Integral Equations for a Planar Crack Subjected to Arbitrary Loadings in a Fluid-Filled, Poroelastic, Infinite Solid*

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Integral equations are derived for a planar crack of arbitrary shape, subjected to both normal and shear tractions as well as fluid pressure, in an infinite, fluid-saturated, poroelastic, isotropic solid. The obtained equations relate the surface tractions and fluid pressure on crack surfaces to relative crack-surface displacements and leak-off flux into the porous medium. It is intended to implement the equations in a 3D simulator of hydraulic fracturing in oil reservoirs. Some discussions are given on reduction of the present solutions to the known ones as special cases and to pure shear mode solutions; it is found that the poroelastic effect does not disappear even for the pure shear mode.

Key Words: Integral Equations, Hydraulic Fracturing, Poroelastic Solid, Arbitrary Loading, Three-Dimensional Planar Crack, Dislocation Segment

1. Introduction

In order to efficiently recover the oil from low-to-moderate permeability reservoirs, it is often necessary to treat a formation to improve the flow of oil and gas through the rock layers. These processes are known as well stimulation treatments, which can be classified into two groups: acidizing and hydraulic fracturing.

The hydraulic fracturing is the process of cracking layers of a formation to allow better flow of petroleum. The fracturing involves injecting high pressure fluids called "frac fluid" into the formation under sufficient pressure to crack the rock formation that holds the petroleum. A "propping agent" or "proppant", like sand, sintered bauxite or other particles, is added to the fluid used to fracture the formation so that, when the pressure is decreased, the frac-

tures will not close completely. This allows for increased flow within a given formation.

The hydraulic fracturing can be used on almost any kind of formation, and it is sometimes combined with acidizing. In some cases, the fracture vertically created is several hundred meters long and several ten meters high. For such a gigantic fracture, the fracturing costs nearly 50% of the well drilling and completion cost.

On such a background, various types of (in most cases, two-dimensional) simulators for hydraulic fracturing have been developed for a long time to simulate the whole process of fracturing and thus to improve the injection program and select appropriate frac fluids and proppants, aiming at minimizing the whole cost. More recently, three-dimensional simulators have been developed by some institutes backed by the majors, many petroleum engineering companies, some national institutes, and some universities, especially by those organizations in the US. In Japan, the Technology Research Center of Japan National Oil Corporation has been ambitiously developing a full three-dimensional simulator which can deal with a curved fracture. See Refs. (1)-(3) as to

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Fig. 1 Inclined three-dimensional hydraulic fracture model

the hydraulic fracturing and related simulators.

Development of a three-dimensional simulator requires an integral equation or equations which relate(s) the pressure or other tractions on crack faces to the crack opening or other quantities; use of the integral equations saves computation time and memory and thus allows for numerical simulations.

Regarding a formation which holds oil or gas as a homogeneous, elastic, infinite solid, Clifton and Abou-Sayed⁽⁴⁾ derived an integral equation which relates the crack opening width to pressure on the crack faces and implanted it into a 3D simulator. Kurashige and Clifton⁽⁵⁾ extended that integral equation to a pair of integral equations for the case of a fluid-saturated, poroelastic, infinite solid, which takes it into account that the oil-bearing formation consists of porous rocks with the porosity being up to 20% and allows flow of oil and leaking frac fluid. In deriving those equations, they used the reciprocal theorem for the fluid-filled, poroelastic solid in the Laplace space and fundamental solutions for dislocation segments. The equations obtained represent the normal traction and fluid pressure on the crack surfaces in terms of integrals over the crack surface whose integrands include the crack opening width and leak-off flux from the surface into the formation. These solutions were also implanted into the same simulator and then some effects of the "back stresses and back pressure" in the porous formation on the fracturing pressure and others were studied in Ref.(6).

In general, the oil-bearing formation is sandwiched by impermeable layers and the fluid flows predominantly in the horizontal directions except at the early time of fracturing. By assuming zero permeability in the vertical direction or ignoring the vertical flow within the reservoir, Kurashige and Clifton⁽⁷⁾ obtained similar integral equations as those in Ref.

(5) following the approach used in Ref.(5); these equations are applicable to the formation of finite thickness, which is the same as the fracture height, although these are for an infinite space. These integral equations are now being implanted in the simulator.

Permeability is governed not only by the intrinsic permeability of the formation rock but also by the in situ tectonic stresses. Thus, the permeability is anisotropic in many formations. It is reported (p. 71 in Ref.(2)) that the anisotropy from 2:1 to 3:1 is considered common and that it is as much as 100:1 in some cases for fissured rocks. Considering this hydraulic anisotropy, Ganbe and Kurashige⁽⁸⁾ gave coupled integral equations for an elastically isotropic, porous solid having transversely isotropic permeability with a vertical anisotropy axis. As limiting cases, these equations include those for an isotropic case⁽⁵⁾ and for the case of zero permeability in one direction⁽⁷⁾; of course, the purely elastic case⁽⁴⁾ as well.

All the above-mentioned integral equations, however, deal only with the normal traction or pressure on the crack faces, that is, the Mode I case. Following the approach used by Kurashige and Clifton⁽⁵⁾, Guo et al.⁽⁹⁾ obtained the integral equations for a crack loaded by both normal and shear tractions in an infinite elastic (non-porous) space.

In the present paper, we derive coupled integral equations for a planar crack subjected to non-uniformly distributed, normal and shear tractions in an isotropic, fluid-saturated, poroelastic, infinite solid, following the approaches employed by Kurashige and Clifton⁽⁵⁾ and Guo et al.⁽⁹⁾ Although the hydraulically induced fractures commonly extend in a plane perpendicular to the minimum in situ principal stress, these integral equations for arbitrary loadings may be useful when the fracture extends along a possible weak plane and when fault slips, fault creep events, and aftershock slip processes are dealt with. Two-dimensional poroelastic analyses and discussions on these various things associated with earthquakes, especially on some effects of the fluid diffusion around the slip, can be seen in many references; for example, see Refs.(10) and (11).

2. Governing Equations

Let us consider an elastically and hydraulically isotropic, homogeneous, fluid-saturated, poroelastic solid. Denote total stress, pore fluid pressure, average strain, pore fluid volume change per unit total volume and pore fluid volume flux by σ_{ij} , p , ε_{ij} , ΔV and q_i , respectively. Constitutive equations and the Darcy law for such a poroelastic material⁽¹²⁾ are given by

$$\begin{aligned}\sigma_{ij} &= L_{ijpq} \varepsilon_{pq} - \alpha p \delta_{ij} \\ &= 2G \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} \right) - \alpha p \delta_{ij}, \\ \Delta V &= \frac{3(\nu_u - \nu)}{2GB(1+\nu_u)(1+\nu)} \left(\sigma_{kk} + \frac{3}{B} p \right),\end{aligned}\quad (2)$$

$$q_i = -\kappa p_{,i}, \quad (3)$$

where α is Biot's coefficient of effective stresses and given by,

$$\alpha = \frac{3(\nu_u - \nu)}{B(1+\nu_u)(1-2\nu)}, \quad (4)$$

and G, B, ν, ν_u, κ are the shear modulus, Skempton's pore pressure coefficient, drained and undrained Poisson's ratios, and permeability. Elasticity tensor L_{ijpq} is defined by Eq. (1). The summation convention is applied and Latin subscript, e.g., i , following a comma implies a partial derivative with respect to Cartesian coordinate x_i . The Cartesian coordinates will be often denoted by (x, y, z) according to convenience in what follows.

