Large Deflection Analysis of Rectangular Plates on Two-parameter Elastic Foundation by the Boundary Strip Method*

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A method is proposed for calculating the large deflection of a rectangular plate on an elastic foundation by the boundary integral equation method. The elastic foundation is assumed to be of the Pasternak type. The large deflection is formulated based on the nonlinear Berger equation. In the analysis, the governing nonlinear partial differential equation for a plate is transformed into an ordinary nonlinear differential equation using the Kantrovich method. The resultant fourth-order nonlinear differential equation was analyzed by the boundary element method. The nonlinear boundary integral equation is solved by using an iteration scheme. A fundamental solution required in the formulation is derived by solving the singular differential equation. It is found that the numerical results obtained by the present method agree well with the analytical solutions. To show the usefulness of the method, several numerical results based on the present formulation are given for various types of boundary and loading conditions.

Key Words: Structural Analysis, Large Deflection, Boundary Element Method, Boundary Strip Method, Plate on Pasternak Foundation, Numerical Analysis

1. Introduction

If the deflection of a plate or beam is of the order of magnitude of its thickness, the differential equations for the deflection and the in-plane displacements can be written as one fourth-order and two second-order nonlinear equations⁽¹⁾. In general, it is difficult to obtain the exact solutions of these equations and hence numerical methods such as the finite element method (FEM) and the boundary element method (BEM) have been employed to solve the nonlinear (i.e., large deflection) equations. Several BEM approaches for solving large deflection problems have been proposed by several investigators^{(2)–(5)}.

On the other hand, the use of beams and plates on an elastic foundation has recently become widespread in engineering. Consequently it is important to clarify the influence of the foundation on the behavior of the beams and plates in practical designs. Although large deflections on an elastic foundation have already been

analyzed by many investigators, most of those works use the Winkler hypothesis^{(6)–(13)}. The Winkler model, which is a closely spaced linear spring model, is simple but interactions of the springs are not considered, so it does not represent the characteristics of practical foundations. To overcome the restriction of the Winkler model, the Pasternak (two-parameter) foundation model which takes into account the shear interactions between springs has been proposed⁽¹⁴⁾. However, few papers have focused on a large deflection on a Pasternak elastic foundation^{(15),(16)}.

In the present paper, a modified boundary integral equation method (BIEM) which analyzes the large deflection of rectangular plates on a Pasternak elastic foundation is proposed. The proposed method is applicable only for rectangular plates just as the boundary strip method⁽¹⁷⁾ is. The present method however has a superior accuracy over other numerical techniques such as the finite element method. To show the effectiveness of the proposed method, several numerical examples are given. Comparisons are also made with the exact solutions in order to check the accuracy of the present formulation.

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2. Basic Theory

2.1 Governing equation

Following the approach of Berger⁽¹⁸⁾, the governing equation for the large deflection of plates on a Pasternak foundation is given by:

$$\nabla^4 w(x, y) - \alpha^2 \nabla^2 w(x, y) - \frac{G_p}{D} \nabla^2 w(x, y) + \frac{k_p}{D} w = \frac{q(x, y)}{D}, \quad (1)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

where $w(x, y)$ is the deflection, G_p is the shear modulus of the shear layer, k_p is the foundation coefficient, $q(x, y)$ is the transverse load per unit area, D is the flexural rigidity of the plate and $\nabla^2(\)$ is the Laplace operator. α^2 is the Berger constant, which is defined by the following equation:

$$\alpha^2 = \frac{6}{Sh^2} \iint_s \left(\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) dS \quad (2)$$

where S is the area and h is the thickness of the plate respectively. It is well known that the Berger solutions can be a good approximation to the rigorous solutions if the in-plane displacements are constrained at the boundary. Equation (1), which is linear in w , can be solved for w in terms of α and by using this solution the constant α in Eq. (2) can be determined. Berger⁽¹⁸⁾ and Sinha⁽¹⁹⁾ have derived the exact solutions based on that iterative calculation process.

