A Higher Order Deformation Theory of Orthotropic Beams*

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If a beam is subjected to impact load or distributed load locally, the height of the cross section in the beam directly under the load is deformed. In order to estimate accurately the deformation of the cross section height and the transversely normal and shear stress components, a consistent higher order deformation theory of orthotropic beams is proposed. In order to verify the validity and effectiveness of the proposed theory, the analytical solution of a simple cantilevered beam problem is obtained and compared with the existed elastic solution. It is found that the displacement and stress components are identical to the corresponding solution of the Airy stress function in elasticity theory, and the deformation of the cross section height and the transversely normal stress are also estimated reasonably.

Key Words : Higher Order Beam Theory, Orthotropic Material, Consistent Condition, Normal Deformation, Shear Deformation

1. Introduction

In general, beam theories are classified into the Euler-Bernoulli and the Timoshenko beam⁽¹⁾. The Euler-Bernoulli beam theory is the simplest one neglecting shear deformation, and applied to a beam having a length that is much greater than its height in engineering practice. The Timoshenko beam theory is referred to as "shear beam theory", and used to a beam accounting for bending as well as shear deformation in case that the height of the cross section is comparatively large⁽²⁾.

These theories have been widely used for design and analysis of a beam made of an isotropic material such as steel. But it is impossible to estimate the contraction or the stretching of the cross section height since these theories all together neglect the deformation of the cross section.

If a beam is subjected to impact load or distributed load locally, the failure of the beam is caused by the excessive bending stress. But in reality, since the cross section directly under the load is deformed and the transversely normal stress is developed, that cross section will be firstly failed before the failure of the

beam is occurred. Also recently, since a beam made of an orthotropic material such as composite material is often used, a theoretical analysis method is urgently required for an orthotropic beam⁽³⁾.

In this paper, therefore, a consistent higher order deformation theory of an orthotropic beam is proposed, which accounts for the deformation of the cross section height as well as the transversely normal and shear stresses. The validity and effectiveness of the proposed theory is verified by a numerical example, which the exact solution can be obtained and compared with the existed elastic solution.

2. Consistent Higher Order Beam Theory

2.1 Displacement

The beam shown in Fig.1 is subjected to the arbitrary distributed load $q(x)$. The beam of length l has a rectangular cross section of width b and height h .

We assume that the longitudinal displacement u_x is linear in y , and the vertical displacement u_y is parabolic in y as the following⁽²⁾.

$$u_x(x, y) = u(x) - y\theta(x) \quad (1)$$

$$u_y(x, y) = v(x) + y\alpha(x) + (C + y^2)\beta(x) \quad (2)$$

In Eq. (2), C is an arbitrary constant, α and β are arbitrary parameters and assumed to be a function of x . θ is the rotation angle of the cross section. $u(x)$ and $v(x)$ in Eqs. (1) and (2) are not the displacements in the x and y direction at the neutral axis,

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Fig. 1 Beam configuration and loading state

respectively. We assume that $u(x)$ and $v(x)$ are the average displacements in each direction defined by the following.

$$u(x) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u_x(x, y) dy \quad (3)$$

$$v(x) = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} u_y(x, y) dy \quad (4)$$

Then, Eqs.(1) and (3) are automatically satisfied. And substituting Eq.(2) into Eq.(4), C is obtained by

$$C = -\frac{h^2}{12} \quad (5)$$

2.2 Strain and stress

The strain components are obtained from Eqs.(1), (2) and (5).

$$\epsilon_x = \frac{du}{dx} - y \frac{d\theta}{dx} \quad (6.a)$$

$$\epsilon_y = \alpha(x) + 2y\beta(x) \quad (6.b)$$

$$\gamma_{xy} = y \frac{d\alpha}{dx} + \left(y^2 - \frac{h^2}{12}\right) \frac{d\beta}{dx} + \frac{dv}{dx} - \theta(x) \quad (6.c)$$

According to the characteristics of orthogonal materials, the stress components are obtained as the followings.

$$\sigma_x = c_{11}\epsilon_x + c_{12}\epsilon_y \quad (7.a)$$

$$\sigma_y = c_{12}\epsilon_x + c_{22}\epsilon_y \quad (7.b)$$

$$\tau_{xy} = c_{33}\gamma_{xy} \quad (7.c)$$

where

$$\begin{aligned} c_{11} &= \frac{E_{11}}{1 - \nu_{12}\nu_{21}}, & c_{22} &= \frac{E_{22}}{1 - \nu_{12}\nu_{21}} \\ c_{12} &= c_{22}\nu_{12}, & c_{33} &= G_{12} \\ E_{11}\nu_{21} &= E_{22}\nu_{12} \end{aligned} \quad (8)$$

In Eq.(8), E_{11} and E_{22} are known as Young's modulus in each direction. ν_{12} and ν_{21} are known as Poisson's ratio. G_{12} is known as the shear modulus.