For an isothermal and quasi-static process, the equilibrium equation and fluid mass conservation law are given by

$$\sigma_{ij,j} + f_i = 0, \quad (5)$$

$$q_{i,i} + \frac{\partial \Delta V}{\partial t} = r^F, \quad (6)$$

where f_i and r^F denote a body force vector per unit volume and a rate of fluid volume supply per unit volume, respectively. For an infinitesimal deformation, the average strains are given in terms of average displacements u_j by

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}). \quad (7)$$

In the above and in what follows, Latin indices range from 1 to 3, unless mentioned otherwise.

3. Fundamental Solutions

The fundamental solutions for an instantaneous point force and instantaneous point fluid source in the isotropic, homogeneous, fluid-saturated, porous solid are summarized here for later use. They are seen in Ref. (13). These solutions were obtained using the conventional method, i.e., the Laplace and triple Fourier transforms, for the instantaneous point disturbances and proved to be identical with the solutions which had been derived from the known fundamental solutions^{(14),(15)} for step-like disturbances; the latter solutions were cleverly obtained by resorting to the canonical representation theorems.

For the instantaneous fluid source of unit intensity $Q_0=1$ described by

$$r^F = Q_0 \delta(x_1) \delta(x_2) \delta(x_3) \delta(t), \quad f_i = 0 \quad (8)$$

with $\delta(\bullet)$ being the Dirac delta function, the fundamental solutions are summarized as

$$p_{(0)}(\mathbf{r}, t) = \frac{c}{\pi^{3/2} \kappa} \frac{1}{r^3} \xi^3 e^{-\xi^2}, \quad (9)$$

$$\begin{aligned}\sigma_{(0)ij}(\mathbf{r}, t) &= \frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \frac{1}{r^3} \\ &\times \left[\left(\delta_{ij} - 3 \frac{x_i x_j}{r^2} \right) \left(\text{erf}[\xi] - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right) \right. \\ &\quad \left. - \frac{4}{\sqrt{\pi}} \left(\delta_{ij} - \frac{x_i x_j}{r^2} \right) \xi^3 e^{-\xi^2} \right],\end{aligned}\quad (10)$$

where $\text{erf}[\bullet]$ denotes the error function. Note that subscript (0) attached to the pore fluid pressure and total stress implies those for the instantaneous source. In the above solutions, r and ξ are

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \xi = \frac{r}{2\sqrt{ct}}, \quad (11)$$

and c is the diffusivity parameter⁽¹⁴⁾ defined as

$$c = \frac{2\kappa GB^2(1+\nu_u)^2(1-\nu)}{9(\nu_u - \nu)(1-\nu_u)}. \quad (12)$$

Similarly, for the instantaneously applied point force of unit magnitude $P_k=1$ in the x_k direction described by

$$f_k = P_k \delta(x_1) \delta(x_2) \delta(x_3) \delta(t), \quad r^F = 0, \quad (13)$$

the solutions, denoted by subscripts (k), are

$$p_{(k)}(\mathbf{r}, t) = \frac{B(1+\nu_u)}{12\pi(1-\nu_u)} \frac{x_k}{r^3} \left[\delta(t) - \frac{8c}{\sqrt{\pi}} \frac{1}{r^2} \xi^5 e^{-\xi^2} \right], \quad (14)$$

$$\begin{aligned}\sigma_{(k)ij}(\mathbf{r}, t) &= \frac{\delta(t)}{8\pi(1-\nu_u)} \frac{1}{r^2} \left[(1-2\nu_u) \right. \\ &\times \left(\delta_{ij} \frac{x_k}{r} - \delta_{jk} \frac{x_i}{r} - \delta_{ik} \frac{x_j}{r} \right) - 3 \frac{x_i x_j x_k}{r^3} \Big] \\ &+ \frac{B^2 G \kappa (1+\nu_u)^2}{18\pi(1-\nu_u)^2} \frac{1}{r^4} \left[-\frac{8}{\sqrt{\pi}} \left(\delta_{ij} \frac{x_k}{r} - \frac{x_i x_j x_k}{r^3} \right) \xi^5 e^{-\xi^2} \right. \\ &\quad \left. - \left(\delta_{ij} \frac{x_k}{r} + \delta_{jk} \frac{x_i}{r} + \delta_{ik} \frac{x_j}{r} - 5 \frac{x_i x_j x_k}{r^3} \right) \right. \\ &\quad \left. \times \left(3 \text{erf}[\xi] - \frac{2}{\sqrt{\pi}} (3+2\xi^2) \xi e^{-\xi^2} \right) \right].\end{aligned}\quad (15)$$

We can see that the first term on the right hand side of Eq. (15) represents the instantaneous (undrained) elastic response of a fluid-saturated poroelastic solid, which is Kelvin's solution⁽¹⁶⁾ if the Dirac delta $\delta(t)$ is ignored there. The second term is the effect of fluid diffusion on stresses.

4. Reciprocal Theorem

Equations (1) through (7), supplemented by appropriate initial and boundary conditions, constitute a complete system of equations for an isotropic, homogeneous, fluid-saturated, poroelastic solid. Solutions of this linear system of equations satisfy a reciprocal theorem in the Laplace space⁽¹⁴⁾. If we define the fluid displacement Q_i and the total volume of supplied fluid R^F by

$$q_i = \frac{\partial Q_i}{\partial t}, \quad r^F = \frac{\partial R^F}{\partial t}, \quad (16)$$

the reciprocal theorem relates two independent "deformation" fields $\{u_i^{(1)}, Q_i^{(1)}\}$ and $\{u_i^{(2)}, Q_i^{(2)}\}$ and associated "stress" fields $\{\sigma_{ij}^{(1)}, p^{(1)}\}$ and $\{\sigma_{ij}^{(2)}, p^{(2)}\}$ resulting from the "forcing" fields $\{f_i^{(1)}, R^{F(1)}\}$ and $\{f_i^{(2)}, R^{F(2)}\}$.

$R^{F(2)}$ by

$$\begin{aligned} & \int_S (\sigma_{ij}^{(1)*} u_j^{(2)*} - p^{(1)*} Q_i^{(2)*}) n_i dS \\ & + \int_V (f_i^{(1)*} u_i^{(2)*} - R^{F(1)*} p^{(2)*}) dV \\ & = \int_S (\sigma_{ij}^{(2)*} u_j^{(1)*} - p^{(2)*} Q_i^{(1)*}) n_i dS \\ & + \int_V (f_i^{(2)*} u_i^{(1)*} - R^{F(2)*} p^{(1)*}) dV, \end{aligned} \quad (17)$$

where V and S are, respectively, the volume and the surface of the body under consideration; superscript $*$ means the Laplace transform of the variable on which the asterisk lies. In this relation (and in Eq. (3)), we have neglected a body force acting on the pore fluid, because we do not need it to obtain the integral equations that we require.

5. Application of the Reciprocal Theorem

The reciprocal theorem can be used to obtain coupled integral equations for the problem of a crack of arbitrary shape subjected to both normal and shear tractions in the fluid-saturated, poroelastic solid. To this end, consider three solutions of Eqs. (1) to (7). These solutions, designated by subscripts (d), (k) and (0) in place of superscripts (1) and (2) in Eq. (17), are described below.

Solution (d), unknown, corresponds to the sudden appearance of a closed dislocation loop that bounds a surface S_c across which the jump in solid displacement is given by the Burgers vector

$$\mathbf{b} = \mathbf{u}_{(d)+} - \mathbf{u}_{(d)-}, \quad (18)$$

and the jump in fluid displacement is

$$\Delta \mathbf{Q} = \mathbf{Q}_{(d)+} - \mathbf{Q}_{(d)-}, \quad (19)$$

where subscripts “+” and “-” denote evaluation on the lower and upper surface of S_c , which has a unit outward (i.e., pointing into the crack) normal $\mathbf{n} = \mathbf{n}_+ = -\mathbf{n}_-$.