2.2 Transformation of governing equation

Consider a rectangular plate as shown in Fig. 1. The origin is at the center of the plate and a and b are the length and width of the plate. The deflection $w(x, y)$ and the load $q(x, y)$ are assumed to be of the following series form

$$w(x, y) = \sum_{m=1}^{\infty} \varphi_m(x) \psi_m(y), \quad (3)$$

$$q(x, y) = q_0 f(x) g(y)$$

in which $\varphi_m(x)$ and $\psi_m(y)$ are arbitrary functions in

Fig. 1 Coordinate system

the transverse and longitudinal direction respectively. Inserting expressions (3) into governing equation (1), we obtain infinite series equations

$$\sum_{m=1}^{\infty} \left[\varphi_m(x) \psi_m''''(y) - \left(\alpha^2 + \frac{G_p}{D} \right) \varphi_m(x) \psi_m''(y) + \left(\varphi_m''''(x) - \left(\alpha^2 + \frac{G_p}{D} \right) \varphi_m''(x) + \frac{k_p}{D} \varphi_m(x) \right) \psi_m(y) \right] = \frac{q_0}{D} f(x) g(y) \quad (4)$$

in which the prime denotes differentiation with respect to the corresponding coordinates, e.g. $\varphi_m''(x) = d^2 \varphi_m / dx^2$, $\psi_m''(y) = d^2 \psi_m / dy^2$.

2.3 Transverse direction function $\varphi_n(x)$

The function $\varphi_n(x)$, defined in the x -direction, is assumed to satisfy the following differential equation of a vibrating beam

$$\varphi_n''''(x) - \left(\frac{k_n}{a} \right)^4 \varphi_n(x) = 0, \quad (n=1, 2, \dots, \infty) \quad (5)$$

in which k_n are the eigenvalues. The function $\varphi_n(x)$ therefore satisfies the following orthogonality relations:

$$\left. \begin{aligned} \int_{-a/2}^{a/2} \varphi_m(x) \varphi_n(x) dx &\equiv (\varphi_m(x), \varphi_n(x)) = 0, \\ \int_{-a/2}^{a/2} \varphi_m''''(x) \varphi_n(x) dx &\equiv (\varphi_m''''(x), \varphi_n(x)) = 0 \end{aligned} \right\} \quad (m \neq n) \quad (6)$$

Multiplying Eq. (4) by $\varphi_n(x)$, integrating with respect to x over the interval from $x = -a/2$ to $x = a/2$, and using Eq. (6), we obtain a system of simultaneous differential equations:

$$\begin{aligned} \psi_n''''(y) - 2\delta_n^2 \psi_n''(y) + \epsilon_n^4 \psi_n(y) + 2 \sum_{\substack{m=1 \\ (m \neq n)}}^{\infty} \frac{(\varphi_m'', \varphi_n)}{(\varphi_n, \varphi_n)} \psi_m''(y) \\ - \left(\alpha^2 + \frac{G_p}{D} \right) \sum_{\substack{m=1 \\ (m \neq n)}}^{\infty} \frac{(\varphi_m'', \varphi_n)}{(\varphi_n, \varphi_n)} \psi_m(y) = q_n g(y), \end{aligned} \quad (7)$$

$$n=1, 2, \dots, \infty$$

where

$$\begin{aligned} (\varphi_n, \varphi_n) &= \int_{-a/2}^{+a/2} \varphi_n \cdot \varphi_n dx, \\ (\varphi_n'', \varphi_n) &= \int_{-a/2}^{+a/2} \left(\frac{d^2 \varphi_n}{dx^2} \right) \cdot \varphi_n dx, \\ (\varphi_m'', \varphi_n) &= \int_{-a/2}^{+a/2} \left(\frac{d^2 \varphi_m}{dx^2} \right) \cdot \varphi_n dx, \\ q_n &= \frac{q_0 (f, \varphi_n)}{D (\varphi_n, \varphi_n)} \end{aligned} \quad (8)$$

$$\begin{aligned} \delta_n^2 &= \frac{1}{2} \left(\alpha^2 + \frac{G_p}{D} \right) - \frac{(\varphi_n, \varphi_n'')}{(\varphi_n, \varphi_n)}, \\ \epsilon_n^4 &= \left(\frac{k_n}{D} \right)^4 - \left(\alpha^2 + \frac{G_p}{D} \right) \frac{(\varphi_n, \varphi_n'')}{(\varphi_n, \varphi_n)} + \frac{k_p}{D} \end{aligned} \quad (9)$$

The above derivation procedure of Eq. (7) is similar to the method of Kantovich⁽²⁰⁾, reducing the integration of partial differential equations to the integration of systems of ordinary differential equations.

