Integral Equations for a 3D Crack in a Fluid-Saturated Poroelastic Infinite Space of Transversely Isotropic Permeability* (Case of Anisotropy Axis Perpendicular to Crack Face)

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A pair of integral equations are derived for a non-uniformly pressurized vertical planar crack of arbitrary shape in a fluid-saturated, poroelastic, infinite space of transversely isotropic permeability with its anisotropy axis perpendicular to the crack surface, by using the fundamental solutions obtained in a previous paper and the concept of a dislocation segment. This anisotropy assumption is more realistic for many oil reservoirs. The equations obtained relate normal tractions and fluid pressure on the crack faces to crack opening gradients and fluid injection rate per unit fracture area and include the known integral equations for an isotropic permeability as a limiting case. These integral equations are intended to be implanted in a 3D hydraulic fracturing simulator.

Key Words : Crack, Fracture Mechanics, Anisotropy, Integral Equation, Poroelastic Solid, Permeability, Petroleum, Hydraulic Fracturing

1. Introduction

In the oil/gas production, one of the reservoir stimulation methods is hydraulic fracturing, which is intended to enhance the oil productivity. This is the method that they artificially make up a large crack in an oil reservoir by pumping pressurized fluid (frac fluid) with propping agents such as sand or sintered bauxite through an oil/gas production well. In some cases, the crack is several hundred of meters long and dozens of meters high.

The hydraulic fracturing treatment has been performed for many wells and costs almost 50% of the well completion expenses for the case of large-scale fracturing. Therefore, it is important to understand

the whole process of fracturing. So, hydraulic fracturing simulators have been developed for the best program of frac fluid pumping and the best fracture treatment design and furthermore for the reduction in the whole cost.

Nowadays, the three-dimensional simulators which can simulate propagation of a fracture of arbitrary shape for the whole process are being developed in some countries, especially in the US, by institutes sponsored by the oil majors, oil technology service companies, some national laboratories and some universities. In Japan, the Technology Research Center, Japan National Oil Corporation, has been developing a full 3D hydraulic fracturing simulator, which can treat a curved fracture of arbitrary shape.

Among these simulators, TerraFrac™ of Terra-Tek, Co. Ltd. seems to lead the development. Development of this simulator started in the late 1970's and some important papers concerning the modeling of a hydraulic fracture, derivation of the governing integral equations and discretization of them were published from 1979 to 1982. See Ref.(1) for example. After that, various aspects of improvement were made and new functions were added to it.

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Considering the oil or gas reservoir as a fluid-saturated poroelastic solid, Kurashige and Clifton^{(2),(3)} derived, for that simulator, coupled integral equations which relate the crack opening and leak-off flux from the crack surfaces to the total traction and pore fluid pressure at the crack face. The method of derivation is by superposition of dislocation segment solutions. The derived integral equations have the singularity lower by unity than others like Clifton and Abou-Sayed⁽¹⁾, which results in the easiness in numerical calculation. Reference (4) introduced some approximation in the obtained integral equations and implanted them in the simulator; the simulation results made clear some effect of fluid diffusion around the fracture.

Following the approach by Kurashige and Clifton^{(2),(3)}, the pioneering integral equation given by Clifton and Abou-Sayed⁽¹⁾ was extended to various cases⁽⁵⁾⁻⁽⁷⁾; as a recent example, Kurashige and Clifton⁽⁶⁾ obtained the coupled integral equations for the case where a reservoir of infinite poroelastic space has zero permeability in the vertical direction. These integral equations were derived from a viewpoint different from that in the present paper; the real oil or gas reservoir is sandwiched by two impermeable formations so that oil or fracturing fluid flows predominantly in the horizontal direction for large-scale fracturing if the fracture extends with its height as large as the reservoir thickness. Regarding the reservoir as a poroelastic solid of zero permeability in one direction is an excellent model, because these equations are applicable to a reservoir of finite thickness although they treat the reservoir as an infinite poroelastic body.

Anisotropy in the permeability of an oil/gas reservoir is in general governed not only by the intrinsic anisotropy of reservoir rock but also by the difference between three principal tectonic stresses. Thus, the permeability is anisotropic in many reservoirs. It is reported that the permeability anisotropy in the horizontal plane is as much as 100:1. Anisotropy from 2:1 to 3:1 is common, especially in naturally fissured media. The maximum permeability is also reported to be in the direction normal to the minimum tectonic stress⁽⁸⁾.

However, the integral equations which take into account the anisotropy in permeability had not been obtained yet. So, we started the research which was intended to do so; that is, to derive coupled integral equations of a vertical hydraulic fracture (a plane crack of arbitrary shape) in a fluid-saturated poroelastic solid of isotropic elasticity but anisotropic permeability. First of all, in Ref.(9) with its correction in Ref.(10), we derived fundamental solutions for a fluid saturated poroelastic infinite space of trans-

versely isotropic permeability with its anisotropy axis in the vertical direction. Next, applying these fundamental solutions, we obtained the integral equations for a vertical crack with the anisotropy axis in the crack plane⁽¹¹⁾. Because of the introduction of anisotropy in addition to the poroelasticity introduced in Refs.(2), (3) and (6), the derivation of the equations was somewhat complicated. In spite of this complicated procedure, we could successfully obtain the more generalized integral equations, which tend to the solutions given in Ref.(2) if $\epsilon \rightarrow 1$ and with those given in Ref.(6) if $\epsilon \rightarrow 0$, where $\epsilon = \kappa'/\kappa$ with κ' and κ being the permeability in the anisotropy axis direction and in the transverse plane, respectively. The derivation method followed the approach taken by Ref.(6).