Solution (k), known as in Eqs. (14) and (15), is associated with the instantaneous appearance of a point body force in the x_j direction at time t at the point $\mathbf{r}'$, described by

$$f_{(k)j} = P_j \delta(\mathbf{r} - \mathbf{r}') \quad (20)$$

in the Laplace space with its magnitude taken up as $P_j = 1$.

Solution (0), known as in Eqs. (9) and (10), is for the instantaneous appearance of a fluid volume point source at time t at the point $\mathbf{r}'$. From the second equation of Eq. (16), the instantaneous point source is given by

$$R_{(0)}^{F*} = \frac{Q_0}{s} \delta(\mathbf{r} - \mathbf{r}') \quad (21)$$

in the Laplace space, where its intensity has been taken as $Q_0 = 1$ and s denotes the Laplace transform parameter.

Using (d) as the first solution and (k) as the

second solution in the reciprocal theorem (17) with the region of integration S taken as S_c , one obtains a response of the poroelastic infinite solid due to the sudden appearance of dislocation loop as follows:

$$\begin{aligned} u_m^*(\mathbf{r}') &= - \int_{S_c} \{ \sigma_{(m)ij}^*(\mathbf{r}, \mathbf{r}') b_j^*(\mathbf{r}) \\ & - p_{(m)}^*(\mathbf{r}, \mathbf{r}') \Delta Q_i^*(\mathbf{r}) \} n_i(\mathbf{r}) dS. \end{aligned} \quad (22)$$

Similarly, use of (d) as the first solution and (0) as the second solution in the reciprocal theorem (17) yields

$$\begin{aligned} p^*(\mathbf{r}') &= s \int_{S_c} \{ \sigma_{(0)ij}^*(\mathbf{r}, \mathbf{r}') b_j^*(\mathbf{r}) \\ & - p_{(0)}^*(\mathbf{r}, \mathbf{r}') \Delta Q_i^*(\mathbf{r}) \} n_i(\mathbf{r}) dS. \end{aligned} \quad (23)$$

In Eqs. (22) and (23), the subscripts (d) have been omitted from the variables on the left hand sides; $\sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r})$ and $p_{(m)}^*(\mathbf{r}', \mathbf{r})$ are the total stress tensor and pore fluid pressure at the point $\mathbf{r}'$ due to the point body force of unit magnitude acting in the x_m direction at $\mathbf{r}$, while $\sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r})$ and $p_{(0)}^*(\mathbf{r}', \mathbf{r})$ those due to the instantaneous fluid volume source of unit intensity. All these stresses and pressures are defined in the Laplace space, denoted by the asterisk.

Next, in the above equations, interchanging $\mathbf{r}$ and $\mathbf{r}'$ and using the fluid volume supply rate per unit fracture area Δq_n^* defined in the Laplace space by

$$\Delta q_n^* = -s \Delta Q_j^* n_j \quad (24)$$

which comes from the first equation of Eq. (16) and Eq. (19), yields

$$u_m^*(\mathbf{r}) = u_m^{*(b)}(\mathbf{r}) + u_m^{*(q)}(\mathbf{r}), \quad (25)$$

$$p^*(\mathbf{r}) = p^{*(b)}(\mathbf{r}) + p^{*(q)}(\mathbf{r}) \quad (26)$$

with

$$u_m^{*(b)}(\mathbf{r}) = - \int_{S_c} \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) b_j^*(\mathbf{r}') n'_i dS', \quad (27)$$

$$u_m^{*(q)}(\mathbf{r}) = - \frac{1}{s} \int_{S_c} p_{(m)}^*(\mathbf{r}', \mathbf{r}) \Delta q_n^*(\mathbf{r}') dS', \quad (28)$$

$$p^{*(b)}(\mathbf{r}) = s \int_{S_c} \sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r}) b_j^*(\mathbf{r}') n'_i dS', \quad (29)$$

$$p^{*(q)}(\mathbf{r}) = \int_{S_c} p_{(0)}^*(\mathbf{r}', \mathbf{r}) \Delta q_n^*(\mathbf{r}') dS', \quad (30)$$

where superscripts (b) and (q) mean the contributions from the Burgers vector of displacement discontinuity and the jump in fluid flux across the surface S_c .

6. Dislocation Segment Solutions

To obtain the stresses and pressure resulting from an infinitesimal dislocation segment, first of all, consider $u_m^{*(b)}(\mathbf{r})$ and $p^{*(b)}(\mathbf{r})$ given by Eqs. (27) and (29). Differentiating these equations with respect to x_s , one obtains

$$\frac{\partial u_m^{*(b)}(\mathbf{r})}{\partial x_s} = \int_{S_c} b_i^*(\mathbf{r}') \frac{\partial}{\partial x_s} \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) n'_j dS', \quad (31)$$

$$\frac{\partial p^{*(b)}(\mathbf{r})}{\partial x_s} = -s \int_{S_c} b_i^*(\mathbf{r}') \frac{\partial}{\partial x_s} \sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r}) n'_j dS'. \quad (32)$$

In deriving the above equations, use has been made of

$$\frac{\partial}{\partial x_s} \sigma_{(l)ij}^*(\mathbf{r}', \mathbf{r}) = -\frac{\partial}{\partial x_s} \sigma_{(l)ij}^*(\mathbf{r}', \mathbf{r}), \quad (33)$$

where (and in the next equation), exceptionally, index l in the parentheses ranges from zero to three. In Eqs. (31) and (32), use of

$$b_i^*(\mathbf{r}') \frac{\partial}{\partial x'_j} \sigma_{(l)ij}^*(\mathbf{r}', \mathbf{r}) n'_s = 0 \quad (34)$$

obtained from the equilibrium equation (5) with no body force and application of the Stokes theorem gives

$$\frac{\partial u_m^{*(b)}(\mathbf{r})}{\partial x_s} = \varepsilon_{jsn} \oint b_i^* \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) dx'_n, \quad (35)$$

$$\frac{\partial p^{*(b)}(\mathbf{r})}{\partial x_s} = -s \varepsilon_{jsn} \oint b_i^* \sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r}) dx'_n, \quad (36)$$

where ε_{jsn} is the permutation symbol. For a dislocation loop in a purely elastic, infinite space, Eq.(35) becomes the Mura formula⁽¹⁷⁾ and Eq.(36) vanishes. Note that these solutions are valid for an infinite space which is elastically and hydraulically anisotropic.

From Eqs.(35) and (36), and Eq.(1), the stresses and pressure gradient caused by the Burgers vector b_i^* along line segment $\Delta x'_n$ follow as

$$\Delta \sigma_{pq}^{*(b)}(\mathbf{r}) + \alpha \Delta p^{*(b)}(\mathbf{r}) = L_{pqms} \varepsilon_{jsn} b_i^* \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) \Delta x'_n, \quad (37)$$

$$\frac{\partial}{\partial x_s} \Delta p^{*(b)}(\mathbf{r}) = -s \varepsilon_{jsn} b_i^* \sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r}) \Delta x'_n, \quad (38)$$

where L_{pqms} are the elastic moduli given in Eq.(1).

