An Application of the Method of Arbitrary Lines to 3D Elastic Stress Analysis*

Ken KAMINISHI** and Ryuma ANDO**

The MAL (Method of Arbitrary Lines) is a technique of reducing a partial differential equation to a system of ordinary differential equations. It is known that relevant use of this procedure yields high accuracy in some problems of two-dimensional elasticity and elastoplasticity. Since the basic concept of MAL is simple and based on generality, it is expected that many problems in other fields will be effectively solvable by this method. In this study, we consider the application of MAL to 3D (three-dimensional) elasticity analysis. We first give a MAL formulation of 3D elasticity problems, and demonstrate its effectiveness and accuracy for a typical problem. The reported numerical results are compared with the exact solution or that of the finite element method (FEM).

Key Words: Numerical Analysis, Computational Mechanics, Method of Arbitrary Lines, Elasticity, Finite Element Method, Three-Dimensional Analysis

1. Introduction

The Method of Arbitrary Lines (MAL) is a new general-purpose numerical analysis method which uses a dimensional reduction approach for partial differential equations over arbitrary domains by solving a system of a boundary value problem (BVP) for ordinary differential equations (ODEs). A high level of accuracy can be expected by the MAL analysis, since it does not require discretization in the curve direction. It has already been applied to two-dimensional elastic and elastoplastic problems⁽¹⁾⁻⁽³⁾, and good results have been reported. In this work, we consider the extension of the MAL to 3D (three-dimensional) elastic stress analysis. We first derive the principle of discrete virtual work by 3D MAL and propose a method for solving the discrete systems. Furthermore, the MAL solutions are compared with FEM ones and the effectiveness and accuracy of the

MAL is revealed.

2. The Principle of Discrete Virtual Work by the MAL

The principle of 3D virtual work for a 3D elasticity problem is

$$\begin{aligned} & \iiint_{\Omega} \left\{ \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta u \right. \\ & \quad + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \delta v \\ & \quad + \left. \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) \delta w \right\} dx dy dz \\ & = - \iiint_{\Omega} (f_x \delta u + f_y \delta v + f_z \delta w) dx dy dz. \quad (1) \end{aligned}$$

Applying Green's formula to Eq.(1), the following equation is obtained:

$$\begin{aligned} & \iiint_{\Omega} \left\{ D_{11} \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} + D_{12} \frac{\partial v}{\partial y} \frac{\partial \delta u}{\partial x} + D_{13} \frac{\partial w}{\partial z} \frac{\partial \delta u}{\partial x} \right. \\ & \quad + D_{21} \frac{\partial u}{\partial x} \frac{\partial \delta v}{\partial y} + D_{22} \frac{\partial v}{\partial y} \frac{\partial \delta v}{\partial y} + D_{23} \frac{\partial w}{\partial z} \frac{\partial \delta v}{\partial y} \\ & \quad + D_{31} \frac{\partial u}{\partial x} \frac{\partial \delta w}{\partial z} + D_{32} \frac{\partial v}{\partial y} \frac{\partial \delta w}{\partial z} + D_{33} \frac{\partial w}{\partial z} \frac{\partial \delta w}{\partial z} \\ & \quad + D_{44} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left(\frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) \\ & \quad + D_{55} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \left(\frac{\partial \delta v}{\partial z} + \frac{\partial \delta w}{\partial y} \right) \\ & \quad + \left. D_{66} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \left(\frac{\partial \delta w}{\partial x} + \frac{\partial \delta u}{\partial z} \right) \right\} dx dy dz \end{aligned}$$

* Received 21th March, 2001. Japanese original: Trans Jpn. Soc. Mech. Eng., Vol.66, No.644, A (2000), pp.685-691 (Received 23rd April, 1999)

** Yamaguchi University, Department of Mechanical Engineering, 2-16-1 Tokiwadai, Ube, Yamaguchi 755-8611, Japan. E-mail: kaminisi@po.cc.yamaguchi-u.ac.jp

$$= \iiint_{\Omega} (f_x \delta u + f_y \delta v + f_z \delta w) dx dy dz \\ + \iint_{\Gamma_N} (h_x \delta u + h_y \delta v + h_z \delta w) ds, \quad (2)$$

where

$$h_x = \sigma_x \cos(n, x) + \tau_{xy} \cos(n, y) + \tau_{zx} \cos(n, z) \\ h_y = \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{yz} \cos(n, z) \\ h_z = \tau_{zx} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_z \cos(n, z).$$

u, v, w are the displacements of x, y, z , respectively, $\delta u, \delta v, \delta w$ are the corresponding virtual displacements and f_x, f_y, f_z are the corresponding body forces. D_{11} , etc. are components of the elastic stress-strain matrix.