The transverse direction function $\varphi_n(x)$ must satisfy the boundary conditions given at two sides, $x = -a/2$ and $x = a/2$. The functions and their integrals under various boundary conditions are listed as

follows:

1) Both ends simply supported (S-S)

$$\begin{aligned}\varphi_n(x) &= \sin\left(k_n\left(\frac{x}{a} + \frac{1}{2}\right)\right), \\ \frac{(\varphi_n'', \varphi_n)}{(\varphi_n, \varphi_n)} &= -\left(\frac{k_n}{a}\right)^2, \quad \frac{(\varphi_n''', \varphi_n)}{(\varphi_n, \varphi_n)} = \left(\frac{k_n}{a}\right)^4, \\ \frac{(\varphi_m'', \varphi_n)}{(\varphi_n, \varphi_n)} &= 0, \quad k_n = \pi, 2\pi, 3\pi, \dots\end{aligned}\quad (10)$$

2) One end simply supported and the other end clamped (S-C)

$$\begin{aligned}\varphi_n(x) &= \frac{\sinh\left(k_n\left(\frac{x}{a} + \frac{1}{2}\right)\right)}{\sinh k_n} - \frac{\sin\left(k_n\left(\frac{x}{a} + \frac{1}{2}\right)\right)}{\sin k_n}, \\ \frac{(\varphi_n'', \varphi_n)}{(\varphi_n, \varphi_n)} &= -\left(\frac{k_n}{a}\right)^2 \coth k_n \left(\coth k_n - \frac{1}{k_n}\right), \\ \frac{(\varphi_n''', \varphi_n)}{(\varphi_n, \varphi_n)} &= \left(\frac{k_n}{a}\right)^4, \\ \frac{(\varphi_m'', \varphi_n)}{(\varphi_n, \varphi_n)} &= 4\left(\frac{k_n}{a}\right)^2 \frac{k_m^2}{(k_m^4 - k_n^4)} (k_m \cot k_m - k_n \cot k_n), \\ k_n &= 3.926602, 7.068583, 10.210176, \dots\end{aligned}\quad (11)$$

where k_n are the roots of the transcendental equation $\tan k_n + \tanh k_n = 0$

3) Both ends clamped (C-C)

$$\begin{aligned}\varphi_n(x) &= \frac{\cosh\left(k_n \frac{x}{a}\right)}{\sinh \frac{k_n}{2}} - \frac{\cos\left(k_n \frac{x}{a}\right)}{\sin \frac{k_n}{2}}, \\ \frac{(\varphi_n'', \varphi_n)}{(\varphi_n, \varphi_n)} &= -\left(\frac{k_n}{a}\right)^2 \tanh^2 \frac{k_n}{2} \left(1 - \frac{2}{k_n} \coth \frac{k_n}{2}\right), \\ \frac{(\varphi_n''', \varphi_n)}{(\varphi_n, \varphi_n)} &= \left(\frac{k_n}{a}\right)^4, \\ \frac{(\varphi_m'', \varphi_n)}{(\varphi_n, \varphi_n)} &= 8\left(\frac{k_n}{a}\right)^2 \frac{k_m^2}{(k_m^4 - k_n^4)} \\ &\quad \times \tanh^2 \frac{k_n}{2} \left(k_n \cot \frac{k_m}{2} - k_m \cot \frac{k_n}{2}\right), \\ k_n &= 4.7300407, 10.995608, 17.278760, \dots\end{aligned}\quad (13)$$

where k_n are the roots of the transcendental equation

$$\tan \frac{k_n}{2} + \tanh \frac{k_n}{2} = 0 \quad (14)$$

When the two sides at $x = \pm a/2$ are simply supported (S-S), the summation terms in the left-hand side of Eq.(7) vanish because of the orthogonality relation $(\varphi_m'', \varphi_n) = 0$, shown in Eq.(10). Hence unknown functions $\psi_m(y)$ and $\psi_n(y)$ uncouple. We can therefore solve the simultaneous ordinary differential equations (7) term by term, ignoring the summation terms, whereas under the other boundary conditions (S-C) or (C-C), the orthogonal relation $(\varphi_m'', \varphi_n) = 0$ does not hold. This indicates that unknown functions $\psi_m(y)$ and $\psi_n(y)$ couple together. As a result it is difficult to solve the coupled simultaneous ordinary differential equations (7). However, if we neglect the summation terms in the left-hand side of Eq.(7) regardless of the boundary condition, we obtain the uncoupled approximate governing equations:

$$\begin{aligned}\psi_n''''(y) - 2\delta_n^2 \psi_n''(y) + \epsilon_n^4 \psi_n(y) &= q_n g(y), \\ n &= 1, 2, \dots, \infty\end{aligned}\quad (15)$$

Fig. 2 Relative values of neglected terms $(a/k_n)^2 |(\varphi_m'', \varphi_n) / (\varphi_n, \varphi_n)|$ (S-C)

We can solve each term $\psi_n(y)$ of the series individually, just as we do when the two sides are simply supported.