In the present report, we derive integral equations for a non-uniformly pressurized, vertical crack of arbitrary shape in a poroelastic infinite solid having the transversely isotropic permeability with the anisotropy axis perpendicular to the crack plane. In deriving the solutions, the fundamental solutions⁽⁹⁾ are here used after being rewritten by exchanging the coordinate axes.

2. Modeling of a Hydraulic Fracture

We shall take the y axis in the vertical direction and the crack face on the x - y ($z=0$) plane (Fig.1). We shall model a hydraulic fracture in an oil/gas reservoir as follows:

- (1) Take the oil/gas reservoir as a fluid-saturated poroelastic infinite body.
- (2) The reservoir is elastically isotropic but hydraulically anisotropic. This anisotropy is of transverse isotropy with its anisotropy axis being normal to the crack plane, that is, the permeability in the z direction κ' is different from those in the x and y directions κ .
- (3) The hydraulic fracture on the $z=0$ plane is of

Fig. 1 3D hydraulic fracture model

arbitrary shape.

For this model of a hydraulic fracture, we will obtain the integral equations which relate total stresses and pore fluid pressure acting on the crack faces to the crack opening displacement and fluid supply per unit time per unit area of the crack surfaces.

3. Method of Integral Equation Derivation

Figure 2 depicts the flow of analysis in deriving the integral equations for the hydraulic fracture. To obtain the final solutions, let us consider the following three kinds of solutions (*d*), (*f*) and (*s*): Solution (*d*) is the solution when a closed dislocation loop bounding a surface S_c suddenly appears in the poroelastic infinite space. Solution (*f*) is that for which an instantaneous point force $P_j \delta(x) \delta(y) \delta(z) \delta(t)$ acts at the origin in that space, where $\delta(\cdot)$ is the Dirac delta function and P_j is an intensity of the point force in the x_j direction. Solution (*s*) is the solution for an instantaneous point fluid source given by $Q_0 \delta(x) \delta(y) \delta(z) \delta(t)$, where Q_0 is the intensity of the source.

Here, let us use these three solutions in the reciprocal theorem for a poroelastic medium⁽¹¹⁾; that is, use is made of Solutions (*d*) and (*f*) combined and Solutions (*d*) and (*s*) combined in the theorem in the Laplace transform space. Since Solutions (*d*) and (*s*) are known in Ref.(9), we can express transformed displacement $u_j^*(\mathbf{r}')$ and pore fluid pressure $p^*(\mathbf{r}')$ in terms of integrals over S_c including the Burgers vector $b_j^*(\mathbf{r})$ and jump in fluid displacement $\Delta Q^*(\mathbf{r})$

across S_c in the integrands. Here, the asterisks on the variables denote the Laplace transform and $\mathbf{r}$, $\mathbf{r}'$ are the position vectors in the three-dimensional space. The following expressions are obtained:

$$P_j u_j^*(\mathbf{r}') = - \int_{S_c} [\sigma_{ij}^{*(f)}(\mathbf{r}, \mathbf{r}') b_j^*(\mathbf{r}) - p^{*(f)}(\mathbf{r}, \mathbf{r}') \Delta Q_i^*(\mathbf{r})] n_i(\mathbf{r}) dS, \quad (1)$$

$$Q_0 p^*(\mathbf{r}') = S \int_{S_c} [\sigma_{ij}^{*(s)}(\mathbf{r}, \mathbf{r}') b_j^*(\mathbf{r}) - p^{*(s)}(\mathbf{r}, \mathbf{r}') \Delta Q_i^*(\mathbf{r})] n_i(\mathbf{r}) dS, \quad (2)$$

where superscripts (*d*) on the left hand sides have been omitted. In these equations, $\sigma_{ij}^{*(f)}$, $\sigma_{ij}^{*(s)}$, $p^{*(f)}$, $p^{*(s)}$ are the total stresses and pore fluid pressure caused by the point force (*f*) and fluid source (*s*); n_j is the unit vector normal to the surface S_c . In what follows, we take $P_j P_j = 1$, $Q_0 = 1$.

Exchanging $\mathbf{r}$ and $\mathbf{r}'$ in Eqs.(1) and (2) and decomposing the resulting equations into the part stemming from the Burgers vector (*b*) and the part from the fluid supply (*q*), we obtain

$$u_m^*(\mathbf{r}) = u_m^{*(b)}(\mathbf{r}) + u_m^{*(q)}(\mathbf{r}), \quad (3)$$

$$p^*(\mathbf{r}) = p^{*(b)}(\mathbf{r}) + p^{*(q)}(\mathbf{r}). \quad (4)$$

If we denote the total stress and pore fluid pressure at $\mathbf{r}'$ due to the point force acting at $\mathbf{r}$ in the x_m direction by $\sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r})$, $p_{(m)}^*(\mathbf{r}', \mathbf{r})$ and those due to the point fluid source by $\sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r})$, $p_{(0)}^*(\mathbf{r}', \mathbf{r})$, we can rewrite the right hand sides of Eqs.(3) and (4) into

$$u_m^{*(b)}(\mathbf{r}) = - \int_{S_c} \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) b_j^*(\mathbf{r}') n'_i dS', \quad (5)$$

$$u_m^{*(q)}(\mathbf{r}) = - \frac{1}{S} \int_{S_c} p_{(0)}^*(\mathbf{r}', \mathbf{r}) \Delta q_n^*(\mathbf{r}') dS', \quad (6)$$