We express Ω as the sum of subregions $\Omega = \bigcup_{i=1}^N \Omega_i$ and assume a suitable mapping Φ_i between master element $\bar{\Omega} = (-1, 1)^3$ -domain and Ω_i in (x, y, z) -domain.

$$\Omega_i = \Phi_i(\bar{\Omega}) \\ x = x(\xi, \eta, \zeta) \\ y = y(\xi, \eta, \zeta) \\ z = z(\xi, \eta, \zeta)$$

Furthermore, $u, v, w, \delta u, \delta v$ and δw for each element Ω_i will be approximated by the following forms without discretization in the ζ direction.

$$u|_{\Omega_i} = u(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p X_{xj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{X}_x^{iT} \mathbf{Q} \quad (3)$$

$$v|_{\Omega_i} = v(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p X_{yj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{X}_y^{iT} \mathbf{Q} \quad (4)$$

$$z|_{\Omega_i} = w(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p X_{zj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{X}_z^{iT} \mathbf{Q} \quad (5)$$

$$\delta u|_{\Omega_i} = \delta u(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p \Psi_{xj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{\Psi}_x^{iT} \mathbf{Q} \quad (6)$$

$$\delta v|_{\Omega_i} = \delta v(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p \Psi_{yj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{\Psi}_y^{iT} \mathbf{Q} \quad (7)$$

$$\delta w|_{\Omega_i} = \delta w(\Phi_i(\xi, \eta, \zeta)) = \sum_j^p \Psi_{zj}^i(\zeta)^T Q_j(\xi, \eta) \\ = \mathbf{\Psi}_z^{iT} \mathbf{Q} \quad (8)$$

where p is the number of lines which constitute Ω_i . Substituting Eqs. (3)–(8) into Eq. (2), the left-hand side of Eq. (2) can be arranged as follows.

$$\int_{-1}^1 \left(\{ \mathbf{\Psi}_{x\zeta}^i, \mathbf{\Psi}_{y\zeta}^i, \mathbf{\Psi}_{z\zeta}^i \} \begin{bmatrix} \mathbf{A}_{11}^i & \mathbf{A}_{12}^i & \mathbf{A}_{13}^i \\ \mathbf{A}_{21}^i & \mathbf{A}_{22}^i & \mathbf{A}_{23}^i \\ \mathbf{A}_{31}^i & \mathbf{A}_{32}^i & \mathbf{A}_{33}^i \end{bmatrix} \begin{Bmatrix} \mathbf{X}_{x\zeta}^i \\ \mathbf{X}_{y\zeta}^i \\ \mathbf{X}_{z\zeta}^i \end{Bmatrix} \right. \\ + \{ \mathbf{\Psi}_x^i, \mathbf{\Psi}_y^i, \mathbf{\Psi}_z^i \} \begin{bmatrix} \mathbf{B}_{11}^i & \mathbf{B}_{12}^i & \mathbf{B}_{13}^i \\ \mathbf{B}_{21}^i & \mathbf{B}_{22}^i & \mathbf{B}_{23}^i \\ \mathbf{B}_{31}^i & \mathbf{B}_{32}^i & \mathbf{B}_{33}^i \end{bmatrix} \begin{Bmatrix} \mathbf{X}_x^i \\ \mathbf{X}_y^i \\ \mathbf{X}_z^i \end{Bmatrix} \\ \left. + \{ \mathbf{\Psi}_x^i, \mathbf{\Psi}_y^i, \mathbf{\Psi}_z^i \} \begin{bmatrix} \mathbf{C}_{11}^i & \mathbf{C}_{12}^i & \mathbf{C}_{13}^i \\ \mathbf{C}_{21}^i & \mathbf{C}_{22}^i & \mathbf{C}_{23}^i \\ \mathbf{C}_{31}^i & \mathbf{C}_{32}^i & \mathbf{C}_{33}^i \end{bmatrix} \begin{Bmatrix} \mathbf{X}_{x\zeta}^i \\ \mathbf{X}_{y\zeta}^i \\ \mathbf{X}_{z\zeta}^i \end{Bmatrix} \right)$$