In order to investigate the influence of the neglect on accuracy, we evaluated the magnitude of the coefficient of the neglected terms $\psi_m''(y)$, i.e. $(a/k_n)^2 |(\varphi_m'', \varphi_n) / (\varphi_n, \varphi_n)|$. Figure 2 shows the values of the coefficient when one end is simply supported and the other end is clamped (S-C) in the x -direction. The figure shows that the diagonal term (φ_n'', φ_n) is dominant compared with the other non-diagonal one. Therefore, it is evident that the omission of the summation terms causes only a little error. And this fact is numerically confirmed by the previous paper⁽¹¹⁾. Figure 3 shows the evaluation of the neglected terms when both ends are clamped (C-C) in the x -direction. The results are similar to the case of (S-C).

3. Boundary Strip Method Formulation

3.1 Boundary integral equation for $\psi_n(y)$

In the previous section, we have shown that Eq. (15) is the governing equation for the large deflection of the plates on a Pasternak foundation. To obtain the boundary integral representation of Eq.(15), we introduce the following fundamental solutions $\psi_n^*(y, \eta)$ defined by:

$$\begin{aligned}\frac{d^4 \psi_n^*(y, \eta)}{dy^4} - 2\delta_n^2 \frac{d^2 \psi_n^*(y, \eta)}{dy^2} + \epsilon_n^4 \psi_n^*(y, \eta) \\ = \delta(y, \eta), \quad (n=1, \dots, \infty)\end{aligned}\quad (16)$$

where η is a field variable and $\delta(y, \eta)$ is the Dirac delta function. Multiplying both sides of Eq.(15) by a fundamental function $\psi_n^*(y, \eta)$ and integrating over the range from $-b/2$ to $b/2$, it can be shown that

$$\int_{-b/2}^{b/2} (\psi_n''''(y) - 2\delta_n^2 \psi_n''(y) + \epsilon_n^4 \psi_n(y)) \psi_n^*(y, \eta) dy$$

Fig. 3 Relative values of neglected terms $(a/k_n)^2 |(\phi_n'')/(\phi_n, \phi_n)|$ (C-C)

$$= q_n \int_{-b/2}^{b/2} g(y) \phi_n^*(y, \eta) dy, \quad n=1, 2, \dots, \infty \quad (17)$$

and after integrating Eq.(17) by parts four times, we obtain the integral equation:

$$\begin{aligned} \phi_n(\eta) + & \left[\left(\frac{d^3 \phi_n(y)}{dy^3} - (2-\nu) \delta_n^2 \frac{d\phi_n(y)}{dy} \right) \phi_n^*(y, \eta) \right]_{-b/2}^{+b/2} \\ & - \left[\left(\frac{d^2 \phi_n(y)}{dy^2} - \nu \delta_n^2 \phi_n(y) \right) \frac{d\phi_n^*(y, \eta)}{dy} \right]_{-b/2}^{+b/2} \\ & + \left[\frac{d\phi_n(y)}{dy} \left(\frac{d^2 \phi_n^*(y, \eta)}{dy^2} - \nu \delta_n^2 \phi_n^*(y, \eta) \right) \right]_{-b/2}^{+b/2} \\ & - \left[\phi_n(y) \left(\frac{d^3 \phi_n^*(y, \eta)}{dy^3} - (2-\nu) \delta_n^2 \frac{d\phi_n^*(y, \eta)}{dy} \right) \right]_{-b/2}^{+b/2} \\ & = q_n \int_{-b/2}^{+b/2} g(y) \phi_n^*(y, \eta) dy, \quad n=1, 2, \dots, \infty \quad (18) \end{aligned}$$

where ν is Poisson's ratio.