Equations (37) and (38) are the general dislocation segment solutions. For applications in hydraulic fracturing and fault slip process, our interest is in tractions $\sigma_{3k}^{(b)}$ and pore fluid pressure $p^{(b)}$ on the plane $z=0$ caused by a dislocation segment in the plane $z'=0$. For definiteness, consider a "horizontal" dislocation segment with the Burgers vector b_i^h along the line segment $\Delta x'=\Delta x'_1$; later, a "vertical" dislocation segment with the Burgers vector b_i^v along the line segment $\Delta y'=-\Delta x'_2$ will be included to allow representation of a general dislocation segment in the $z'=0$. (As for the sign convention, see Fig. 1 in Ref.(7).) For the horizontal dislocation segment, using $\Delta x'=\Delta x'_1$ and b_i^{*h} in place of b_i^* in Eqs.(37) and (38) and making Laplace inversion of the resulting equation in addition to integrating the pressure gradient from $-\infty$ to 0 with respect to z yields

$$\Delta \sigma_{33}^{(b)h}(\mathbf{R}, t) + \alpha \Delta p^{(b)h}(\mathbf{R}, t) = \left(\frac{2G}{1-2\nu} \right) \int_0^t b_i^h(\tau) \times \{ (1-\nu) \sigma_{(3)iz}(\mathbf{R}', \mathbf{R}, t-\tau) - \nu \sigma_{(2)iz}(\mathbf{R}', \mathbf{R}, t-\tau) \} d\tau \Delta x', \quad (39)$$

$$\Delta \sigma_{3a}^{(b)h}(\mathbf{R}, t) = G \int_0^t b_i^h(\tau) \{ \sigma_{(a)iz}(\mathbf{R}', \mathbf{R}, t-\tau) - \sigma_{(3)iz}(\mathbf{R}', \mathbf{R}, t-\tau) \delta_{2a} \} d\tau \Delta x', \quad (40)$$

$$\Delta p^{(b)h}(\mathbf{R}, t) = - \left[b_i^h(t) \int_{-\infty}^0 \sigma_{(0)iz}(\mathbf{R}', \mathbf{r}, 0) dz \right.$$

$$\left. + \int_0^t b_i^h(\tau) \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)iz}(\mathbf{R}', \mathbf{r}, t-\tau) dz d\tau \right] \Delta x', \quad (41)$$

where $\mathbf{R}$ and $\mathbf{R}'$ are the position vectors in the planes $z=0$ and $z'=0$; the Greek indices range from one to two. For the vertical dislocation segment, one can obtain the tractions and pressure in the similar manner:

$$\Delta \sigma_{33}^{(b)v}(\mathbf{R}, t) + \alpha \Delta p^{(b)v}(\mathbf{R}, t) = \left(\frac{2G}{1-2\nu} \right) \int_0^t b_i^v(\tau) \times \{ (1-\nu) \sigma_{(3)i1}(\mathbf{R}', \mathbf{R}, t-\tau) - \nu \sigma_{(1)i3}(\mathbf{R}', \mathbf{R}, t-\tau) \} d\tau \Delta y', \quad (42)$$

$$\Delta \sigma_{3a}^{(b)v}(\mathbf{R}, t) = G \int_0^t b_i^v(\tau) \{ \sigma_{(a)i1}(\mathbf{R}', \mathbf{R}, t-\tau) - \sigma_{(3)i3}(\mathbf{R}', \mathbf{R}, t-\tau) \delta_{1a} \} d\tau \Delta y', \quad (43)$$

$$\Delta p^{(b)v}(\mathbf{R}, t) = - \left[b_i^v(t) \int_{-\infty}^0 \sigma_{(0)i1}(\mathbf{R}', \mathbf{r}, 0) dz + \int_0^t b_i^v(\tau) \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)i1}(\mathbf{R}', \mathbf{r}, t-\tau) dz d\tau \right] \Delta y'. \quad (44)$$

Next, consider relations between the Burgers vectors and relative crack-surface displacements $u_i(x', y', z'=0^+) - u_i(x', y', z=0^-)$. Denote this relative crack-surface displacement by $u_i(x', y')$. In the similar manner as in the opening mode^{(5),(7)}, the relations can be written as

$$b_i^h = - \frac{\partial u_i(x', y')}{\partial y'} \Delta y', \quad b_i^v = - \frac{\partial u_i(x', y')}{\partial x'} \Delta x'. \quad (45)$$

Finally, substituting the above relations into Eqs. (39)-(41) for the horizontal dislocation segment and into Eqs.(42)-(44) for the vertical segment, combining the contributions from the both segments, and then integrating the resulting equations over the entire fracture area Sc with area element $dS'=\Delta x' \Delta y'$ yields

$$\sigma_{33}^{(b)}(\mathbf{R}, t) + \alpha p^{(b)}(\mathbf{R}, t) = - \left(\frac{2G}{1-2\nu} \right) \int_{sc} \int_0^t \frac{\partial u_i(\mathbf{R}', \tau)}{\partial x'_\beta} \times \{ (1-\nu) \sigma_{(3)i\beta}(\mathbf{R}', \mathbf{R}, t-\tau) - \nu \sigma_{(\beta)i3}(\mathbf{R}', \mathbf{R}, t-\tau) \} d\tau dS', \quad (46)$$

$$\sigma_{3a}^{(b)}(\mathbf{R}, t) = - G \int_{sc} \int_0^t \left\{ \frac{\partial u_i(\mathbf{R}', \tau)}{\partial x'_\beta} \sigma_{(a)i\beta}(\mathbf{R}', \mathbf{R}, t-\tau) - \frac{\partial u_i(\mathbf{R}', \tau)}{\partial x'_a} \sigma_{(3)i3}(\mathbf{R}', \mathbf{R}, t-\tau) \right\} d\tau dS', \quad (47)$$

$$p^{(b)}(\mathbf{R}, t) = \int_{sc} \left[\frac{\partial u_i(\mathbf{R}', t)}{\partial x'_\beta} \int_{-\infty}^0 \sigma_{(0)i\beta}(\mathbf{R}', \mathbf{r}, 0) dz + \int_0^t \frac{\partial u_i(\mathbf{R}', \tau)}{\partial x'_\beta} \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)i\beta}(\mathbf{R}', \mathbf{r}, t-\tau) dz d\tau \right] dS', \quad (48)$$

where $\sigma_{(m)ij}(\mathbf{R}', \mathbf{R}, t-\tau)$, ($m=0, 1, 2, 3$) are given by Eqs.(9), (10), and (14), (15). The integrals of $\sigma_{(0)i\beta}(\mathbf{R}', \mathbf{r}, t-\tau)$ from $z=-\infty$ to $z=0$ appearing in Eq.(48) can be readily estimated by use of a software

"Mathematica"⁽¹⁸⁾ as an algebraic tool; the results are given in **Appendix A**. Substituting these expressions of stresses and estimates of integration into Eqs. (46) to (48), one obtains the following integral equations for the tractions and pressure on the crack surface; these are the contributions from the Burgers vectors.

$$\begin{aligned} \sigma_{33}^{(b)}(\mathbf{R}, t) + \alpha p^{(b)}(\mathbf{R}, t) &= -\frac{2G}{(1-2\nu)} \int_{sc} \left[\frac{-(1-2\nu_u)}{8\pi(1-\nu_u)} \frac{(x'_\beta - x_\beta)}{R^3} \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} \right. \\ &+ \frac{GB^2 \kappa (1+\nu_u)^2}{18\pi(1-\nu_u)^2} \int_0^t \left\{ (1-2\nu) \left(-3 \operatorname{erf}[\xi] \right. \right. \\ &+ \left. \left. \frac{6}{\sqrt{\pi}} \xi e^{-\xi^2} + \frac{4}{\sqrt{\pi}} \xi^3 e^{-\xi^2} \right) \right. \\ &\left. \left. - \nu \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \right\} \frac{(x'_\beta - x_\beta)}{R^5} \frac{\partial u_3(\mathbf{R}', \tau)}{\partial x'_\beta} d\tau \right] dS', \end{aligned} \quad (49)$$