2.3 Consistent condition

The shear strain γ_{xy} of Eq.(6.c) can be expressed by the sum of the following two equations.

$$\frac{\partial u_x}{\partial y} = -\theta(x) \quad (9.a)$$

$$\frac{\partial u_y}{\partial x} = \frac{dv}{dx} + y \frac{d\alpha}{dx} + \left(y^2 - \frac{h^2}{12}\right) \frac{d\beta}{dx} \quad (9.b)$$

Equation(9.a), the change rate of the displacement in the x direction with respect to the y direction, is constant in y . Whereas, Eq.(9.b), the change rate of the displacement in the y direction with respect to the x direction, is expressed by a parabolic function of y .

Thus, since the distribution type of the shear strain can be changed according to load direction, a consistent condition should be given to Eq.(9.a) in order to ensure the independence of it⁽⁴⁾.

$$\frac{\partial u_x^*}{\partial y} = \sum_{n=0}^2 a_n y^n \quad (10)$$

After the modified shear strain γ_{xy}^* is obtained using Eqs.(9.b) and (10), the coefficient a_n should be determined in order to make it satisfying the following conditions.

$$\gamma_{xy}^* \Big|_{y=\pm \frac{h}{2}} = 0 \quad (11.a)$$

$$\min \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} (\gamma_{xy}^* - \gamma_{xy})^2 dy \right) \quad (11.b)$$

Equation(11.a) denotes the condition that shear stress at the top and bottom surface should be zero. And Eq.(11.b) means the least-squares procedure with respect to the coefficient a_n to minimize the sum of the squares of the differences between the modified shear strain γ_{xy}^* and the shear strain γ_{xy} over the section height. If the coefficient, a_n is obtained using Eqs.(11.a) and (11.b), the modified shear strain γ_{xy}^* is expressed by

$$\gamma_{xy}^* = \frac{6}{h^2} \left(y^2 - \frac{h^2}{4} \right) \left(\theta - \frac{dv}{dx} \right) \quad (12)$$

Since Eq.(12) denotes that the distribution of the modified shear strain γ_{xy}^* is parabolic in y , it is coincide with the concept of mechanics of materials⁽⁵⁾.

Meanwhile, the change rate of stress components in the y direction of the equilibrium equations for neglecting body force can be expressed by the followings.

$$\frac{\partial \tau_{xy}^*}{\partial x} = c_{33} \frac{6}{h^2} \left(y^2 - \frac{h^2}{4} \right) \frac{d}{dx} \left(\theta - \frac{dv}{dx} \right) \quad (13.a)$$

$$\frac{\partial \sigma_y}{\partial y} = \left(-c_{12} \frac{d\theta}{dx} + 2c_{22}\beta \right) \quad (13.b)$$

Equation(13.a), the change rate of the shear stress in the x direction, is expressed by a parabolic function of y . Whereas, Eq.(13.b), the change rate of the normal stress in the y direction, is constant in y . Thus, since the equilibrium equations can be changed due to the distribution of the stress components, a consistent condition should be given by the following⁽⁴⁾.

$$\epsilon_y^* = \sum_{n=0}^3 b_n y^n \quad (14)$$

Equation(14) is assumed from the fact that the normal stress σ_y is affected the most by the normal strain ϵ_y . If Eq.(14) is differentiated with respect to y , a parabolic function of y can be obtained like Eq.(13.a). The coefficient b_n should be determined in order to make it satisfying the following conditions.

$$\left[\frac{\partial \tau_{xy}^*}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} \right]_{y=\pm \frac{h}{2}} = 0 \quad (15.a)$$

$$\min\left(\int_{-\frac{h}{2}}^{\frac{h}{2}}(\varepsilon_y^* - \varepsilon_y)^2 dy\right) \quad (15.b)$$

Equation(15.a) denotes the condition that the equilibrium condition at the top and bottom surface should be satisfied. And Eq.(15.b) means the least-squares procedure with respect to the coefficient b_n to minimize the sum of the squares of the differences between the modified normal strain ε_y^* and the normal strain ε_y over the section height. If the coefficient, b_n of Eq. (14) is obtained using Eqs.(15.a) and (15.b), the modified normal strain ε_y^* is expressed by

$$\varepsilon_y^* = \alpha + \left(\frac{5}{2}\beta - \frac{1}{4}v_{12}\frac{d\theta}{dx}\right)y + \frac{5}{3h^2}\left(v_{12}\frac{d\theta}{dx} - 2\beta\right)y^3 \quad (16)$$

Equation(16) means the distribution of the modified normal strain ε_y^* , and is coincide with the distribution of elasticity theory⁽⁵⁾.