To formulate sufficient equations for the solution of the present problem, it is necessary to have another equation which is the derivative of Eq.(18) with respect to η and it may be expressed as

$$\begin{aligned} \tilde{\phi}_n(\eta) + & \left[\left(\frac{d^3 \phi_n(y)}{dy^3} - (2-\nu) \delta_n^2 \frac{d\phi_n(y)}{dy} \right) \tilde{\phi}_n^*(y, \eta) \right]_{-b/2}^{+b/2} \\ & - \left[\left(\frac{d^2 \phi_n(y)}{dy^2} - \nu \delta_n^2 \phi_n(y) \right) \frac{d\tilde{\phi}_n^*(y, \eta)}{dy} \right]_{-b/2}^{+b/2} \\ & + \left[\frac{d\phi_n(y)}{dy} \left(\frac{d^2 \tilde{\phi}_n^*(y, \eta)}{dy^2} - \nu \delta_n^2 \tilde{\phi}_n^*(y, \eta) \right) \right]_{-b/2}^{+b/2} \\ & - \left[\phi_n(y) \left(\frac{d^3 \tilde{\phi}_n^*(y, \eta)}{dy^3} - (2-\nu) \delta_n^2 \frac{d\tilde{\phi}_n^*(y, \eta)}{dy} \right) \right]_{-b/2}^{+b/2} \\ & = q_n \int_{-b/2}^{+b/2} g(y) \tilde{\phi}_n^*(y, \eta) dy, \quad n=1, 2, \dots, \infty \quad (19) \end{aligned}$$

where $\tilde{}$ denotes the derivative with respect to η , e.g. $\tilde{\phi}_n^*(y, \eta) = d\phi_n^*(y, \eta)/d\eta$.

Equations (18) and (19) provide the solutions $\phi_n(\eta)$ and $d\phi_n(\eta)/d\eta$ at any interior point under the specified loading $g(y)$ in the y -direction.

Taking the field point η to the boundaries such that $\eta = -b/2 + \epsilon$ and $\eta = b/2 - \epsilon$ (ϵ is a very small value) in Eqs.(18) and (19), we can determine all the

boundary data from the four sets of boundary equations⁽²¹⁾. Once all the boundary data are known, the deflection of the plate at any interior point is evaluated from Eq.(3).

We call the present approach the *boundary strip method* just as Cheung called a similar approach the *finite strip method*⁽¹⁷⁾.

3.2 Fundamental solutions

In this section, we derive the fundamental solutions $\phi_n^*(y, \eta)$ as follows. The fundamental solutions must satisfy Eq.(16). The general solutions for Eq.(16) are obtained as follows. First we divide the infinite range along the y -direction into two ranges at the point $y = \eta$ and express the solutions in the range $y < \eta$ as $\phi_n^*_{-}(y, \eta)$ and in the range $y > \eta$ as $\phi_n^*_{+}(y, \eta)$. The governing equation for $\phi_n^*_{-}(y, \eta)$ is, from Eq.(16),

$$\frac{d^4 \phi_n^*_{-}(y, \eta)}{dy^4} - 2\delta_n^2 \frac{d^2 \phi_n^*_{-}(y, \eta)}{dy^2} + \epsilon_n^4 \phi_n^*_{-}(y, \eta) = 0, \quad (n=1, \dots, \infty) \quad (20)$$

The general solutions for Eq.(16) are obtained as follows, depending on whether δ_n is equal, greater or less than ϵ_n :

1) Case of $\delta_n = \epsilon_n$

$$\begin{aligned} \phi_n^*_{-}(y, \eta) = & A_1 \cosh \delta_n(b+y-\eta) \\ & + A_2 \delta_n(b+y-\eta) \sinh \delta_n(b+y-\eta) \\ & + A_3 \sinh \delta_n(b+y-\eta) \\ & + A_4 \delta_n(b+y-\eta) \cosh \delta_n(b+y-\eta) \end{aligned} \quad (21)$$

2) Case of $\delta_n > \epsilon_n$

$$\begin{aligned} \phi_n^*_{-}(y, \eta) = & B_1 \sinh m_\delta(b+y-\eta) \\ & + B_2 \cosh m_\delta(b+y-\eta) + B_3 \sinh n_\delta(b+y-\eta) \\ & + B_4 \cosh n_\delta(b+y-\eta) \end{aligned} \quad (22)$$

where

$$\begin{aligned} m_\delta = & \frac{1}{\sqrt{2}} (\sqrt{\delta_n^2 + \epsilon_n^2} + \sqrt{\delta_n^2 - \epsilon_n^2}), \\ n_\delta = & \frac{1}{\sqrt{2}} (\sqrt{\delta_n^2 + \epsilon_n^2} - \sqrt{\delta_n^2 - \epsilon_n^2}) \end{aligned} \quad (23)$$