$$p^{*(b)}(\mathbf{r}) = s \int_{S_c} \sigma_{(0)ij}^*(\mathbf{r}', \mathbf{r}) b_j^*(\mathbf{r}') n'_i dS', \quad (7)$$

$$p^{*(q)}(\mathbf{r}) = \int_{S_c} p_{(0)}^*(\mathbf{r}', \mathbf{r}) \Delta q_n^*(\mathbf{r}') dS', \quad (8)$$

where s is the Laplace transformation parameter; Δq_n is the fluid volume flux through the crack surface into the poroelastic reservoir, which is related to ΔQ^* by

$$\Delta q_n^* = -s \Delta Q_j^* n_j. \quad (9)$$

In the following sections, we will separately derive the integral equations from Eqs.(5) and (7) which the Burgers vector contributes to and from Eqs.(6) and (8) which the fluid point supply does; those will be superposed after all.

4. Dislocation Segment Solutions and Crack Opening

We have to express the first part of the required integral equations in terms of the crack opening displacement. In what follows, we will show the procedure to do so. First of all, we differentiate the both sides of Eqs.(5) and (7). We obtain

$$\frac{\partial u_m^{*(b)}(\mathbf{r})}{\partial x_s} = \int_{S_c} b_i^*(\mathbf{r}') \frac{\partial}{\partial x_s} \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) n'_j dS', \quad (10)$$

Fig. 2 Flowchart for derivation of integral equations

$$\frac{\partial p^{*(b)}(\mathbf{r})}{\partial x_s} = -S \int_{sc} b_i^*(\mathbf{r}') \frac{\partial}{\partial x_s} \sigma_{(b)ij}^*(\mathbf{r}', \mathbf{r}) n_j dS', \quad (11)$$

where we have used the following identity :

$$\frac{\partial}{\partial x_s} \sigma_{(b)ij}^*(\mathbf{r}', \mathbf{r}) = -\frac{\partial}{\partial x_s'} \sigma_{(b)ij}^*(\mathbf{r}', \mathbf{r}) \quad (\beta = m, 0) \quad (12)$$

Next, adding the following equation

$$b_i^*(\mathbf{r}') \frac{\partial}{\partial x_j'} \sigma_{(b)ij}^*(\mathbf{r}', \mathbf{r}) n_s' = 0 \quad (\beta = m, 0) \quad (13)$$

which comes from the equilibrium equation, to the right hand sides of Eqs.(10) and (11) and applying the Stokes theorem to the resulting equations, we can convert the surface integrals in Eqs.(10) and (11) into the line integrals as

$$\frac{\partial u_m^{*(b)}(\mathbf{r})}{\partial x_s} = \varepsilon_{jsn} \oint b_i^* \sigma_{(m)ij}^*(\mathbf{r}', \mathbf{r}) dx_n', \quad (14)$$

$$\frac{\partial p^{*(b)}(\mathbf{r})}{\partial x_s} = -S \varepsilon_{jsn} \oint b_i^* \sigma_{(b)ij}^*(\mathbf{r}', \mathbf{r}) dx_n', \quad (15)$$

where, in order to obtain the dislocation segment solutions next, we have assumed that Burgers vector b_i^* is independent from its position. In the above, ε_{jsn} implies the permutation symbol.

As a next step, using Eqs.(14) and (15) in the strain-displacement relations and constitutive equations (see the governing equations in the previous paper⁽⁹⁾), we can obtain the total stress and the gradient of pore fluid pressure generated by a dislocation segment along line segment $\Delta x_n'$ with Burgers vector b_i^* . The obtained solutions are the general dislocation segment solutions. We reduce these general solutions to the total stress and pore fluid pressure on the $z=0$ plane, $\Delta \sigma_{zz}^{(b)}$ and $\Delta p^{(b)}$, due to the dislocation segment along the horizontal line segment $\Delta x'$ on the $z'=0$ plane with Burgers vector b_z^h after performing the Laplace inversion and integrating $\partial \Delta p^{(b)} / \partial z$ from $-\infty$ to zero with respect to z . The results are

$$\begin{aligned} \Delta \sigma_{zz}^{(b)h}(\mathbf{R}, t) + a \Delta p^{(b)h}(\mathbf{R}, t) \\ = -\frac{2G}{1-2\nu} \int_0^t b_z^h(\tau) [(1-\nu)(\sigma_{(z)zy}(\mathbf{R}', \mathbf{R}, t-\tau) \\ - \nu(\sigma_{(y)zz}(\mathbf{R}', \mathbf{R}, t-\tau))] d\tau \Delta x', \end{aligned} \quad (16)$$

$$\begin{aligned} \Delta p^{(b)h}(\mathbf{R}, t) = - \left[b_z^h(t) \int_{-\infty}^0 \sigma_{(0)zy}(\mathbf{R}', \mathbf{R}, 0) dz \right. \\ \left. + \int_0^t b_z^h(\tau) \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)zy}(\mathbf{R}', \mathbf{R}, t-\tau) dz d\tau \right] \Delta x', \end{aligned} \quad (17)$$

where $\mathbf{R}, \mathbf{R}'$ are the position vectors on the $z=0$ and $z'=0$ planes, respectively; superscript h on the right hand sides corresponds to the "horizontal" line segment.

In the similar manner, we can obtain the total stress and pore fluid pressure due to the dislocation segment along the line segment $\Delta y'$ with Burgers vector b_z^v . Here, superscript v implies the "vertical segment."

