.
- (84) Rudd, M. J., *Nature*, Vol. 224 (1969), p. 587.
- (85) Achia, B. U. and Thompson, D. W., *Proc. Int. Conf. Drag Reduction*, (1974), p. A 2-23.
- (86) Tomita, Y. and Mochimaru, Y., *Colloques Internationaux du CNRS Polymeres et Lubrification*, Vol. 233 (1978), p. 305.
- (87) Kato, H. and Sibanuma, H., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 45, No. 399, B (1979), p. 1669, p. 1678.
- (88) Tomita, Y. and Yokoyama, T., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 51, No. 463, B (1985), p. 806.
- (89) White, D. A., *Proc. Symp. Viscous Drag Reduction* (1969), p. 351, Plenum Press.
- (90) Canham, H. J. S., et al., *Trans. Inst. Naval Architects*, Vol. 113, No. 3 (1971), p. 187.
- (91) Granville, P. S., *NSRDC, Rep.*, 3711 (1972).
- (92) Mochimaru, Y. and Tomita, Y., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 38, No. 310 (1972), p. 1416.
- (93) Kato, H., Watanabe, K. and Ueda, K., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 38, No. 306 (1972), p. 310.
- (94) Bilgen, E. and Vasseur, P., *Trans. ASME, Ser. E*, Vol. 41, No. 1 (1974), p. 45.
- (95) Davies, A. R., Walters, K. and Webster, M. F., *J. Non-Newtonian Fluid Mech.*, Vol. 4 (1979), p. 325.
- (96) Kim, S., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 46, No. 405, B(1980), p. 873.
- (97) Horikawa, A., Mori, N. and Mimura, N., *Trans. Jpn. Soc. Mech. Eng.*, (in Japanese), Vol. 47, No. 422, B(1981), p. 1971.
- (98) Tanner, R. I., Nickell, R. E. and Bilger, R. W., *Comput. Methods Appl. Mech. and Eng.*, Vol. 6 (1975), p. 155.
- (99) Viriyayuthakorn, M. and Caswell, B., *J. Non-Newtonian Fluid Mech.*, Vol. 6 (1980), p. 245.
- (100) Coleman, C. J., *J. Non-Newtonian Fluid Mech.*, Vol. 7 (1980), p. 289.
- (101) Crochet, M. J., *J. Non-Newtonian Fluid Mech.*, Vol. 20 (1986), p. 1.
- (102) Crochet, M. J., Davies, A. R. and Walters, K., *Numerical Simulation of Non-Newtonian Flow, Rheology Ser. Vol. 1* (1984), p. 352, Elsevier.