2.4 Parameters, α and β

The normal stress σ_y^* obtained from Eqs.(16) and (6.a) under the distributed load $q(x)$ should be satisfied for the following conditions.

$$\sigma_y^* \Big|_{y=\frac{h}{2}} = \frac{q(x)}{b} \quad (17.a)$$

$$\sigma_y^* \Big|_{y=-\frac{h}{2}} = 0 \quad (17.b)$$

Using Eqs.(17.a) and (17.b), α and β are expressed by

$$\alpha = \frac{q}{2bc_{22}} - v_{12}\frac{du}{dx} \quad (18.a)$$

$$\beta = \frac{3q}{5bh c_{22}} + \frac{1}{2}v_{12}\frac{d\theta}{dx} \quad (18.b)$$

Parameters, α and β above were introduced in Eq. (2) for assuming the vertical displacement u_y . These are functions of the distributed load $q(x)$ and Poisson's ratio. Especially, it is known that α and β vary due to the longitudinal displacement u and the rotation angle of the cross section θ , respectively.

2.5 Stress resultants and governing differential equations

The modified strain components ε_y^* and γ_{xy}^* by the consistent conditions are replace by ε_y and γ_{xy} and then using the parameters α and β of Eqs.(18.a) and (18.b), the stress components of Eq. (7) are expressed by the followings.

$$\sigma_x = E_{11}\left(\frac{du}{dx} - y\frac{d\theta}{dx}\right) + v_{12}\frac{q}{2b}\left[1 + 3\left(\frac{y}{h}\right) - 4\left(\frac{y}{h}\right)^3\right] \quad (19.a)$$

$$\sigma_y = c_{12}\left(\frac{du}{dx} - y\frac{d\theta}{dx}\right) + \frac{q}{2b}\left[1 + 3\left(\frac{y}{h}\right) - 4\left(\frac{y}{h}\right)^3\right] \quad (19.b)$$

$$\tau_{xy} = \frac{6}{h^2}G_{12}\left(y^2 - \frac{h^2}{4}\right)\left(\theta - \frac{dv}{dx}\right) \quad (19.c)$$

And the stress resultants become

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x(bdy) = bh\left[E_{11}\frac{du}{dx} + \frac{1}{2}v_{12}\left(\frac{q}{b}\right)\right] \quad (20.a)$$

$$N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y(bdy) = \frac{h}{2}q \quad (20.b)$$

$$Q = \int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy}(bdy) = bhG_{12}\left(\frac{dv}{dx} - \theta\right) \quad (20.c)$$

$$M_z = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_x(-y)(bdy) = \frac{bh^3}{12}\left[E_{11}\frac{d\theta}{dx} - \frac{6}{5bh}v_{12}q\right] \quad (20.d)$$

It is found that N_x expressed by Eq.(20.a) includes the Poisson's effect due to the distributed load $q(x)$ in the y direction. And N_y expressed by Eq.(20.b) equals to area multiplied by the average distributed load $q(x)$.

In addition, $M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_y(-y)(bdy) = -\frac{q}{10}h^2$ and $M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xy}(-y)(bdy) = 0$ are also obtained.

The equilibrium equations for neglecting body force become

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \quad (21.a)$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \quad (21.b)$$

If the equilibrium equations above are integrated over the cross section, the equilibrium equations for the stress resultants are obtained by the followings.

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}\right)(bdy) = 0; \frac{dN_x}{dx} = 0 \quad (22.a)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y}\right)(-y)(bdy) = 0; \frac{dM_z}{dx} + Q = 0 \quad (22.b)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y}\right)(bdy) = 0; \frac{dQ}{dx} + q = 0 \quad (22.c)$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y}\right)(-y)(bdy) = 0; \frac{dM_{xy}}{dx} + N_y - \frac{h}{2}q = 0 \quad (22.d)$$

Substituting Eq.(20) into Eq.(22), Eq.(22.d) is automatically satisfied. Thus, the following three governing differential equations are obtained.

$$\frac{d}{dx}\left[A\left(E_{11}\frac{du}{dx} + \frac{1}{2}v_{12}\frac{q}{b}\right)\right] = 0 \quad (23.a)$$

$$\frac{d}{dx}\left[I\left(E_{11}\frac{d\theta}{dx} - \frac{6}{5}\frac{v_{12}}{A}q\right)\right] + AG_{12}\left(\frac{dv}{dx} - \theta\right) = 0 \quad (23.b)$$

$$\frac{d}{dx}\left[AG_{11}\left(\frac{dv}{dx} - \theta\right)\right] + q = 0 \quad (23.c)$$

$$\text{where } I = \frac{bh^3}{12}, A = bh$$

Equation(23.a) denotes that the longitudinal displacement u varies with Poisson's ratio v_{12} due to the distributed load $q(x)$ in the y direction. If v_{12} is zero, Eqs.(23.b) and (23.c) become identical with the governing differential equations of the Timoshenko's beam^{(1),(2)}.