3) Case of $\delta_n < \epsilon_n$

$$\begin{aligned} \phi_n^*_{-}(y, \eta) = & C_1 \cosh m_\epsilon(b+y-\eta) \sin n_\epsilon(b+y-\eta) \\ & + C_2 \cosh m_\epsilon(b+y-\eta) \cos n_\epsilon(b+y-\eta) \\ & + C_3 \sinh m_\epsilon(b+y-\eta) \cos n_\epsilon(b+y-\eta) \\ & + C_4 \sinh m_\epsilon(b+y-\eta) \sin n_\epsilon(b+y-\eta) \end{aligned} \quad (24)$$

where

$$m_\epsilon = \sqrt{\frac{\epsilon_n^2 + \delta_n^2}{2}}, \quad n_\epsilon = \sqrt{\frac{\epsilon_n^2 - \delta_n^2}{2}} \quad (25)$$

Here, $A_1, \dots, A_4, B_1, \dots, B_4$ and $C_1, \dots, C_4$ are unknown constants determined from the boundary conditions. We set the following boundary conditions so as to make the fundamental solutions possess a simple form:

$$\begin{aligned} \text{at } y = \eta - b; \quad & \phi_n^*_{-}(y, \eta) = 0, \quad \frac{d^2 \phi_n^*_{-}(y, \eta)}{dy^2} = 0 \\ \text{at } y = \eta; \quad & \frac{d\phi_n^*_{-}(y, \eta)}{dy} = 0, \quad \frac{d^3 \phi_n^*_{-}(y, \eta)}{dy^3} = -\frac{1}{2} \end{aligned} \quad (26)$$

Substitution of these boundary conditions into the general solutions Eqs.(21), (22) and (24) leads to

$$\begin{aligned} \phi_{n-}^*(y, \eta) &= \frac{1}{4\delta_n^3} \left[(1 + (\delta_n b) \tanh(\delta_n b)) \frac{\sinh(\delta_n(b - \eta + y))}{\cosh(\delta_n b)} \right. \\ &\quad \left. - \delta_n(b - \eta + y) \frac{\cosh(\delta_n(b - \eta + y))}{\cosh(\delta_n b)} \right], \\ (\text{for } \delta_n = \varepsilon_n) & \end{aligned} \quad (27)$$

$$\begin{aligned} \phi_{n-}^*(y, \eta) &= \frac{1}{2(m_\delta^2 - n_\delta^2)} \left[\frac{\sinh(n_\delta(b - \eta + y))}{n_\delta \cosh(n_\delta b)} \right. \\ &\quad \left. - \frac{\sinh(m_\delta(b - \eta + y))}{m_\delta \cosh(m_\delta b)} \right], \\ (\text{for } \delta_n > \varepsilon_n) & \end{aligned} \quad (28)$$

$$\begin{aligned} \phi_{n-}^*(y, \eta) &= \frac{1}{4\varepsilon_n^2(\cosh^2(m_\varepsilon b) - \sin^2(n_\varepsilon b))} \\ &\times \left[\left(\frac{\sinh(m_\varepsilon b) \sin(n_\varepsilon b)}{m_\varepsilon} - \frac{\cosh(m_\varepsilon b) \cos(n_\varepsilon b)}{n_\varepsilon} \right) \right. \\ &\times \cosh(m_\varepsilon(b - \eta + y)) \sin(n_\varepsilon(b - \eta + y)) \\ &+ \left(\frac{\cosh(m_\varepsilon b) \cos(n_\varepsilon b)}{m_\varepsilon} + \frac{\sinh(m_\varepsilon b) \sin(n_\varepsilon b)}{n_\varepsilon} \right) \\ &\times \sinh(m_\varepsilon(b - \eta + y)) \cos(n_\varepsilon(b - \eta + y)) \Big], \\ (\text{for } \delta_n < \varepsilon_n) & \end{aligned} \quad (29)$$

The solution $\phi_{n+}^*(y, \eta)$ in the right-hand range is derived in the same manner. Or, we can also derive the solution $\phi_{n+}^*(y, \eta)$ by making use of the reciprocity of the fundamental solution, $\phi_{n+}^*(y, \eta) = \phi_{n-}^*(\eta, y)$. Introducing $\rho_y = |y - \eta|$, we can summarize the above results as follows.