$$+ \{ \mathbf{\Psi}_{x\zeta}^i, \mathbf{\Psi}_{y\zeta}^i, \mathbf{\Psi}_{z\zeta}^i \} \begin{bmatrix} \mathbf{C}_{11}^i & \mathbf{C}_{12}^i & \mathbf{C}_{13}^i \\ \mathbf{C}_{21}^i & \mathbf{C}_{22}^i & \mathbf{C}_{23}^i \\ \mathbf{C}_{31}^i & \mathbf{C}_{32}^i & \mathbf{C}_{33}^i \end{bmatrix} \begin{Bmatrix} \mathbf{X}_x^i \\ \mathbf{X}_y^i \\ \mathbf{X}_z^i \end{Bmatrix} \Bigg) d\zeta$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are matrices consisting of an elastic matrix as follows.

$$\mathbf{A}_{11}^i = \int_{-1}^1 \int_{-1}^1 (D_{11} \Delta_{31}^2 + D_{44} \Delta_{32}^2 + D_{66} \Delta_{33}^2) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{12}^i = \int_{-1}^1 \int_{-1}^1 (D_{12} \Delta_{31} \Delta_{32} + D_{44} \Delta_{32} \Delta_{31}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{13}^i = \int_{-1}^1 \int_{-1}^1 (D_{13} \Delta_{31} \Delta_{33} + D_{66} \Delta_{33} \Delta_{31}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{21}^i = \int_{-1}^1 \int_{-1}^1 (D_{44} \Delta_{31} \Delta_{32} + D_{21} \Delta_{32} \Delta_{31}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{22}^i = \int_{-1}^1 \int_{-1}^1 (D_{44} \Delta_{31}^2 + D_{22} \Delta_{32}^2 + D_{55} \Delta_{33}^2) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{23}^i = \int_{-1}^1 \int_{-1}^1 (D_{23} \Delta_{32} \Delta_{33} + D_{55} \Delta_{33} \Delta_{32}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{31}^i = \int_{-1}^1 \int_{-1}^1 (D_{66} \Delta_{31} \Delta_{33} + D_{31} \Delta_{33} \Delta_{31}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{32}^i = \int_{-1}^1 \int_{-1}^1 (D_{55} \Delta_{32} \Delta_{33} + D_{32} \Delta_{33} \Delta_{32}) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{A}_{33}^i = \int_{-1}^1 \int_{-1}^1 (D_{66} \Delta_{31}^2 + D_{55} \Delta_{32}^2 + D_{33} \Delta_{33}^2) \mathbf{Q} \mathbf{Q}^T |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{11}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{11} \Delta_{11}^2 + D_{44} \Delta_{12}^2 + D_{66} \Delta_{13}^2) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{11} \Delta_{11} \Delta_{21} + D_{44} \Delta_{12} \Delta_{22} + D_{66} \Delta_{13} \Delta_{23}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{11} \Delta_{11} \Delta_{21} + D_{44} \Delta_{12} \Delta_{22} + D_{66} \Delta_{13} \Delta_{23}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{11} \Delta_{21}^2 + D_{44} \Delta_{22}^2 + D_{66} \Delta_{23}^2) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{12}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{12} \Delta_{11} \Delta_{12} + D_{44} \Delta_{12} \Delta_{11}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{12} \Delta_{11} \Delta_{22} + D_{44} \Delta_{12} \Delta_{21}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{12} \Delta_{21} \Delta_{12} + D_{44} \Delta_{22} \Delta_{11}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{12} \Delta_{21} \Delta_{22} + D_{44} \Delta_{22} \Delta_{21}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{13}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{13} \Delta_{11} \Delta_{13} + D_{66} \Delta_{13} \Delta_{11}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{13} \Delta_{11} \Delta_{23} + D_{66} \Delta_{13} \Delta_{21}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{13} \Delta_{21} \Delta_{13} + D_{66} \Delta_{23} \Delta_{11}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{13} \Delta_{21} \Delta_{23} + D_{66} \Delta_{23} \Delta_{21}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{21}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{44} \Delta_{11} \Delta_{12} + D_{21} \Delta_{12} \Delta_{11}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{44} \Delta_{11} \Delta_{22} + D_{21} \Delta_{12} \Delta_{21}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{44} \Delta_{21} \Delta_{12} + D_{21} \Delta_{22} \Delta_{11}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{44} \Delta_{21} \Delta_{22} + D_{21} \Delta_{22} \Delta_{21}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{22}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{44} \Delta_{11}^2 + D_{22} \Delta_{12}^2 + D_{55} \Delta_{13}^2) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{44} \Delta_{11} \Delta_{21} + D_{22} \Delta_{12} \Delta_{22} + D_{55} \Delta_{13} \Delta_{23}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{44} \Delta_{11} \Delta_{21} + D_{22} \Delta_{12} \Delta_{22} + D_{55} \Delta_{13} \Delta_{23}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{44} \Delta_{21}^2 + D_{22} \Delta_{22}^2 + D_{55} \Delta_{23}^2) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{23}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{23} \Delta_{12} \Delta_{13} + D_{55} \Delta_{13} \Delta_{12}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{23} \Delta_{12} \Delta_{23} + D_{55} \Delta_{13} \Delta_{22}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{23} \Delta_{22} \Delta_{13} + D_{55} \Delta_{23} \Delta_{12}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{23} \Delta_{22} \Delta_{23} + D_{55} \Delta_{23} \Delta_{22}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\mathbf{B}_{31}^i = \int_{-1}^1 \int_{-1}^1 \{ (D_{66} \Delta_{11} \Delta_{13} + D_{31} \Delta_{13} \Delta_{11}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\ + (D_{66} \Delta_{11} \Delta_{23} + D_{31} \Delta_{13} \Delta_{21}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\ + (D_{66} \Delta_{21} \Delta_{13} + D_{31} \Delta_{23} \Delta_{11}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\ + (D_{66} \Delta_{21} \Delta_{23} + D_{31} \Delta_{23} \Delta_{21}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta$$