$$\begin{aligned} \sigma_{3\gamma}^{(b)}(\mathbf{R}, t) &= \frac{G}{4\pi(1-\nu_u)} \\ &\times \int_{sc} \left[\left\{ 3 \frac{(x'_\alpha - x_\alpha)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{2R^5} \right. \right. \\ &- \left. \left. (1-2\nu_u) \frac{(x'_\gamma - x_\gamma) \delta_{\alpha\beta} - (x'_\beta - x_\beta) \delta_{\alpha\gamma}}{2R^3} \right\} \frac{\partial u_\alpha(\mathbf{R}', t)}{\partial x'_\beta} \right. \\ &- \frac{2GB^2 \kappa (1+\nu_u)^2}{9(1-\nu_u)} \int_0^t \left\{ \frac{(x'_\gamma - x_\gamma)}{R^5} \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \delta_{\alpha\beta} \right. \\ &- \frac{(x'_\beta - x_\beta) \delta_{\alpha\gamma} + (x'_\gamma - x_\gamma) \delta_{\alpha\beta}}{R^5} \\ &\times \left(3 \operatorname{erf}[\xi] - \frac{6}{\sqrt{\pi}} \xi e^{-\xi^2} - \frac{4}{\sqrt{\pi}} \xi^3 e^{-\xi^2} \right) \\ &+ \left. \frac{(x'_\alpha - x_\alpha)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{R^7} \right. \\ &\times \left. \left(15 \operatorname{erf}[\xi] - \frac{30}{\sqrt{\pi}} \xi e^{-\xi^2} - \frac{20}{\sqrt{\pi}} \xi^3 e^{-\xi^2} \right. \right. \\ &\left. \left. - \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \right) \right\} \frac{\partial u_\alpha(\mathbf{R}', \tau)}{\partial x'_\beta} d\tau \right] dS', \end{aligned} \quad (50)$$

$$\begin{aligned} p^{(b)}(\mathbf{R}, t) &= -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \\ &\times \int_{sc} \left[\left\{ \frac{1}{R^3} \delta_{\alpha\beta} - 2 \frac{(x'_\alpha - x_\alpha)(x'_\beta - x_\beta)}{R^4} \right\} \right. \\ &\times \frac{\partial u_\alpha(\mathbf{R}', t)}{\partial x'_\beta} + \frac{(x'_\beta - x_\beta)}{R^3} \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} \\ &+ c \int_0^t \left\{ 4 \frac{\xi^4}{R^4} \left(e^{-\xi^2} \delta_{\alpha\beta} - 2\xi^2 e^{-\xi^2} \delta_{\alpha\beta} \right. \right. \\ &+ \left. \left. 2 \frac{(x'_\alpha - x_\alpha)(x'_\beta - x_\beta)}{R^2} \xi^2 e^{-\xi^2} \right\} \frac{\partial u_\alpha(\mathbf{R}', \tau)}{\partial x'_\beta} \right. \\ &\left. \left. - \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \frac{(x'_\beta - x_\beta)}{R^5} \frac{\partial u_3(\mathbf{R}', \tau)}{\partial x'_\beta} \right\} d\tau \right] dS' \end{aligned} \quad (51)$$

with R and ξ being re-defined as

$$R = \sqrt{(x' - x)^2 + (y' - y)^2}, \quad \xi = \frac{R}{2\sqrt{c(t - \tau)}}. \quad (52)$$

Here, we can easily show that the first and third terms in the bracket on the right hand side of Eq. (51) identically vanish (See **Appendix B**). Thus, Eq. (51) is simplified into

$$p^{(b)}(\mathbf{R}, t) = -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \int_{sc} \left[\frac{(x'_\beta - x_\beta)}{R^3} \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} \right]$$

$$- c \frac{8}{\sqrt{\pi}} \int_0^t \xi^5 e^{-\xi^2} \frac{(x'_\beta - x_\beta)}{R^5} \frac{\partial u_3(\mathbf{R}', \tau)}{\partial x'_\beta} d\tau \Big] dS'. \quad (53)$$

7. Contributions from Fluid Supply

To the results given by Eqs. (49), (50) and (53), further contributions from the fluid supply must be added. Let us return to Eqs. (28) and (30) and seek stresses and pressure caused by the fluid volume supply through the crack surfaces. After Laplace inversion of Eq. (28) with $z=0$ and $z'=0$, using Eqs. (1) and (7) in the resulting equation, one obtains

$$\begin{aligned} \sigma_{ij}^{(q)}(\mathbf{R}, t) + \alpha p^{(q)}(\mathbf{R}, t) \delta_{ij} &= 2G \int_{sc} \int_0^t \int_0^\tau \left[\frac{1}{2} \left(\frac{\partial p_{(i)}(\mathbf{R}', \mathbf{R}, \tau - s)}{\partial x'_i} \right) \right. \\ &+ \left. \frac{\nu}{1-2\nu} \frac{\partial p_{(k)}(\mathbf{R}', \mathbf{R}, \tau - s)}{\partial x'_k} \delta_{ij} \right] \Delta q_n(s) ds d\tau dS'. \end{aligned} \quad (54)$$

More simply, the inverse Laplace transform of Eq. (30) with $z=0$ and $z'=0$ can be obtained as

$$\begin{aligned} p^{(q)}(\mathbf{R}, t) &= \int_{sc} \int_0^t p_{(0)}(\mathbf{R}', \mathbf{R}, t - \tau) \Delta q_n \\ &\times (\mathbf{R}', \tau) d\tau dS'. \end{aligned} \quad (55)$$

To convert the double integration of Eq. (54) with respect to s and τ into a single one, one must first interchange the order of integrations in this equation, where the unknown fluid supply rate Δq_n is a function of the inner integration variable s , and then interchange the notation of integral variables. The interchange of the integration order changes the original range of integration from $s=0$ to $s=\tau$ and then from $\tau=0$ to $\tau=t$ into a new range from $\tau=s$ to $\tau=t$ and then from $s=0$ to $s=t$. Subsequent interchange of the notation of integration variables τ and s allows the inner integration with respect to s after substituting $p_{(i)}(\mathbf{R}', \mathbf{R}, \tau - s)$; the results of these integrations are shown in **Appendix C**. Thus, the normal and shear tractions and pressure on the crack faces due to the fluid supply are

$$\begin{aligned} \sigma_{33}^{(q)}(\mathbf{R}, t) + \alpha p^{(q)}(\mathbf{R}, t) &= \frac{GB(1+\nu_u)}{6\pi^{3/2}(1-\nu_u)} \int_{sc} \frac{1}{R^3} \int_0^t \left(\sqrt{\pi} \operatorname{erf}[\xi] - 2\xi e^{-\xi^2} \right. \\ &+ \left. 4 \frac{\nu}{1-2\nu} \xi^3 e^{-\xi^2} \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS', \end{aligned} \quad (56)$$

$$\sigma_{3\alpha}^{(q)}(\mathbf{R}, t) = 0, \quad (57)$$

$$p^{(q)}(\mathbf{R}, t) = \frac{c}{\pi^{3/2}\kappa} \int_{sc} \int_0^t \frac{\xi^3}{R^3} e^{-\xi^2} \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \quad (58)$$

It may be natural that the shear tractions on the crack surfaces are not caused by the fluid supply from the crack as seen in Eq. (57).