Fig. 3 Sign convention for dislocation segments $\Delta x'$ and $\Delta y'$ and crack opening $w(x', y')$

The Burgers vector associated with the horizontal and vertical segments can be related with crack opening displacement $w(x', y')$ as follows (see Fig. 3) :

$$b_z^h = -\frac{\partial w(x', y')}{\partial y'} \Delta y', \quad b_z^v = -\frac{\partial w(x', y')}{\partial x'} \Delta x', \quad (18)$$

where we have put minus signs on the right hand sides because we defined that b_z^h and b_z^v are positive when the upper and lower surfaces of a crack overlap.

Thus, by superposing the total stresses and pore fluid pressures corresponding to the horizontal and vertical dislocation segments, we can reach the integral equations associated with the Burgers vector :

$$\begin{aligned} \sigma_{zz}^{(b)}(\mathbf{R}, t) + a p^{(b)}(\mathbf{R}, t) \\ = -\frac{2G}{1-2\nu} \int_{sc} \int_0^t \frac{\partial w(\mathbf{R}', \tau)}{\partial x_\beta'} [(1-\nu) \\ \times (\sigma_{(z)z\beta}(\mathbf{R}', \mathbf{R}, t-\tau)) \\ - \nu(\sigma_{(\beta)zz}(\mathbf{R}', \mathbf{R}, t-\tau))] d\tau dS', \end{aligned} \quad (19)$$

$$\begin{aligned} p^{(b)}(\mathbf{R}, t) = \int_{sc} \left[\frac{\partial w(\mathbf{R}', t)}{\partial x_\beta'} \int_{-\infty}^0 \sigma_{(0)z\beta}(\mathbf{R}', \mathbf{R}, 0) dz \right. \\ \left. + \int_0^t \frac{\partial w(\mathbf{R}', t)}{\partial x_\beta'} \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)z\beta} \right. \\ \left. \times (\mathbf{R}', \mathbf{R}, t-\tau) dz d\tau \right] dS', \end{aligned} \quad (20)$$

where dS' is an area element on the crack surface given by $dS' = \Delta x' \Delta y'$; Greek β takes the values 1 and 2.

5. Total Stress and Pore Fluid Pressure due to Fluid Supply

Next, let us move on to the part of the integral equations due to the fluid supply. Since Eq.(8) is expressed in terms of the fluid flux from the crack surface $\Delta q_n^*(\mathbf{r}')$, we can use this equation as it is. With respect to Eq.(6), substituting it into the strain-displacement relations and constitutive equations

yields

$$\begin{aligned} & \sigma_{zz}^{*(q)}(\mathbf{r}) + \alpha p^{*(q)}(\mathbf{r}) \\ &= \frac{2G}{s} \int_{sc} \left[\frac{\partial p_{(z)}^*(\mathbf{r}', \mathbf{r})}{\partial z'} \right. \\ & \left. + \frac{\nu}{1-2\nu} \frac{\partial p_{(m)}^*(\mathbf{r}', \mathbf{r})}{\partial x'_m} \right] \Delta q_n(\mathbf{r}') dS'. \end{aligned} \quad (21)$$

6. Fundamental Solutions

We are seeking the integral equations for the case where the anisotropy axis of transversely isotropic permeability is perpendicular to the crack plane ($z=0$). In the following, after changing the coordinates like $y \rightarrow z$, $z \rightarrow x$, $x \rightarrow y$ in the fundamental solutions in Ref. (9), we present only the components of stress and

pressure which are necessary to derive the required integral equations. There, a subscript in parentheses shows the direction (x, y or z) of the point force acting in the poroelastic infinite space or the fluid point source when it is 0.

$$\sigma_{(z)zx}(\mathbf{R}, t) = xT(R, t), \quad \sigma_{(z)zy}(\mathbf{R}, t) = yT(R, t), \quad (22)$$

$$\sigma_{(x)zz}(\mathbf{R}, t) = xU(R, t), \quad \sigma_{(y)zz}(\mathbf{R}, t) = yU(R, t), \quad (23)$$

$$\sigma_{(0)zx}(\mathbf{r}, t) = xV(R, z, t), \quad \sigma_{(0)zy}(\mathbf{r}, t) = yV(R, z, t), \quad (24)$$

$$p_{(x)}(\mathbf{r}, t) = xX(R, z, t), \quad p_{(y)}(\mathbf{r}, t) = yX(R, z, t), \quad (25)$$

where

$$\begin{aligned} T(R, t) &= \frac{\alpha_3 c \varepsilon (1-\varepsilon)(4ct-R^2)}{32\pi(ct)^3 \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} + \frac{\varepsilon(1-3\varepsilon)\alpha_3 c}{16(ct)^2 \pi \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} + \frac{\delta(t)(1-2\nu_u)}{8\pi R^3(\nu_u-1)} \\ & - \frac{\alpha_3 c(1-\varepsilon)ct\varepsilon}{2\pi R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^5 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho \\ & - \frac{\alpha_3 c(1-3\varepsilon)}{4\pi R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^3 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho, \end{aligned} \quad (26)$$

$$\begin{aligned} U(R, t) &= \frac{\alpha_3 c \varepsilon (1-\varepsilon)(8ct-R^2)}{32(ct)^3 \pi \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} + \frac{\delta(t)(1-2\nu_u)}{8\pi(1-\nu_u)R^3} \\ & + \frac{\alpha_3 c(1+\varepsilon)}{4\pi R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^3 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho \\ & - \frac{\alpha_3 c(1-\varepsilon)ct\varepsilon}{2\pi R} \int_0^\infty \rho^5 J_1[R\rho] e^{-(1-\varepsilon)\rho^2 ct} (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho, \end{aligned} \quad (27)$$