$$\begin{aligned}
B_{32}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{55}\Delta_{12}\Delta_{13} + D_{32}\Delta_{13}\Delta_{12}) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\
&\quad + (D_{55}\Delta_{12}\Delta_{23} + D_{32}\Delta_{13}\Delta_{22}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\
&\quad + (D_{55}\Delta_{22}\Delta_{13} + D_{32}\Delta_{23}\Delta_{12}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\
&\quad + (D_{55}\Delta_{22}\Delta_{23} + D_{32}\Delta_{23}\Delta_{22}) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
B_{33}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{66}\Delta_{11}^2 + D_{55}\Delta_{12}^2 + D_{33}\Delta_{13}^2) \mathbf{Q}_\xi \mathbf{Q}_\xi^T \\
&\quad + (D_{66}\Delta_{11}\Delta_{21} + D_{55}\Delta_{12}\Delta_{22} + D_{33}\Delta_{13}\Delta_{23}) \mathbf{Q}_\xi \mathbf{Q}_\eta^T \\
&\quad + (D_{66}\Delta_{11}\Delta_{21} + D_{55}\Delta_{12}\Delta_{22} + D_{33}\Delta_{13}\Delta_{23}) \mathbf{Q}_\eta \mathbf{Q}_\xi^T \\
&\quad + (D_{66}\Delta_{21}^2 + D_{55}\Delta_{22}^2 + D_{33}\Delta_{23}^2) \mathbf{Q}_\eta \mathbf{Q}_\eta^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{11}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{11}\Delta_{11}\Delta_{31} + D_{44}\Delta_{12}\Delta_{32} \\
&\quad + D_{66}\Delta_{13}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{11}\Delta_{21}\Delta_{31} + D_{44}\Delta_{22}\Delta_{32} \\
&\quad + D_{66}\Delta_{23}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{12}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{12}\Delta_{12}\Delta_{31} + D_{44}\Delta_{11}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{12}\Delta_{22}\Delta_{31} + D_{44}\Delta_{21}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{13}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{13}\Delta_{13}\Delta_{31} + D_{66}\Delta_{11}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{13}\Delta_{23}\Delta_{32} + D_{66}\Delta_{21}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{21}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{21}\Delta_{11}\Delta_{32} + D_{44}\Delta_{12}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{21}\Delta_{21}\Delta_{32} + D_{44}\Delta_{22}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{22}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{22}\Delta_{12}\Delta_{32} + D_{44}\Delta_{11}\Delta_{31} \\
&\quad + D_{55}\Delta_{13}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{22}\Delta_{22}\Delta_{32} + D_{44}\Delta_{21}\Delta_{31} \\
&\quad + D_{55}\Delta_{23}\Delta_{33}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{23}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{23}\Delta_{13}\Delta_{32} + D_{55}\Delta_{12}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{23}\Delta_{23}\Delta_{32} + D_{55}\Delta_{22}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{31}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{31}\Delta_{11}\Delta_{33} + D_{66}\Delta_{13}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{31}\Delta_{21}\Delta_{33} + D_{66}\Delta_{23}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{32}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{32}\Delta_{12}\Delta_{33} + D_{55}\Delta_{13}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{32}\Delta_{22}\Delta_{33} + D_{55}\Delta_{23}\Delta_{32}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta \\
C_{33}^i &= \int_{-1}^1 \int_{-1}^1 \{ (D_{33}\Delta_{13}\Delta_{33} + D_{55}\Delta_{12}\Delta_{32} \\
&\quad + D_{66}\Delta_{11}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \\
&\quad + (D_{33}\Delta_{23}\Delta_{33} + D_{55}\Delta_{22}\Delta_{32} \\
&\quad + D_{66}\Delta_{21}\Delta_{31}) \mathbf{Q} \mathbf{Q}^T \} |\mathbf{J}|^{-1} d\xi d\eta,
\end{aligned}$$

