Intensity of a Plastic Singular Stress Field at the Notch Tip*

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In the present study, the relation between the plastic and the elastic stress intensity factors, $K_p$ and $K_e$, for various $V$-shaped notch problems is studied by using the finite element analysis. Detailed results for mode I under plane stress and plane strain conditions are investigated. Based on the numerical results, it is found that there is a one-to-one correspondence between $K_p$ and $K_e$, even when the load is so large that the condition of the small scale yielding is not satisfied. Also, it is found that the values of $K_p^{1-\alpha}/K_e^{1-\alpha}$ for various notch problems depend only on the notch opening angle $\gamma$ and hardening exponent $\alpha$. This enables one to evaluate the factor $K_p$ for various notch problems without any plastic stress analysis.

**Key Words**: Plasticity, Nonlinear Problem, Stress Intensity Factor, Notch, Stress Singularity, Plane Problem

1. Introduction

A plastic singular stress field at a tip of $V$-shaped notch is generally written as this form,

$$\sigma_i(r, \theta) = \frac{K_p}{r^{1-\alpha}} \delta_0(\theta) \quad (ij = r, \theta, r\theta),$$

($r, \theta$: Polar coordinates centered at the notch tip; $\alpha$: The order of plastic stress singularity; $\delta_0(\theta)$: The angular distribution of the singular stresses).

A lot of parameter trackings are necessary to analyze some factors which have influence on the plastic stress intensity factor $K_p$. For the plastic stress intensity factor $K_p$ which shows the strength of plastic singular stress field, we have already studied by using the finite element method (FEM), and discussed the relation between $K_p$ and the elastic stress intensity factor $K_e$ under plane stress condition. From this study, it is found that the parameter $K_p^{1-\alpha}/K_e^{1-\alpha}$ is independent of the strength of singular stress field, where $1-\alpha$ is the order of the elastic stress singularity. Table 1 shows the values of $\lambda_e$ and $\lambda_p$ for some cases of $V$-notch obtained by asymptotic analysis. In Table 1, the terms in ( ) show the results under plane strain condition, and the others show the results under plane stress condition.

From the above conclusion, we can say that if the factor $K_p$ for one loading level has already been obtained, the variation of $K_p$ for some loading levels.

Table 1 $\lambda_e$ and $\lambda_p$ for various notches

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\lambda_e$</th>
<th>$\lambda_p$ (n=3)</th>
<th>$\lambda_p$ (n=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$30^\circ$</td>
<td>0.5015</td>
<td>(0.7508)</td>
<td>0.9097</td>
</tr>
<tr>
<td></td>
<td>0.5122</td>
<td>0.7511</td>
<td>(0.9098)</td>
</tr>
<tr>
<td></td>
<td>0.5448</td>
<td>0.7646</td>
<td>0.9093</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>(0.5015)</td>
<td>(0.7520)</td>
<td>(0.9097)</td>
</tr>
<tr>
<td></td>
<td>(0.5122)</td>
<td>(0.7589)</td>
<td>(0.9114)</td>
</tr>
<tr>
<td></td>
<td>(0.5448)</td>
<td>(0.7750)</td>
<td>(0.9155)</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>0.5448</td>
<td>0.7646</td>
<td>0.9093</td>
</tr>
</tbody>
</table>


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can be estimated by using the constant value of $K_p^{\gamma=\theta}/K_x^{\gamma=\theta}$ and $K_e$ obtained by linear elastic analysis. This has big meaning for parameter survey to establish the assessment of material strength based on stress singularity.

It is however, that our previous paper only shows the results of V-notch problems under plane stress condition, and it is unknown whether the above conclusion is also satisfied under plane strain condition. Moreover, we have not discussed that the parameter $K_p^{\gamma=\theta}/K_x^{\gamma=\theta}$ also keeps constant value regardless of notch geometry apart from the notch tip, such as a notch depth. If the parameter $K_p^{\gamma=\theta}/K_x^{\gamma=\theta}$ is also independent of the notch depth, the plastic stress intensity factor $K_p$ for a V-notch can be estimated precisely by using the value of $K_p^{\gamma=\theta}/K_x^{\gamma=\theta}$ which is obtained previously for another V-notch with the same notch opening angle $\gamma$, hardening exponent $n$ and elastic stress intensity factor $K_e$.

In this study, the elastic-plastic stress analysis for V-notch strips with various notch depth is studied under plane strain condition, as well as plane stress condition by using MSC. Marc, a software for finite element analysis. Our attention is paid to investigate the relation between $K_p$ and $K_e$ for various notch problems.

During our investigation, the plastic and the elastic stress intensity factors, $K_p$ and $K_e$, are related with the stress component $\sigma_{\theta\theta}$ obtained from the elastic-plastic and the linear elastic stress analysis respectively, as follows:

$$
\sigma_{\theta\theta} = \frac{2(1+\nu)K_p\sigma_{\theta\theta}}{E} \left\{ \frac{\lambda_e \cos 2\pi - \gamma + \cos(\lambda_e(2\pi - \gamma))}{2\lambda_e(1 - \cos \gamma)} + \frac{(\lambda_e - 1)\theta}{1 - \cos(\lambda_e(2\pi - \gamma))} \right\}.
$$

(4)

Here,

$$
\lambda_e = \frac{(3 - \nu)(1 + \nu)}{3 - 4\nu} \quad \text{plane stress}
$$

(5)

This circular model enables one to investigate the relation between the plastic stress intensity factor $K_p$ and the elastic stress intensity factor $K_e$, with a small number of division. Here, $E$ is Young's modulus, $\nu$ is Poisson's ratio (set to 0.3), and the parameter $\lambda_e$ is the order of elastic stress singularity obtained from theoretical analysis.