8. Integral Equations

Summing up of Eqs. (49), (50), (53) and (56) to

(58) yields the desired integral equations, which relate the relative crack-surface displacements and the fluid supply (or leak-off) rate per unit fracture area to the normal and shear tractions and the fluid pressure acting on the crack surface. The final equations are

$$\sigma_{3k}(\mathbf{R}, t) = \frac{G}{4\pi(1-\nu_u)} \int_{Sc} \left[S_{i\beta k}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_i(\mathbf{R}', t)}{\partial x'_\beta} + \int_0^t \left\{ T_{i\beta k}(\mathbf{R}' - \mathbf{R}, t - \tau) \frac{\partial u_i(\mathbf{R}', \tau)}{\partial x'_\beta} + \delta_{3k} U(\mathbf{R}' - \mathbf{R}, t - \tau) \Delta q_n(\mathbf{R}', \tau) \right\} d\tau \right] dS', \quad (59)$$

$$p(\mathbf{R}, t) = -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \int_{Sc} \left[S_{3\beta 3}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} + \int_0^t \left\{ H_\beta(\mathbf{R}' - \mathbf{R}, t - \tau) \frac{\partial u_3(\mathbf{R}', \tau)}{\partial x'_\beta} + V(\mathbf{R}' - \mathbf{R}, t - \tau) \Delta q_n(\mathbf{R}', \tau) \right\} d\tau \right] dS', \quad (60)$$

where the non-zero components of $S_{i\beta k}(\mathbf{R}' - \mathbf{R})$ are given by

$$S_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}) = \left\{ 3 \frac{(x'_a - x_a)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{2R^5} - (1 - 2\nu_u) \frac{(x'_\gamma - x_\gamma)\delta_{a\beta} - (x'_\beta - x_\beta)\delta_{a\gamma}}{2R^3} \right\}, \quad (61)$$

$$S_{3\beta 3}(\mathbf{R}' - \mathbf{R}) = \frac{(x'_\beta - x_\beta)}{R^3}. \quad (62)$$

These coefficients exactly correspond to the coefficient S_{3k}^{ij} appearing in Ref.(9), which treated a planar crack subjected to normal and shear tractions in an elastic, infinite solid. Non-zero evolution functions in the final results are given by

$$T_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}, t - \tau) = -\frac{c(\nu_u - \nu)}{(1 - \nu)} \left\{ \frac{(x'_\gamma - x_\gamma)}{R^5} \times \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \delta_{a\beta} - \frac{(x'_\beta - x_\beta)\delta_{a\gamma} + (x'_\gamma - x_\gamma)\delta_{a\beta}}{R^5} D(\xi) + \frac{(x'_a - x_a)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{R^7} \times \left(5D(\xi) - \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \right) \right\}, \quad (63)$$

$$T_{3\beta 3}(\mathbf{R}' - \mathbf{R}, t - \tau) = \frac{2c(\nu_u - \nu)}{(1 - \nu)} \frac{(x'_\beta - x_\beta)}{R^5} \left(D(\xi) - \frac{8}{\sqrt{\pi}} \xi^5 e^{-\xi^2} \right), \quad (64)$$

$$H_\beta(\mathbf{R}' - \mathbf{R}, t - \tau) = -\frac{8c}{\sqrt{\pi}} \frac{(x'_\beta - x_\beta)}{R^5} \xi^5 e^{-\xi^2}, \quad (65)$$

$$U(\mathbf{R}' - \mathbf{R}, t - \tau) = \frac{2B(1+\nu_u)}{3} \frac{1}{R^3} \left(\text{erf}[\xi] - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} - \frac{4}{\sqrt{\pi}} \xi^3 e^{-\xi^2} \right), \quad (66)$$

$$V(\mathbf{R}' - \mathbf{R}, t - \tau) = -\frac{4B(1+\nu_u)(1+\nu)}{3(\nu_u - \nu)} \frac{1}{\sqrt{\pi}} \frac{\xi^3}{R^3} e^{-\xi^2} \quad (67)$$

with

$$D(\xi) = 3\text{erf}[\xi] - \frac{6}{\sqrt{\pi}} \xi e^{-\xi^2} - \frac{4}{\sqrt{\pi}} \xi^3 e^{-\xi^2} = \frac{8}{\sqrt{\pi}} \int_0^\xi s^4 e^{-s^2} ds. \quad (68)$$

9. Discussion

To obtain the integral equations which govern the mechanical responses of a planar crack subjected to non-uniformly distributed, normal and shear tractions and pressure, we have followed the approach used by Kurashige and Clifton⁽⁵⁾, that is, the approach resorting to superposition of the fundamental solutions of dislocation segments. Thus, the obtained singular integral equations (59) and (60) are expressed in terms of the "gradients" of the relative crack-surface displacements. As stressed in some places, for example, in Refs.(9) and (19), we can expect that the singularity of these integral equations is one order less than for the integral equations in terms of the relative crack-surface displacements themselves so that no special numerical techniques are required in obtaining numerical solution of the integral equations⁽²⁰⁾. In order to show that this is the case, we have to explore how the evolution functions behave as $R \rightarrow 0$. The exploration shows that evolution functions $T_{i\beta k}(\mathbf{R}' - \mathbf{R}, t - \tau)$ & $H_\beta(\mathbf{R}' - \mathbf{R}, t - \tau)$ and $U(\mathbf{R}' - \mathbf{R}, t - \tau)$ & $V(\mathbf{R}' - \mathbf{R}, t - \tau)$ have singularities of $O(1/R^2)$ and $O(1/R)$ at this limit, respectively; see **Appendix D**. Boundedness of integrands at the integral limits are also shown in this appendix. Thus, it is possible to solve numerically the derived integral equations (D.1) and (D.2) in the exactly same way as in Ref.(6), which treated a mode I crack in an infinite, fluid-saturated, isotropic, poroelastic solid and showed some effects of poroelasticity on crack opening width and others.

Putting $B=0$ and $\nu_u=\nu$, that is, considering the purely elastic case, the present result of Eq.(59) reduces to that by Guo et al.⁽⁹⁾ and Eq.(60) vanishes. However, it may be better to mention that $(x'_a + x_a)$ in the first parentheses of the first term and the plus sign in front of the second term in Eq.(13) in Ref.(9) should be read as $(x'_a - x_a)$ and a minus sign.

Next, for the case of no shear traction, the present integral equations reduce to those given in Ref.(5). That is, putting $k=3$ in Eq.(59), it reduces to

$$\sigma_{33}(\mathbf{R}, t) = \frac{G}{4\pi(1-\nu_u)} \int_{Sc} \left[S_{3\beta 3}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} + \int_0^t \left\{ T_{3\beta 3}(\mathbf{R}' - \mathbf{R}, t - \tau) \frac{\partial u_3(\mathbf{R}', \tau)}{\partial x'_\beta} + U(\mathbf{R}' - \mathbf{R}, t - \tau) \Delta q_n(\mathbf{R}', \tau) \right\} d\tau \right] dS' \quad (69)$$

and Eq.(60) remains unchanged. If $\sigma_{33}(\mathbf{R}, t)$ and $p(\mathbf{R}, t)$ are given on the crack surface, Eqs.(69) and (60) constitute a complete system of integral equations with respect to $u_3(\mathbf{R}', \tau)$ and $\Delta q_n(\mathbf{R}', \tau)$. Here again, some errors have been found in Ref.(5); they are the erroneous sign in front of the right hand side of Eqs.

(70) and (72), which are due to the error that a minus sign was missed out on the left hand side of Eq.(19) and, consequently, also of Eqs.(23) and (25).