where

$$\Delta_{11} = y_\eta z_\xi - y_\xi z_\eta$$

$$\begin{aligned}
\Delta_{12} &= -x_\eta z_\xi + x_\xi z_\eta \\
\Delta_{13} &= x_\eta y_\xi - x_\xi y_\eta \\
\Delta_{21} &= -y_\xi z_\xi - y_\xi z_\xi \\
\Delta_{22} &= x_\xi z_\xi - x_\xi z_\xi \\
\Delta_{23} &= -x_\xi y_\xi + x_\xi y_\xi \\
\Delta_{31} &= y_\xi z_\eta - y_\eta z_\xi \\
\Delta_{32} &= -x_\xi z_\eta + x_\eta z_\xi \\
\Delta_{33} &= x_\xi y_\eta - x_\eta y_\xi \\
\mathbf{J} &= \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix}
\end{aligned}$$

x_ζ , etc. denote the partial differential by ζ of x . The first term on the right-hand side of Eq.(2) is expressed as

$$\begin{aligned}
&\iiint_{\Omega} (f_x \delta u + f_y \delta v + f_z \delta w) dx dy dz \\
&= \int_{-1}^1 \Psi_x^{iT}(\zeta) \int_{-1}^1 \int_{-1}^1 f_x(\Phi_i(\xi, \eta, \zeta)) |\mathbf{J}| \mathbf{Q}(\xi, \eta) d\xi d\eta d\zeta \\
&\quad + \int_{-1}^1 \Psi_y^{iT}(\zeta) \int_{-1}^1 \int_{-1}^1 f_y(\Phi_i(\xi, \eta, \zeta)) |\mathbf{J}| \mathbf{Q}(\xi, \eta) d\xi d\eta d\zeta \\
&\quad + \int_{-1}^1 \Psi_z^{iT}(\zeta) \int_{-1}^1 \int_{-1}^1 f_z(\Phi_i(\xi, \eta, \zeta)) |\mathbf{J}| \mathbf{Q}(\xi, \eta) d\xi d\eta d\zeta
\end{aligned}$$

and the second term on the right-hand side of Eq.(2) is expressed as

$$\begin{aligned}
&\iint_{\Gamma_N} (h_x \delta u + h_y \delta v + h_z \delta w) ds \\
&= \int_{-1}^1 \int_{-1}^1 h_x(\Phi_i(\xi, \eta, \zeta)) \Psi_x^{iT} \mathbf{Q}(\xi, \eta) ds|_{\xi=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_y(\Phi_i(\xi, \eta, \zeta)) \Psi_y^{iT} \mathbf{Q}(\xi, \eta) ds|_{\xi=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_z(\Phi_i(\xi, \eta, \zeta)) \Psi_z^{iT} \mathbf{Q}(\xi, \eta) ds|_{\xi=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_x(\Phi_i(\xi, \eta, \zeta)) \Psi_x^{iT} \mathbf{Q}(\xi, \eta) ds|_{\eta=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_y(\Phi_i(\xi, \eta, \zeta)) \Psi_y^{iT} \mathbf{Q}(\xi, \eta) ds|_{\eta=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_z(\Phi_i(\xi, \eta, \zeta)) \Psi_z^{iT} \mathbf{Q}(\xi, \eta) ds|_{\eta=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_x(\Phi_i(\xi, \eta, \zeta)) \Psi_x^{iT} \mathbf{Q}(\xi, \eta) ds|_{\zeta=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_y(\Phi_i(\xi, \eta, \zeta)) \Psi_y^{iT} \mathbf{Q}(\xi, \eta) ds|_{\zeta=\pm 1} \\
&\quad + \int_{-1}^1 \int_{-1}^1 h_z(\Phi_i(\xi, \eta, \zeta)) \Psi_z^{iT} \mathbf{Q}(\xi, \eta) ds|_{\zeta=\pm 1}
\end{aligned}$$