1) Case of $\delta_n = \varepsilon_n$

$$\begin{aligned} \phi_n^*(y, \eta) &= \frac{1}{4\delta_n^3} \left[(1 + (\delta_n b) \tanh(\delta_n b)) \frac{\sinh(\delta_n(b - \rho_y))}{\cosh(\delta_n b)} \right. \\ &\quad \left. - \delta_n(b - \rho_y) \frac{\cosh(\delta_n(b - \rho_y))}{\cosh(\delta_n b)} \right] \end{aligned} \quad (30)$$

2) Case of $\delta_n > \varepsilon_n$

$$\begin{aligned} \phi_n^*(y, \eta) &= \frac{1}{2(m_\delta^2 - n_\delta^2)} \left[\frac{\sinh(n_\delta(b - \rho_y))}{n_\delta \cosh(n_\delta b)} \right. \\ &\quad \left. - \frac{\sinh(m_\delta(b - \rho_y))}{m_\delta \cosh(m_\delta b)} \right] \end{aligned} \quad (31)$$

3) Case of $\delta_n < \varepsilon_n$

$$\begin{aligned} \phi_n^*(y, \eta) &= \frac{1}{4\varepsilon_n^2(\cosh^2(m_\varepsilon b) - \sin^2(n_\varepsilon b))} \\ &\times \left[\left(\frac{\sinh(m_\varepsilon b) \sin(n_\varepsilon b)}{m_\varepsilon} - \frac{\cosh(m_\varepsilon b) \cos(n_\varepsilon b)}{n_\varepsilon} \right) \right. \\ &\times \cosh(m_\varepsilon(b - \rho_y)) \sin(n_\varepsilon(b - \rho_y)) \\ &+ \left(\frac{\cosh(m_\varepsilon b) \cos(n_\varepsilon b)}{m_\varepsilon} + \frac{\sinh(m_\varepsilon b) \sin(n_\varepsilon b)}{n_\varepsilon} \right) \\ &\times \sinh(m_\varepsilon(b - \rho_y)) \cos(n_\varepsilon(b - \rho_y)) \Big] \end{aligned} \quad (32)$$

As to the differential quantities $\phi_n^{*'}(y, \eta), \dots, \phi_n^{*'''}(y, \eta)$ in Eq.(19), we can easily derive the following relations regardless of $\delta_n = \varepsilon_n$, $\delta_n > \varepsilon_n$, or $\delta_n < \varepsilon_n$:

$$\begin{aligned} \tilde{\phi}_n^*(y, \eta) &= -\phi_n^{*'}(y, \eta), \quad \tilde{\phi}_n^{*'}(y, \eta) = -\phi_n^{*''}(y, \eta), \\ \tilde{\phi}_n^{*''}(y, \eta) &= -\phi_n^{*'''}(y, \eta), \\ \tilde{\phi}_n^{*'''}(y, \eta) &= -2\delta_n^2 \phi_n^{*''}(y, \eta) + \varepsilon_n^4 \phi_n^*(y, \eta) \end{aligned} \quad (33)$$

From the above discussion, it is necessary to employ different fundamental solutions depending on the magnitude of δ_n and ε_n . However, this is not troublesome for computer programs.

4. Estimation of the Magnitude of α by the Iterative Scheme

Berger constant α^2 , defined by Eq.(2), is unknown for the moment. In order to determine the constant, the following iterative procedure is proposed in this paper.

(1) Set the boundary conditions and input the transverse load and an arbitrary Berger constant B_θ .

(2) Solve the system equations (18) and (19).

(3) Calculate the new Berger constant B_i by the following equation using Gaussian numerical integration:

$$B_i = \frac{6}{Sh^2} \iint_S \left(\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right) dS \quad (34)$$

(4) If $|(B_\theta - B_i)/B_\theta| > \varepsilon$ (ε = a very small value), then the Berger constant is updated by

$$B_\theta \leftarrow (B_\theta + B_i)/2 \quad (35)$$

and go to step (2).

(5) Repeat steps (2) to (4) until $|(B_\theta - B_i)/B_\theta| < \varepsilon$. When the condition is satisfied, set $\alpha^2 \leftarrow B_\theta$ and output the calculated results.