3. Numerical Example

In order to verify the validity and effectiveness of the suggested consistent higher order deformation theory of an orthotropic beam expressed by Eq. (23), it is applied to a cantilevered beam, and then the solution is compared with that of elasticity theory.

A cantilevered beam is subjected to uniform distributed load q_0 . The beam of length l has a rectangular cross section of width b and height h . The solution of the governing differential equation (23) for the beam is obtained as follows. That is, the displacements at the neutral axis for $\xi = x/l$ become

$$u = -\frac{v_{12}q_0l}{2bE_{11}}\xi \quad (24.a)$$

$$v = \frac{q_0l^4}{2bh^3E_{11}}\left[(\xi^4 - 4\xi^3 + 6\xi^2) + \frac{6}{5}v_{12}\left(\frac{h}{l}\right)^2\xi^2\right] + \frac{q_0l^2}{2bhG_{12}}(-\xi^2 + 2\xi) \quad (24.b)$$

$$\theta = \frac{q_0l^3}{bh^3E_{11}}\left[2\xi(\xi^2 - 3\xi + 3) + \frac{6}{5}\left(\frac{h}{l}\right)^2v_{12}\xi\right] \quad (24.c)$$

The solution above is identical to the corresponding result of the Euler-Bernoulli's beam theory⁽¹⁾ if $E_{11} = E$ and $v_{12} = 0$. u in Eq. (24.a) means the extension of the neutral axis and is in proportional to the uniformly distributed load q_0 . Using these equations the stress components are obtained as follows.

$$\sigma_x = -\frac{3q_0}{bh^2}l^2(1-\xi)^2\eta - \frac{v_{12}}{4b}q_0\eta^3 + \frac{3v_{12}}{20b}q_0\eta \quad (25.a)$$

$$\sigma_y = \frac{q_0}{4b}(2 + 3\eta - \eta^3) \quad (25.b)$$

$$\tau_{xy} = \frac{3q_0l}{2bh}(1-\xi)(1-\eta^2) \quad (25.c)$$

$$\text{where } \eta = \frac{2y}{h}$$

Equation (25.a) denotes the bending stress. The first term is the bending stress caused by the bending moment and identical with that obtained by the flexure formula in mechanics of materials⁽¹⁾. The second and third terms denotes the stress caused by the Poisson's effect due to the normal strain in the y direction. Equation (25.b) is the transversely normal stress of the cross section and impossible to obtain by general beam theories⁽⁵⁾. It is identical with the solution of the Airy stress function in elasticity theory, and its distribution is cubic in y .

The shear stress in Eq. (25.c) is identical with that obtained by the shear stress formula in mechanics of materials⁽¹⁾.

The deformation of the cross section height is dependent of the parameters α and β as we assumed in Eqs. (1) and (2). They are obtained as follows.

$$\alpha = \frac{q_0}{2bE_{22}} \quad (26.a)$$

$$\beta = \frac{3q_0l^2v_{12}}{bh^3E_{11}}(1-\xi)^2 + \frac{3q_0}{5bhE_{22}} \quad (26.b)$$

α in Eq. (26.a) is a constant regardless of x , but β in Eq. (26.b) is parabolic in x .

In order to obtain the stretching of the height of the cross section, the average normal strain in the y direction is defined by

$$\bar{\epsilon}_y = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \epsilon_y dy \quad (27)$$

Then, the deformation of the cross section height is obtained by

$$\Delta h = \frac{q_0h}{2bE_{22}} \quad (28)$$

4. Conclusion

In order to estimate the deformation of the cross section height and the transversely normal stress when a beam is subjected to impact load or distributed load locally, a consistent higher order deformation theory of orthotropic beams is proposed. And the validity and effectiveness of the proposed theory is verified by a numerical example, which the exact solution can be obtained and compared with the existed elastic solution. As the result of this study, the following conclusions are obtained.

- (1) The longitudinal stress σ_x is expressed by the sum of bending stress and Poisson's effect of the transversely normal stress σ_y .
- (2) The transversely normal and shear stress components obtained from the proposed theory are identical with those of elasticity theory.
- (3) The deformation of the cross section height can be reasonably evaluated.

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