$$V(R, z, t) = -\frac{G\alpha_0}{4\pi R\alpha_2} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^2 J_1[R\rho] D_1^-(z, t; \rho) d\rho, \quad (28)$$

$$\begin{aligned} X(R, z, t) &= \frac{\alpha_0}{4\pi\alpha_2} \left(\frac{\delta(t)}{(R^2+z^2)^{\frac{3}{2}}} - \frac{c\varepsilon}{4(ct)^2 \sqrt{ct\varepsilon\pi}} e^{-\frac{z^2}{4ct\varepsilon}} e^{-\frac{R^2}{4ct}} \right) \\ & - \frac{\alpha_0}{\alpha_2} \frac{c(1-\varepsilon)}{8\pi R} \int_0^\infty \rho^3 e^{-(1-\varepsilon)\rho^2 ct} J_1[R\rho] D_1^+(z, t; \rho) d\rho. \end{aligned} \quad (29)$$

Furthermore,

$$\begin{aligned} p_{(z)}(\mathbf{r}, t) &= \frac{\alpha_0 z}{4\pi\alpha_2} \left(\frac{\delta(t)}{(R^2+z^2)^{\frac{3}{2}}} - \frac{c}{4(tc)^2 \sqrt{ct\varepsilon\pi}} e^{-\frac{z^2}{4ct\varepsilon}} e^{-\frac{R^2}{4ct}} \right) \\ & - \frac{\alpha_0}{\alpha_2} \frac{c(1-\varepsilon)}{8\pi} \int_0^\infty \rho^3 e^{-(1-\varepsilon)\rho^2 ct} J_0[R\rho] D_1^-(z, t; \rho) d\rho, \end{aligned} \quad (30)$$

$$p_{(0)}(\mathbf{r}, t) = \frac{G}{8\pi\alpha_2\alpha} \frac{e^{-\frac{z^2}{4ct\varepsilon}} e^{-\frac{R^2}{4ct}}}{ct\sqrt{\varepsilon ct\pi}}. \quad (31)$$

In the above equations, $J_n[\cdot]$ denote the first kind Bessel function of order n and $\operatorname{erfc}[\cdot]$ the complementary error function; $R=|\mathbf{R}|$ and

$$D_1^\pm(z, t; \rho) = e^{-z\rho} \operatorname{erfc}\left[\frac{2ct\varepsilon\rho-z}{2\sqrt{ct\varepsilon}}\right] \pm e^{z\rho} \operatorname{erfc}\left[\frac{2ct\varepsilon\rho+z}{2\sqrt{ct\varepsilon}}\right]. \quad (32)$$

Furthermore, α is the Biot effective stress coefficient, B the Skempton pore pressure coefficient, G the shear modulus, ν and ν_u the Poisson ratios in the drained and undrained conditions. Using these moduli, α_j ($j=0, 1, 2, 3$) and c are given by

$$\begin{aligned} \alpha_0 &= \frac{1-2\nu}{2(1-\nu)}, \quad \alpha_1 = \frac{3(1-2\nu_u)}{2B(1+\nu_u)}, \quad \alpha_2 = \frac{3(1-2\nu)(1-\nu_u)}{2B(1+\nu_u)(1-\nu)}, \\ \alpha_3 &= \frac{\nu_u-\nu}{2(1-\nu_u)(1-\nu)}, \quad c = \frac{2GB^2(1+\nu_u)^2(1-\nu)}{9(\nu_u-\nu)(1-\nu_u)} \kappa. \end{aligned} \quad (33)$$

7. Calculation in Analysis Concerning Burgers Vector

Substituting Eq. (24) into Eq. (20), we can perform the integrations with respect to z as follows:

$$\int_{-\infty}^0 \sigma_{(0)zx}(\mathbf{r}, t) dz = xY(R, t), \quad \int_{-\infty}^0 \sigma_{(0)zy}(\mathbf{r}, t) dz = yY(R, t) \quad (34)$$

where

$$Y(R) = \frac{Ga_0}{2\pi R a_2} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho J_1[R\rho] (erfc[\sqrt{ct\varepsilon}\rho]) d\rho. \quad (35)$$

For the above integrations with respect to z , we followed the next procedure. With the help of Eqs. (28) and (32), we expressed the integrals with respect to z in Eq. (20) in terms of $D_1(z, t, \rho)$. Then, we rewrote this integrand into a differential form with respect to τ and exchanged the order of integrations with respect to z and τ . Finally, we obtained the result in the closed form. That is,

$$\begin{aligned} \int_{-\infty}^0 D_1(z, t; \rho) dz &= \int_{-\infty}^0 \int_{-\infty}^t \frac{\partial}{\partial \tau} \left(e^{-\rho z} erfc\left[\frac{2c\tau\varepsilon\rho - z}{2\sqrt{c\tau\varepsilon}}\right] - e^{\rho z} erfc\left[\frac{2c\tau\varepsilon\rho + z}{2\sqrt{c\tau\varepsilon}}\right] \right) d\tau dz \\ &= \int_{-\infty}^t \int_{-\infty}^0 \left(-\frac{c\varepsilon z}{c\varepsilon\sqrt{c\varepsilon\pi}} e^{-\frac{z^2}{4c\tau\varepsilon} - c\varepsilon\rho^2} \right) dz d\tau = -\frac{2}{\rho} erfc[\sqrt{c\varepsilon}\rho]. \end{aligned} \quad (36)$$