The iteration process is illustrated in Fig. 4.

5. Numerical Results

To ascertain the accuracy and the efficiency of the proposed formulation, several numerical examples are presented.

The integral terms of the right-hand side of Eqs. (18) and (19), which include $f(x)$ and $g(y)$

$$\begin{aligned} q_n \int_{-b/2}^{+b/2} g(y) \left\{ \frac{\phi_n^*(y, \eta)}{\tilde{\phi}_n^*(y, \eta)} \right\} dy \\ = \frac{q_0}{D} \frac{(f, \varphi_n)}{(\varphi_n, \varphi_n)} \int_{-b/2}^{+b/2} g(y) \left\{ \frac{\phi_n^*(y, \eta)}{\tilde{\phi}_n^*(y, \eta)} \right\} dy \end{aligned} \quad (36)$$

were evaluated according to the following two cases of loading:

(a) The plate is subjected to a concentrated load P_0 at $x = x_i$, $y = y_i$, so $q(x, y) = P_0 \delta(x - x_i) \delta(y - y_i)$. In this case the integrals become

$$\frac{P_0}{D} \frac{\varphi_n(x_i)}{(\varphi_n, \varphi_n)} \left\{ \frac{\phi_n^*(y_i, \eta)}{\tilde{\phi}_n^*(y_i, \eta)} \right\} \quad (37)$$

(b) The plate is subjected to a uniformly distributed load q_0 ($q(x, y) = q_0$). The integrals become

$$\frac{q_0}{D} \frac{(1, \varphi_n)}{(\varphi_n, \varphi_n)} \int_{-b/2}^{+b/2} \left\{ \frac{\phi_n^*(y, \eta)}{\tilde{\phi}_n^*(y, \eta)} \right\} dy \quad (38)$$

Numerical calculations are performed with the following conditions:

Poisson's ratio $\nu = 0.3$

Number of terms of Eq.(3) $n = 6$

Convergence criterion $|(B_\theta - B_i)/B_\theta| \leq \varepsilon = 1.0 \times 10^{-5}$

Fig. 4 Determination of Berger constant

Number of Gaussian integral points=64.

For the presentation of the numerical results, the following dimensionless foundation quantities are employed :

$$\lambda_k = k_p a^4 / D, \quad \lambda_p = G_p a^2 / D$$

First, a simply supported square plate ($b/a=1$) under uniformly-distributed loading is considered. Foundation parameters are taken as $\lambda_k=100$, $\lambda_p=0$, 20, 40, 60. Figure 5 presents the variation of the maximum deflection $w_{\max}/h$ against dimensionless uniform load $q_0 a^4/(Dh)$ for the square plate, along with the results of Sinha⁽¹⁹⁾. Numerical results obtained from the present method closely agree with the corresponding Sinha's solution, which is obtained from the conventional equations for finite deflections.

Next, to show the wide applicability of the present method, the following examples under various boundary and loading conditions are analyzed. In Fig. 6, the maximum deflection of an all-edges-clamped square plate are plotted versus the uniformly distributed load.

Figure 7 shows the relationship between the uniform load and the maximum deflection of the all-edges-simply-supported plate under different values of aspect ratio b/a .

Figure 8 shows the relationships between the dimensionless concentrated center load $P_0 a^2/(Dh)$ and the maximum deflection $w_{\max}/h$ of a two-edges-simply-supported, other-edges-clamped plate under different values of shear foundation coefficient λ_p .

Fig. 5 Relationship between load and deflection of all-edges-supported plate on Pasternak foundation subjected to uniform load

Fig. 6 Relationship between load and deflection of all-edges-clamped plate on Pasternak foundation subjected to uniform load

Fig. 7 Relationship between load and deflection of all-edges-supported plate subjected to uniform load

From the figures it is found that the deflections differ considerably from the corresponding values for a Winkler type foundation.

Fig. 8 Relationship between load and deflection of two-edges-supported, other-edges-clamped plate subjected to concentrated load

6. Conclusions

The present paper presented a boundary integral equation approach to calculate the large deflection of plates on a Pasternak foundation. The nonlinear boundary integral equations were derived and analyzed employing the iteration method. To show the validity of the present method, some numerical calculations were carried out. The numerical results demonstrated the accuracy and efficiency of the proposed method.

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