where

$$\begin{aligned}
ds|_{\xi=\pm 1} &= \sqrt{\left(\frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \xi} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \xi}\right)^2} d\eta d\zeta \Big|_{\xi=\pm 1} \\
ds|_{\eta=\pm 1} &= \sqrt{\left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \xi} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \xi}\right)^2 + \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \xi}\right)^2} d\xi d\zeta \Big|_{\eta=\pm 1} \\
ds|_{\zeta=\pm 1} &= \sqrt{\left(\frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta}\right)^2 + \left(\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}\right)^2} d\xi d\eta \Big|_{\zeta=\pm 1}
\end{aligned}$$

Therefore, the principle of discrete virtual work by the MAL can be expressed by the following equation:

$$\int_{-1}^1 [\Psi_\xi^T \mathbf{A} \mathbf{X}_\xi + \Psi^T \mathbf{C}^T \mathbf{X}_\xi + \Psi_\xi^T \mathbf{C} \mathbf{X} + \Psi^T \mathbf{B} \mathbf{X}] d\zeta = \int_{-1}^1 \Psi^T(\zeta) \mathbf{F} d\zeta + \Psi^T(\zeta) \mathbf{q}, \quad (9)$$

where

$$\begin{aligned} X &= \sum_i \begin{Bmatrix} X_x^i \\ X_y^i \\ X_z^i \end{Bmatrix}, \quad \Psi = \sum_i \begin{Bmatrix} \Psi_x^i \\ \Psi_y^i \\ \Psi_z^i \end{Bmatrix}, \\ f &= \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix}, \quad h = \begin{Bmatrix} h_x \\ h_y \\ h_z \end{Bmatrix}, \\ A &= \sum_i \begin{bmatrix} A_{11}^i & A_{12}^i & A_{13}^i \\ A_{21}^i & A_{22}^i & A_{23}^i \\ A_{31}^i & A_{32}^i & A_{33}^i \end{bmatrix}, \\ B &= \sum_i \begin{bmatrix} B_{11}^i & B_{12}^i & B_{13}^i \\ B_{21}^i & B_{22}^i & B_{23}^i \\ B_{31}^i & B_{32}^i & B_{33}^i \end{bmatrix}, \\ C &= \sum_i \begin{bmatrix} C_{11}^i & C_{12}^i & C_{13}^i \\ C_{21}^i & C_{22}^i & C_{23}^i \\ C_{31}^i & C_{32}^i & C_{33}^i \end{bmatrix}, \\ F &= \sum_i \left\{ \int_{-1}^1 \int_{-1}^1 f(\Phi_i(\xi, \eta, \zeta)) |J| Q(\xi, \eta) d\xi d\eta \right. \\ &\quad + \int_{-1}^1 \int_{-1}^1 h(\Phi_i(\xi, \eta, \zeta)) Q(\xi, \eta) ds|_{\xi=\pm 1} \\ &\quad + \left. \int_{-1}^1 \int_{-1}^1 h(\Phi_i(\xi, \eta, \zeta)) Q(\xi, \eta) ds|_{\eta=\pm 1} \right\} \\ q &= \sum_i \left\{ \int_{-1}^1 \int_{-1}^1 h(\Phi_i(\xi, \eta, \zeta)) Q(\xi, \eta) ds|_{\xi=\pm 1} \right\}. \end{aligned}$$

3. The Solving of the Discretization System

Integrating by parts the left-hand side of Eq. (9), we have

$$\begin{aligned} &\int_{-1}^1 \Psi^T \{ [-AX_\xi]' + C^T A_\xi \\ &\quad - [CX]' + BX \} d\xi + [\Psi^T AX_\xi + \Psi^T CX]|_{-1}^1 \\ &= \int_{-1}^1 \Psi^T F d\xi + \Psi^T q|_{\xi=\pm 1}. \end{aligned} \quad (10)$$

Considering Ψ , which becomes zero at both edges ($\xi = \pm 1$) in Eq. (10), the following equation is obtained.