The integral with respect to ρ in Eqs. (24) with Eq. (28) can be evaluated when $t=0$. The evaluation resulted in

$$\int_{-\infty}^0 \sigma_{(0)zx}(\mathbf{r}, 0) dz = xZ(R, z, 0), \quad \int_{-\infty}^0 \sigma_{(0)zy}(\mathbf{r}, 0) dz = yZ(R, z, 0), \quad (37)$$

with

$$Z(R, z, 0) = \frac{Ga_0}{2\pi a_2} \frac{1}{R^3}. \quad (38)$$

Furthermore, the differentiations with respect to t in Eq. (20) can be carried out with the aid of Eqs. (34) as

$$\frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)zx}(\mathbf{r}, t) dz = xS(R, t), \quad \frac{\partial}{\partial t} \int_{-\infty}^0 \sigma_{(0)zy}(\mathbf{r}, t) dz = yS(R, t), \quad (39)$$

where

$$S(R) = -\frac{Ga_0}{8\pi a_2} \frac{c\varepsilon}{(ct)^2 \sqrt{c\varepsilon\pi}} e^{-\frac{R^2}{4ct}} - \frac{Ga_0}{2\pi a_2} \frac{c(1-\varepsilon)}{R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^3 J_1[R\rho] (erfc[\sqrt{ct\varepsilon}\rho]) d\rho. \quad (40)$$

In the above, many mathematical manipulations were done by use of an algebraic tool “Mathematica”⁽¹³⁾.

As a result, we can obtain the first pair of coupled integral equations which the Burgers vector contributes to, by substituting Eqs. (22), (23), (37) and (39) into Eqs. (19) and (20), substituting again $p^{(b)}(\mathbf{R}, t)$ given by Eq. (20) into Eq. (19) and rearranging the resulting equations.

8. Calculations in Analysis Concerning Fluid Supply

Let us consider the Laplace inversion of Eqs. (8) and (21). It is easy to inverse Eq. (8) in the following convolution form:

$$q^{(q)}(\mathbf{r}) = \int_{sc} \int_0^t p_{(0)}(\mathbf{r}', \mathbf{r}, t-\tau) \Delta q_n(\mathbf{r}', \tau) d\tau dS'. \quad (41)$$

If we substitute the fundamental solution $p_{(0)}(\mathbf{r}', \mathbf{r}, t-\tau)$ into this equation, we obtain the final result; this will not be shown here.

The Laplace inversion of Eq. (21) is in the form of double integrals; the first integral from zero to t with respect to τ is caused by the factor $1/s$, while the second one stems from the convolution and is from zero to τ with respect to ζ (see the first line in the below). In these double integrals, interchanging the order of integrations (resulting in the change in the range of integration) and then exchanging the notations of integral variables τ and ζ , we obtain the form of integrals in which the integral with respect to ζ is integrable because the unknown function $\Delta q_n(\zeta)$ has been expelled out from this integral (see the second line in the below). These are shown in the following:

$$\begin{aligned} \sigma_{zz}^{(q)}(\mathbf{R}, t) + ap^{(q)}(\mathbf{R}, t) &= 2G \int_{sc} \int_0^t \int_0^\tau \left(\frac{\partial p_{(z)}(\mathbf{r}', \mathbf{r}, \tau-\zeta)}{\partial z'} + \frac{\nu}{1-2\nu} \frac{\partial p_{(v)}(\mathbf{r}', \mathbf{r}, \tau-\zeta)}{\partial x'_i} \right) \Delta q_n(\mathbf{R}', \zeta) d\zeta d\tau dS' \\ &= 2G \int_{sc} \int_0^t \int_\tau^t \left(\frac{\partial p_{(z)}(\mathbf{r}', \mathbf{r}, \zeta-\tau)}{\partial z'} + \frac{\nu}{1-2\nu} \frac{\partial p_{(v)}(\mathbf{r}', \mathbf{r}, \zeta-\tau)}{\partial x'_i} \right) d\zeta \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \end{aligned} \quad (42)$$

To evaluate the integral with respect to ζ in the last line of Eq. (42), differentiating the fundamental solutions appearing in the integrand and substituting the resulting equations, we obtain

$$\begin{aligned} \sigma_{zz}^{(q)}(\mathbf{R}, t) + ap^{(q)}(\mathbf{R}, t) &= 2G \int_{sc} \int_0^t \int_\tau^t \left\{ \frac{a_0 \delta(\zeta-\tau)}{4\pi a_2} \frac{1}{R^3} + \frac{a_0}{a_2} \frac{c(1-\varepsilon)}{4\pi} \int_0^\infty \rho^4 e^{-(1-\varepsilon)\rho^2 c(\zeta-\tau)} J_0[R\rho] \left(erfc\left[\frac{2c(\zeta-\tau)\varepsilon\rho}{2\sqrt{c(\zeta-\tau)\varepsilon}}\right] \right) d\rho \right. \\ &\quad - \frac{((1-2\nu)(3-2\varepsilon)+3\nu)}{(1-2\nu)} \frac{c a_0}{4\pi a_2} \frac{e^{-\frac{R^2}{4c(\zeta-\tau)}}}{4(c(\zeta-\tau))^2 \sqrt{c(\zeta-\tau)\varepsilon\pi}} \\ &\quad \left. + \frac{((1-2\nu)(1-\varepsilon)+\nu)}{(1-2\nu)} \frac{c a_0}{4\pi a_2} \frac{e^{-\frac{R^2}{4c(\zeta-\tau)}} R^2}{8(c(\zeta-\tau))^3 \sqrt{c(\zeta-\tau)\varepsilon\pi}} \right\} d\zeta \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \end{aligned} \quad (43)$$