Further, consider the case of no normal traction, which yields a new result. Taking $k=\gamma$ in Eq.(59) yields

$$\sigma_{3\gamma}(\mathbf{R}, t) = \frac{G}{4\pi(1-\nu_u)} \int_{sc} \left[S_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_a(\mathbf{R}', t)}{\partial x'_\beta} + \int_0^t \left\{ T_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}, t - \tau) \frac{\partial u_a(\mathbf{R}', \tau)}{\partial x'_\beta} \right\} d\tau \right] dS'. \quad (70)$$

In this case, does Eq.(60) remain unchanged? It may be natural to consider that $u_3(\mathbf{R}', \tau)=0$ (no opening) and $\Delta q_n(\mathbf{R}', \tau)=0$ (no fluid supply), and thus $p(\mathbf{R}, t)=0$. This may be the case, if the pore fluid pressure distribution is antisymmetric with respect to the crack plane (and this plane is permeable). See Refs. (9) and (10) for the two-dimensional case. If $\sigma_{31}(\mathbf{R}, t)$ and $\sigma_{32}(\mathbf{R}, t)$ are given on the crack face, Eq.(70) is coupled integral equations with respect to $u_1(\mathbf{R}', \tau)$ and $u_2(\mathbf{R}', \tau)$. Note that the poroelastic effect does not disappear even for the pure shear mode.

Finally, we will consider the integral equation for the undrained and drained conditions. If we take $t=0$ in Eq.(59) or Eq.(D.1), we easily obtain the solution at the undrained state, which is the same as that for an elastic infinite space with undrained Poisson's ratio ν_u . In contrast to this case, it seems very difficult to prove without any specific condition that Eq.(59) or Eq.(D.1) reduces to that for an elastic infinite space with drained Poisson's ratio ν and Eq.(60) or Eq.(D.2) vanishes for the limit of drained state. Thus, in addition to the conditions of steady state $t \rightarrow \infty$ and no leak-off $\Delta q_n(\mathbf{R}', \xi)=0$, we assume

$$u_i(\mathbf{R}', \xi) = u_i(\mathbf{R}', t = \infty) \quad (71)$$

for all ξ in Eqs.(D.1) and (D.2) in **Appendix D**; this enables us to perform integration with respect to ξ (see **Appendix E**), and thus the integral equation for an elastic reservoir, given in Ref.(9), and $p(\mathbf{R}, t)=0$ are recovered from Eqs.(D.1) and (D.2), respectively.

10. Concluding Remarks

We derived four coupled integral equations for a planar crack of arbitrary shape, subjected to the normal and shear tractions and fluid pressure, in an elastically and hydraulically isotropic, fluid-saturated, poroelastic, infinite solid. Taking the shear tractions into account was intended to apply the obtained integral equations to the case that a hydraulically created fracture extends along an inclined weak surface in an oil reservoir in tectonically more active regions.

We reduced the integral equations for arbitrary loadings to those for pure shear traction case. It was expected that those equations could be applied to the

analysis of fault slip events associated with earthquakes.

We confirmed identification of the present solutions with the known ones for the purely elastic case⁽⁹⁾ and the poroelastic case with no shear traction⁽⁵⁾ as special cases.

Appendix A

The integrals of $\sigma_{(0)i\beta}(\mathbf{R}', \mathbf{r}, t - \tau)$ from $z = -\infty$ to $z=0$ and their time derivatives appearing in Eq.(48) in the text can be easily estimated by use of a software "Mathematica"⁽¹⁸⁾ as an algebraic tool; the results are

$$\int_{-\infty}^0 \sigma_{(0)a\beta}(\mathbf{R}', \mathbf{r}, 0) dz = -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \left\{ \frac{1}{R^2} \delta_{a\beta} - 2 \frac{(x'_a - x_a)(x'_\beta - x_\beta)}{R^4} \right\}, \quad (A.1)$$

$$\int_{-\infty}^0 \sigma_{(0)3a}(\mathbf{R}', \mathbf{r}, 0) dz = -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \frac{(x'_a - x_a)}{R^3}, \quad (A.2)$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)a\beta}(\mathbf{R}', \mathbf{r}, t - \tau) dz \\ = -\frac{2GBc(1+\nu_u)}{3\pi(1-\nu_u)} \frac{\xi^4}{R^4} \left\{ e^{-\xi^2} \delta_{a\beta} - 2\xi^2 e^{-\xi^2} \delta_{a\beta} + 2 \frac{(x'_a - x_a)(x'_\beta - x_\beta)}{R^2} \xi^2 e^{-\xi^2} \right\}, \end{aligned} \quad (A.3)$$

$$\begin{aligned} \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)3a}(\mathbf{R}', \mathbf{r}, t - \tau) dz \\ = \frac{4GBc(1+\nu_u)}{3\pi^{3/2}(1-\nu_u)} \frac{\xi^5}{R^5} (x'_a - x_a) e^{-\xi^2}. \end{aligned} \quad (A.4)$$

Appendix B

Let us show that the first and third terms in the bracket on the right hand side of Eq.(51) identically vanish. Integrals of the first and third terms with respect to area can be written as

$$I^{(\gamma)} = \int_{sc} H_{a\beta}^{(\gamma)}(\mathbf{R}' - \mathbf{R}, \xi) \frac{\partial u_a(\mathbf{R}', \tau)}{\partial x'_\beta} dS', \quad (B.1)$$

where $\gamma=1, 2$ correspond to the first and third terms, respectively. With the help of the divergence theorem, these integrals are

$$I^{(\gamma)} = \oint H_{a\beta}^{(\gamma)} u_a n_\beta ds - \int_{sc} \frac{\partial H_{a\beta}^{(\gamma)}}{\partial x'_\beta} u_a dS', \quad (B.2)$$

the first term of which vanishes because $u_a=0$ on the crack periphery. In order to show $I^{(\gamma)}=0$, it is necessary to prove that identically $\partial H_{a\beta}^{(\gamma)} / \partial x'_\beta = 0$. This proof is easy if one remembers that

$$\frac{\partial}{\partial x'_\beta} \left(\frac{1}{R^2} \right) = -\frac{2(x'_\beta - x_\beta)}{R^4}, \quad \frac{\partial \xi^2}{\partial x'_\beta} = 2 \left(\frac{\xi}{R^2} \right)^2 (x'_\beta - x_\beta) \quad (B.3)$$

and especially that

$$\frac{\partial}{\partial x'_\beta} \{ (x'_a - x_a)(x'_\beta - x_\beta) \} = 3(x'_a - x_a). \quad (B.4)$$

Appendix C

The integrals of inner integration with respect to s from $s=\tau$ to $s=t$ after interchanging the integration order and the notation in Eq. (54) can be readily estimated by use of a software "Mathematica"⁽¹⁸⁾ as an algebraic tool again; the results are

$$\int_{\tau}^t \frac{\partial p_{(3)}(\mathbf{R}', \mathbf{R}, s-\tau)}{\partial x'_3} dS = \frac{B(1+\nu_u)}{12\pi(1-\nu_u)} \frac{1}{R^3} \left[\text{erf}[\xi] - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right], \quad (\text{C. 1})$$

$$\int_{\tau}^t \frac{\partial p_{(3)}(\mathbf{R}', \mathbf{R}, s-\tau)}{\partial x'_a} dS = 0, \quad (\text{C. 2})$$

$$\int_{\tau}^t \frac{\partial p_{(a)}(\mathbf{R}', \mathbf{R}, s-\tau)}{\partial x'_3} dS = 0, \quad (\text{C. 3})$$

$$\int_{\tau}^t \frac{\partial p_{(k)}(\mathbf{R}', \mathbf{R}, s-\tau)}{\partial x'_k} dS = \frac{B(1+\nu_u)}{3\pi(1-\nu_u)} \frac{1}{R^3} \frac{1}{\sqrt{\pi}} \xi^3 e^{-\xi^2}. \quad (\text{C. 4})$$