$$\begin{aligned} &\int_{-1}^1 \Psi^T \{ [-AX_\xi]' + C^T A_\xi - [CX]' + BX \} d\xi \\ &= \int_{-1}^1 \Psi^T F d\xi \end{aligned}$$

Since the above equation is established for all Ψ except at both edges, we have

$$-AX_{\xi\xi} - (A_\xi - C^T + C)X_\xi + (B - C_\xi)X = F. \quad (11)$$

Substituting Eq. (11) into Eq. (10), the boundary condition is obtained as

$$AX_\xi + CX = \pm q|_{\xi=\pm 1}. \quad (12)$$

These systems are solved effectively using a standard adaptive BVP-ODE solver, e.g. the software called COLSYS⁽⁴⁾. In this work, however, the solver program based on the B-Spline collocation method has been developed by us in order to examine the computational efficiency of MAL. In this solver, the solutions of ordinary differential equations are expressed as a linear combination of normalized B-Spline B_i as

$$X = \sum_i \alpha_i B_i.$$

Therefore, in the solver, BVP-ODE become a simultaneous linear equation used to find the coefficient α_i in the above equation. Finally, solving this simultaneous linear equation, we can obtain displacement and the first-order differential in the optional point on the curve, and using Eqs. (3) - (8), we can also obtained the strain and stress in the domain Ω .

4. Numerical Results and Discussion

In order to verify the calculation accuracy and efficiency of the MAL in the 3D elasticity problem, the uniaxial tension problem was analyzed using the infinite solid model with a hole. In the hole geometry, the radial length in the x and y directions was fixed at 1 and that of z was varied from 0.5 to 2.0. In this model, the length of the edge in the cube was fixed at 20 which can be regarded as an infinite solid. The symmetrical shape of this model enables us to consider only one-eighth of the model for the MAL and FEM analysis, as shown in Fig. 1. In this MAL analysis, since the parabolic shape function Q in Eqs. (3) - (8) was used, each element was constituted of 8 lines. In FEM analysis, we used MSC. Nastran and employed a 20 nodal points isoparametric element. In this analysis, the symmetry planes are restricted in the corresponding direction, and a uniformly distributed load, 98.1 MPa, was given at the top surface in the z direction. Young's modulus used in this analysis was 206 GPa and Poisson ratio was 0.3. The vector processor SX-4 of the Yamaguchi University integrated information processing center was used in the FEM analysis by MSC. Nastran. Since the C++ compiler used by the MAL program has not been improved in SX-4, the computer equipped with an Alpha 21164A 433 MHz processor was used in the present MAL analysis. Figures 2 and 3 show σ_z distributions on the curve AB calculated by MAL and FEM in the case of $a=1.0$. Although, the calculation results of the MAL and FEM generally agree well with the exact solution,

(a) MAL division number of lines: 40 (b) FEM division number of nodes: 15857

Fig. 1 Lines used for MAL analysis and the corresponding FEM mesh

Fig. 2 Comparison of the MAL results with exact solution

Fig. 3 Comparison of FEM results with exact solution

we can see MAL solutions close to exact solution than FEM ones in the vicinity of the stress concentration zone. Thus, the stress concentration factor is taken as representative of calculation results, as shown in Tables 1 and 2, and the accuracy of both methods is examined. The exact solution α_{exact} of the stress concentration factor of an infinite solid with a hole is 2.045, and the relative error is calculated by the following equation.

$$\text{err.} = \left| \frac{\alpha_{\text{Analysis}} - \alpha_{\text{exact}}}{\alpha_{\text{exact}}} \right|$$

In the case of MAL analysis, the stress concentration factor is 2.058 when the number of curves is 40, and 2.056 when the number of curves is 48. The relative error of the stress concentration factor is 0.592% even for 40 curves, and hence, it is clear that highly precise analysis can be carried out by MAL. In the case of FEM analysis, the stress concentration factor is greatly dependent on the number of nodes, and much smaller mesh elements are required to obtain the highly-precise results.