In the below, we will evaluate the integrals with respect to ζ term by term. With the first term in the braces having the Dirac delta function, it is easy to do so:

$$2G \int_{sc} \int_0^t \int_{\tau}^t \left(\frac{a_0 \delta(\zeta - \tau)}{4\pi a_2 R^3} \right) d\zeta \Delta q_n(\tau) d\tau dS' = 2G \int_{sc} \int_0^t \frac{a_0}{4\pi a_2 R^3} \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \quad (44)$$

Next, we integrate the third and last terms in the braces with respect to ζ ; the result is

$$\begin{aligned} & 2G \int_{sc} \int_0^t \int_{\tau}^t \left(-\frac{a_0}{a_2} \frac{c}{4\pi(1-2\nu)} \frac{e^{-\frac{R^2}{4c(\zeta-\tau)}}}{8((\zeta-\tau)c)^3 \sqrt{c(\zeta-\tau)} \varepsilon \pi} \right. \\ & \quad \times ((1-\nu)(6c(\zeta-\tau)-R^2) - \varepsilon(1-2\nu)(4c(\zeta-\tau)-R^2)) \Big) \Delta q_n(\mathbf{R}', \tau) d\zeta d\tau dS' \\ & = -\frac{2Ga_0}{a_2} \frac{1}{4\pi} \int_{sc} \int_0^t \left(\frac{\sqrt{\varepsilon} e^{-\frac{R^2}{4c(t-\tau)}}}{\sqrt{c(t-\tau)} \pi R^2} - \frac{(1-\nu)e^{-\frac{R^2}{4c(t-\tau)}}}{2c(t-\tau) \sqrt{c(t-\tau)} \varepsilon \pi (1-2\nu)} \right. \\ & \quad \left. + \frac{\sqrt{\varepsilon} e^{-\frac{R^2}{4c(t-\tau)}}}{2c(t-\tau) \sqrt{c(t-\tau)} \pi} - \frac{\sqrt{\varepsilon} \operatorname{erf}\left[\frac{R}{2\sqrt{c(t-\tau)}}\right]}{R^3} + \frac{\sqrt{\varepsilon}}{R^3} \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \end{aligned} \quad (45)$$

Last, we carry out the integration of the second term in the braces of the integrand in Eq.(43). We obtain

$$\begin{aligned} & 2G \int_{sc} \int_0^t \int_{\tau}^t \left(\frac{a_0}{a_2} \frac{c(1-\varepsilon)}{4\pi} \int_0^\infty \rho^4 e^{-(1-\varepsilon)\rho^2 c(\zeta-\tau)} J_0[R\rho] (\operatorname{erfc}[\sqrt{c(\zeta-\tau)} \varepsilon \rho]) d\rho \right) \Delta q_n(\mathbf{R}', \tau) d\zeta d\tau dS' \\ & = 2G \int_{sc} \int_0^t \left(\frac{a_0}{a_2} \frac{1}{4\pi} \left(-\frac{1}{R^3} + \frac{\sqrt{\varepsilon}}{R^3} + \frac{e^{-\frac{R^2}{4ct}} \sqrt{\varepsilon}}{\sqrt{ct} \pi R^2} + \frac{e^{-\frac{R^2}{4ct}} \sqrt{\varepsilon}}{2ct \sqrt{ct} \pi} - \frac{\operatorname{erf}\left[\frac{R}{2\sqrt{ct}}\right] \sqrt{\varepsilon}}{R^3} \right) \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS' \\ & + 2G \int_{sc} \int_0^t \left(\frac{a_0}{a_2} \frac{1}{4\pi} \int_0^\infty \rho^2 J_0[R\rho] (-e^{-(1-\varepsilon)\rho^2 c(t-\tau)} \operatorname{erfc}[\sqrt{c(t-\tau)} \varepsilon \rho]) d\rho \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS' \end{aligned} \quad (46)$$

In this last step, we put the above results in order and obtain for $\sigma_{zz}^{(q)}(\mathbf{R}, t) + \alpha p^{(q)}(\mathbf{R}, t)$

$$\begin{aligned} & \sigma_{zz}^{(q)}(\mathbf{R}, t) + \alpha p^{(q)}(\mathbf{R}, t) \\ & = 2G \int_{sc} \int_0^t \left(\frac{B(1+\nu_u)}{3(1-\nu_u)} \frac{1}{4\pi} \left(\frac{(1-\nu)}{2c(t-\tau) \sqrt{c(t-\tau)} \varepsilon \pi (1-2\nu)} e^{-\frac{R^2}{4c(t-\tau)}} \right) \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS' \\ & + 2G \int_{sc} \int_0^t \left(\frac{B(1+\nu_u)}{3(1-\nu_u)} \frac{1}{4\pi} \int_0^\infty \rho^2 J_0[R\rho] (-e^{-(1-\varepsilon)\rho^2 c(t-\tau)} \operatorname{erfc}[\sqrt{c(t-\tau)} \varepsilon \rho]) d\rho \right) \Delta q_n(\mathbf{R}', \tau) d\tau dS'. \end{aligned} \quad (47)$$

In the similar manner as in deriving the first part of the integral equations which the Burgers vector contributes to, we can obtain the second part of the integral equations which comes from the fluid supply, by substituting $p^{(q)}(\mathbf{R}, t)$ resulting from Eq.(41) into Eq.(47) and putting the resulting equations in order. These integral equations are not shown here because they appear in $h^{(1)}(\mathbf{R}, t)$, $h^{(2)}(\mathbf{R}, t)$ given in the next section.