Appendix D

Although the time evolution in Eqs. (59) and (60) are expressed in terms of integration from its initial time 0 to present time t with respect to τ , it may be better to express them in terms of non-dimensional variable ξ defined by the second equation of Eq. (52). This variable transformation yields

$$\sigma_{3k}(\mathbf{R}, t) = \frac{G}{4\pi(1-\nu_u)} \int_{sc} \left[S_{i\beta k}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_i(\mathbf{R}', t)}{\partial x'_\beta} + \int_{R/2\sqrt{ct}}^{\infty} \left\{ \bar{T}_{i\beta k}(\mathbf{R}' - \mathbf{R}, \xi) \frac{\partial u_i(\mathbf{R}', \xi)}{\partial x'_\beta} + \delta_{3k} \bar{U}(\mathbf{R}' - \mathbf{R}, \xi) \Delta q_n(\mathbf{R}', \xi) \right\} d\xi \right] dS', \quad (\text{D. 1})$$

$$p(\mathbf{R}, t) = -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \int_{sc} \left[S_{3\beta 3}(\mathbf{R}' - \mathbf{R}) \frac{\partial u_3(\mathbf{R}', t)}{\partial x'_\beta} + \int_{R/2\sqrt{ct}}^{\infty} \left\{ \bar{H}_\beta(\mathbf{R}' - \mathbf{R}, \xi) \frac{\partial u_3(\mathbf{R}', \xi)}{\partial x'_\beta} + \bar{V}(\mathbf{R}' - \mathbf{R}, \xi) \Delta q_n(\mathbf{R}', \xi) \right\} d\xi \right] dS', \quad (\text{D. 2})$$

where

$$\begin{aligned} \bar{T}_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}, \xi) &= -\frac{(\nu_u - \nu)}{2(1-\nu)} \left\{ \frac{8}{\sqrt{\pi}} \frac{(x'_\gamma - x_\gamma)}{R^3} \right. \\ &\quad \times \xi^2 e^{-\xi^2} \delta_{a\beta} - \frac{(x'_\beta - x_\beta) \delta_{a\gamma} + (x'_\gamma - x_\gamma) \delta_{a\beta}}{R^3} \xi^{-3} D(\xi) \\ &\quad + \frac{(x'_a - x_a)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{R^5} \left(5\xi^{-3} D(\xi) \right. \\ &\quad \left. \left. - \frac{8}{\sqrt{\pi}} \xi^2 e^{-\xi^2} \right) \right\}, \quad (\text{D. 3}) \end{aligned}$$

$$\begin{aligned} \bar{T}_{3\beta 3}(\mathbf{R}' - \mathbf{R}, \xi) &= \frac{(\nu_u - \nu)}{(1-\nu)} \frac{(x'_\beta - x_\beta)}{R^3} \left(\xi^{-3} D(\xi) \right. \\ &\quad \left. - \frac{8}{\sqrt{\pi}} \xi^2 e^{-\xi^2} \right), \quad (\text{D. 4}) \end{aligned}$$

$$\bar{H}_\beta(\mathbf{R}' - \mathbf{R}, \xi) = -\frac{4}{\sqrt{\pi}} \frac{(x'_\beta - x_\beta)}{R^3} \xi^2 e^{-\xi^2}, \quad (\text{D. 5})$$

$$\bar{U}(\mathbf{R}' - \mathbf{R}, \xi) = \frac{B(1+\nu_u)}{3c} \frac{1}{R} \left(\xi^{-3} \text{erf}[\xi] \right.$$

$$\left. - \frac{2}{\sqrt{\pi}} \xi^{-2} e^{-\xi^2} - \frac{4}{\sqrt{\pi}} e^{-\xi^2} \right), \quad (\text{D. 6})$$

$$\bar{V}(\mathbf{R}' - \mathbf{R}, \xi) = -\frac{2B(1+\nu_u)(1-\nu)}{3(\nu_u - \nu)} \frac{1}{\sqrt{\pi}} \frac{1}{R^3} e^{-\xi^2}. \quad (\text{D. 7})$$

Inspection of Eqs. (D. 3) through (D. 7) shows that evolution functions $\bar{T}_{i\beta k}(\mathbf{R}' - \mathbf{R}, \xi)$ & $\bar{H}_\beta(\mathbf{R}' - \mathbf{R}, \xi)$ and $\bar{U}(\mathbf{R}' - \mathbf{R}, \xi)$ & $\bar{V}(\mathbf{R}' - \mathbf{R}, \xi)$ have singularities of $O(1/R^2)$ and $O(1/R)$ at the limit $R \rightarrow 0$, respectively. The former singularity is the same as that of the first term in Eqs. (D. 1) and (D. 2) while the latter has a weaker one; this assures that the numerical integration of the obtained integral equations is rather easy, at least concerning the treatment of singularity.

What to be done next in this appendix is to discuss the behavior of the evolution functions given by Eqs. (D. 3)–(D. 7) at the limits $\xi \rightarrow \infty$ and $\xi \rightarrow 0$ ($t \rightarrow \infty$). It is easy to show that all evolution functions vanish for the limit $\xi \rightarrow \infty$, if we resort to

$$\lim_{\xi \rightarrow \infty} D(\xi) = -\frac{8}{\sqrt{\pi}} \int_0^\infty s^4 e^{-s^2} ds = 3, \quad (\text{D. 8})$$

$$\lim_{\xi \rightarrow \infty} \text{erf}[\xi] = -\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = 1. \quad (\text{D. 9})$$

For the limit $\xi \rightarrow 0$ ($t \rightarrow \infty$),

$$\bar{T}_{i\beta j}(\mathbf{R}' - \mathbf{R}, 0) = 0, \quad \bar{H}_\beta(\mathbf{R}' - \mathbf{R}, 0) = 0, \quad (\text{D. 10})$$

because

$$\lim_{\xi \rightarrow 0} \xi^{-5} D(\xi) = \frac{8}{5\sqrt{\pi}}, \quad (\text{D. 11})$$

and

$$\begin{aligned} \bar{U}(\mathbf{R}' - \mathbf{R}, 0) &= \frac{8B(1+\nu_u)}{9\sqrt{\pi}c} \frac{1}{R}, \\ \bar{V}(\mathbf{R}' - \mathbf{R}, 0) &= -\frac{2B(1+\nu_u)(1-\nu)}{3(\nu_u - \nu)c} \frac{1}{R} \end{aligned} \quad (\text{D. 12})$$

because of

$$\lim_{\xi \rightarrow 0} \left(\xi^{-3} \text{erf}[\xi] - \frac{2}{\sqrt{\pi}} \xi^{-2} e^{-\xi^2} \right) = \frac{4}{3\sqrt{\pi}}. \quad (\text{D. 13})$$

Appendix E

Assumption (71) puts the gradients of relative displacement discontinuity at the crack surface out of the integrals in Eqs. (D. 1) and (D. 2) and thus enables us to carry out the integration:

$$\begin{aligned} \int_0^\infty \bar{T}_{a\beta\gamma}(\mathbf{R}' - \mathbf{R}, \xi) d\xi &= -\frac{c(\nu_u - \nu)}{2(1-\nu)} \left\{ \frac{(x'_\gamma - x_\gamma)}{R^3} \delta_{a\beta} \right. \\ &\quad \left. + 3 \frac{(x'_a - x_a)(x'_\beta - x_\beta)(x'_\gamma - x_\gamma)}{R^5} \right\}, \quad (\text{E. 1}) \end{aligned}$$

$$\int_0^\infty \bar{T}_{3\beta 3}(\mathbf{R}' - \mathbf{R}, \xi) d\xi = \frac{c(\nu_u - \nu)}{1-\nu} \frac{(x'_\beta - x_\beta)}{R^3}, \quad (\text{E. 2})$$

$$\int_0^\infty \bar{H}_\beta(\mathbf{R}' - \mathbf{R}, \xi) d\xi = -\frac{(x'_\beta - x_\beta)}{R^3}. \quad (\text{E. 3})$$

The last equation yields $p(\mathbf{R}, t) = 0$.

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