Next, the MAL analysis is compared with FEM analysis in terms of memory requirements and calculation time. The B-Spline collocation method is used

Table 1 Relative error in MAL analysis

Number of Lines	α_{Analysis}	err. [%]
40	2.058	0.592
48	2.056	0.518

Table 2 Relative error in FEM analysis

Number of Nodes	α_{Analysis}	err. [%]
529	1.956	4.37
1825	2.084	1.93
15857	2.062	0.811

Fig. 4 Relative error versus number of equations

in order to solve the simultaneous ordinary differential equations in the MAL analysis. The solutions of BVP-ODEs are expressed by the linear combination of normalized B-Spline. Using this solver, it can finally become a simultaneous linear equation of the coefficient of the linear combination. The array size of the simultaneous linear equation is determined by the product of the number of collocations and number of lines. Therefore, the size of the matrix of the simultaneous linear equation is primarily determines the memory requirements and calculation time in both methods. The relationship between the number of simultaneous linear equations and relative errors is shown in Fig. 4. Since MAL results are located the lower left-hand side compared with FEM in this figure, MAL analysis can be carried out with high accuracy compared with FEM, in the case of the same memory or computational time. If we want to obtain results with 1% relative error, MAL requires only 1/10 the number of equations and 1/100 the memory capacity in comparison with those of FEM. Since MAL and FEM calculations cannot be performed using the same computer in this work as described in the previous paragraph, a direct comparison of calculation time cannot be carried out. Therefore, the relative calculation time estimated from the ratio of processing speed of both computers is used in this

Fig. 5 Numerical models of one-half of the hourglass BGA solder joint

Fig. 6 Distribution of equivalent stress calculated by MAL

Fig. 7 Distributions of equivalent stress calculated by FEM

work. The relative time in the MAL analysis is less than 1/300 that of the FEM analysis. From the above results, it is confirmed that the MAL is a very effective method.

As an example of application to an engineering problem, a 3D elastic stress analysis of the ball grid array (BGA) model, a method used for obtaining solder joints of electronic component, was performed. Figure 5 shows the model used in this analysis. The bottom of the solder was restricted in the x , y and z directions and the top surface was restricted in the y ,

Fig. 8 Relative error versus number of equations

z directions. Then, an enforcement displacement $\delta = +0.01$ mm was introduced on the top surface in the x axial direction. Young's modulus used in this analysis was 35 GPa and the Poisson ratio was 0.3. Actually, the plastic deformation may occur in the solder material when enforcement displacement of $\delta = +0.01$ mm is given. However, the accuracy of elastic analysis is very important in this case, because nonlinear analysis based on the incremental theory essentially becomes the addition of the elasticity solution in the minute strain increment. Therefore, the perfect elasticity of the solder material was assumed in the computation, and the consideration of accuracy and comparison with FEM were performed. The number of lines was varied among 27, 41 or 55 in the MAL, and the number of nodes was among 2 910, 7 238 or 11 190 in the FEM. MSC. Nastran with a 20 node isoparametric element was used as in the previous problem. Figures 6 and 7 show the equivalent stress distribution along the curve AB for the MAL and FEM, respectively. Computational results by the MAL are independent of the number of lines, and the results in the case of 27 lines are satisfactory as regards the accuracy in this problem. In FEM, however, equivalent stresses depend largely on the number of nodes at the top and bottom faces of the solder. The relationship between the number of equations and relative errors of equivalent stress at $z = 1.12$ mm is examined. The MAL solution using 55 lines is taken as a reference value, because the exact solution to this problem does not exist. From Fig. 8, it is seen that the MAL solution has high accuracy compared with the FEM one in spite of the fact that the number of equations of the MAL is less than that of FEM. Therefore, it is confirmed that the MAL is an effective method for three dimensional elastoplastic stress analysis.

5. Conclusions

A 3D elastic MAL program was developed, and the effectiveness of the proposed method was

examined in comparison with the exact solutions or FEM results. As a result, the high performance of MAL on the effectiveness and accuracy was confirmed through some typical problems. Furthermore, the MAL is easy to combine with CAD software, since the MAL may divide the object domain into arbitrary curves. The MAL has potential for application to the nonlinear problem, because the theory of the MAL is based on the principle of virtual work. Therefore, the MAL will be applied to the singularity problem and nonlinear problem in the future.

References

- (1) Kaminishi, K. and Miyoshi, T., Proc. HERMIS '94, (1994), pp. 236-245.
- (2) Kaminishi, K. and Miyoshi, T., Proc. ICES '95, (1995), pp. 1298-1303.
- (3) Kaminishi, K., Trans. Jpn. Soc. Mech. Eng., (in Japanese), Vol. 61, No. 592, A (1995), pp. 2677-2683.
- (4) Ascher, U., Christiansen, J. and Russell, R.D., ACM Trans. Math. Software, Vol. 7 (1981), pp. 209-222.