9. Final Integral Equations

Superposing the two parts of the integral equations obtained in the previous sections, we obtain a required pair of the integral equations. These final integral equations are those for a vertical fracture of arbitrary shape in a fluid-saturated poroelastic infinite reservoir of transversely isotropic permeability. They relate the crack opening displacement $w(\mathbf{R}, t)$ and the fluid supply flux through the crack surface $\Delta q(\mathbf{R}, t)$ to the total stress $\sigma_{zz}(\mathbf{R}, t)$ and the pore fluid pressure $p(\mathbf{R}, t)$ acting on the crack surface. The results are

$$\begin{aligned} \sigma_{zz}(\mathbf{R}, t) &= \frac{G}{4\pi(1-\nu_u)} \int_{sc} \left\{ -\nabla' \left(\frac{1}{R} \right) \cdot \nabla' w(\mathbf{R}', t) \right. \\ & \quad \left. + \int_0^t [\mathbf{H}^{(1)}(\mathbf{R}' - \mathbf{R}, t - \tau) \cdot \nabla' w(\mathbf{R}', \tau) + h^{(1)}(\mathbf{R}' - \mathbf{R}, t - \tau) \Delta q_n(\mathbf{R}', \tau)] d\tau \right\} dS', \end{aligned} \quad (48)$$

$$\begin{aligned} p(\mathbf{R}, t) &= -\frac{GB(1+\nu_u)}{6\pi(1-\nu_u)} \int_{sc} \left\{ -\nabla' \left(\frac{1}{R} \right) \cdot \nabla' w(\mathbf{R}', t) \right. \\ & \quad \left. + \int_0^t [\mathbf{H}^{(2)}(\mathbf{R}' - \mathbf{R}, t - \tau) \cdot \nabla' w(\mathbf{R}', \tau) + h^{(2)}(\mathbf{R}' - \mathbf{R}, t - \tau) \Delta q_n(\mathbf{R}', \tau)] d\tau \right\} dS', \end{aligned} \quad (49)$$

with $R = \sqrt{(x' - x)^2 + (y' - y)^2}$, where $\mathbf{H}^{(1)}(\mathbf{R}, t)$, $\mathbf{H}^{(2)}(\mathbf{R}, t)$, $h^{(1)}(\mathbf{R}, t)$, $h^{(2)}(\mathbf{R}, t)$ are defined as

$$\begin{aligned} \mathbf{H}^{(1)}(\mathbf{R}, t) &= \frac{(\nu_u - \nu)}{(1-\nu)} \mathbf{R} \left\{ \frac{c\varepsilon(1-\varepsilon)R^2}{8(ct)^3 \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} - \frac{c\varepsilon(1-2\varepsilon)}{(1-2\nu)} \frac{1}{4(ct)^2 \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} - \frac{c\varepsilon(1+3(1-\varepsilon))}{4(ct)^2 \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} \right. \\ & \quad \left. + \frac{2c\varepsilon(1-\varepsilon)ct}{R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^5 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho \right. \\ & \quad \left. - \frac{c(1+\varepsilon)}{R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^3 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho \right\} d\tau dS', \end{aligned} \quad (50)$$

$$h^{(1)}(\mathbf{r}, t) = -\frac{2B(1+\nu_u)}{3} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^2 J_0[R\rho] \operatorname{erfc}[\sqrt{ct\varepsilon}\rho] d\rho, \quad (51)$$

$$H^{(2)}(\mathbf{R}, t) = -\mathbf{R} \left(\frac{c\varepsilon}{4(ct)^2 \sqrt{ct\varepsilon\pi}} e^{-\frac{R^2}{4ct}} + \frac{c(1-\varepsilon)}{R} \int_0^\infty e^{-(1-\varepsilon)\rho^2 ct} \rho^3 J_1[R\rho] (\operatorname{erfc}[\sqrt{ct\varepsilon}\rho]) d\rho \right), \quad (52)$$

$$h^{(2)}(\mathbf{r}) = -\frac{B(1+\nu_u)(1-\nu)}{6(\nu_u-\nu)} \frac{e^{-\frac{R^2}{4ct}}}{ct\sqrt{\varepsilon ct\pi}}. \quad (53)$$

10. Conclusions

The obtained integral equations are those for a non-uniformly pressurized, vertical fracture of arbitrary shape in a fluid-saturated poroelastic infinite reservoir of transversely isotropic permeability. They relate the crack opening displacement and the fluid supply flux through the crack surface to the total stress and the pore fluid pressure acting on the crack surface. Here, we have treated the case where the anisotropy axis of transversely isotropic permeability is normal to the crack surface. Since the previous paper⁽¹⁰⁾ treated the case of the anisotropy axis existing in the crack plane, the present and previous papers can be applied to a transversely isotropic reservoir according to its situation. If these integral equations are implanted in a hydraulic fracturing simulator, some effect of transversely isotropic permeability on fracture extensions or others will be made clear.

Finally, it should be mentioned that we have confirmed that the present solutions tend to those for the isotropic permeability⁽²⁾ when making $\varepsilon \rightarrow 1$.

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