In order to calculate the singular stress field precisely by using these notch models, first we divide near the notch tip into an arrangement of 6-node second-order isoparametric triangular elements. At the notch tip, the dimension of innermost ring compared to radial distance $R$ is about $10^{-8}$, as seen in Fig. 2(a), (b), and the total number of elements in these models is set about 6 000.

In our study, the material is described by $J_2$-deformation theory of Mises' yield criterion for a Ramberg-Osgood stress strain behavior. In uniaxial tension, the material deforms according to

$$
\varepsilon = \sigma + \frac{\sigma}{\sigma_0} \left( \frac{\sigma}{\sigma_0} \right)^n,
$$

(6)

where $\sigma_0$ and $\varepsilon_0$ are the yield stress and strain respectively, $a$ is the material constant and $n$ is the strain-hardening exponent. In this study, $\sigma_0/E$ and $a$ are taken to be $10^{-8}$ and 0.1 respectively. Moreover, in our plastic calculation, the proportional increment of displacements are applied to these models, and Newton-Raphson method is used to solve the present nonlinear problem.
3. Numerical Results

3.1 Calculation of $K_p$

Figures 3(a) and (b) show plastic zones developed around the V-notch of a circular model for $\gamma = 30^\circ$ and $n=3$. Here, Fig. 3(a) shows the result under plane stress condition and $\theta = 0^\circ$ (under plane stress condition) and $\theta = 90^\circ$ (under plane strain condition). Considering these spreading directions of plastic zones, the plastic stress intensity factor $K_p$ determined by Eq. (2) is calculated from $\sigma_{\text{eff}}\sigma_{\text{pl}}^\text{eff}$ under plane stress condition, and the $K_p$ under plane strain condition is calculated by using the component of effective stress $\sigma_{\text{eff}}$ for $\theta = 60^\circ$ (the ratio $\sigma_{\text{eff}}/\sigma_{\text{pl}}\sigma_{\text{eff}}$ obtained from asymptotic analysis$^{(a)}$). In Fig. 4, the angular distribution of stresses $\sigma_{\text{eff}}$ obtained from asymptotic analysis$^{(a)}$ are shown as solid lines.

Figure 5 shows the distribution of stress $\sigma_{\text{eff}}$ on direction of $\theta = 60^\circ$ ($\gamma = 30^\circ$ and $n=3$).

Figure 5 shows the distribution of stress $\sigma_{\text{eff}}$ on direction of $\theta = 60^\circ$ under plane strain condition. In this figure, the marks $\bigcirc$ and $\bullet$ distinguish between the values of $\sigma_{\text{eff}}$ in the plastic zone and the values in the elastic zone, respectively. From this figure, it is found that the stress $\sigma_{\text{eff}}$ increases in proportion to the value of $K_p$, and the slopes in the plastic and the elastic region are almost the same values of $\lambda_p$ and $\lambda_e$ obtained from asymptotic analysis$^{(a)}$. This feature is also seen under plane stress condition$^{(a)}$. Moreover, the angular distribution of stresses $\sigma_{\text{eff}}/\sigma_{\text{pl}}\text{eff}$ at distance of $r = 2.371 \times 10^{-5}$ is shown in Fig. 4 as $\bullet$ mark under $K_p/\sigma_{\text{pl}} = 0.2$. It is found from Fig. 4 that both analysis results are in a good correspond-
Fig. 6 Variation of $K_e$ and $K_p$ with $U_o$ ($\gamma=30^\circ$ and $n=3$)

dence with each other. Therefore, from these figures, we can say that our finite element analysis under plane strain condition simulates the plastic singular stress field which has the same $\lambda_0$ and $\delta_0(\theta)$ obtained from asymptotic analysis, as well as the case under plane stress condition.

3.2 Relation between $K_p$ and $K_e$ for V-notch with various notch depth

Figure 6 shows the variation of $K_e$ and $K_p$ for various applied displacement $U_o/(\ell_0/E)$ by using the notch model of Fig. 1(b). As seen in Fig. 6, the value of $K_e$ grows linearly, and the value of $K_p$ grows nonlinearly in proportion to the applied displacement $U_o/(\ell_0/E)$. Also, these values of $K_e$ and $K_p$ increases in proportion to the notch depth $t$.

Figure 7 shows the relation between $K_e$ and $K_p$ by using our numerical results shown in Fig. 6. It is found from Fig. 7 that the relation between $K_e$ and $K_p$ seems to be expressed by a curve regardless of notch depth $t$. That is, one can say that the relation between $K_e$ and $K_p$ is independent of notch depth $t$. Here, one should note that this relationship may be also satisfied for a problem over a small scale yielding. Figure 8 shows the extent of plastic zone for $t/w=0.1$ and $t/w=0.2$ corresponding to 'A' in Fig. 7. It is found from Fig. 8 that these plastic regions are different from each other, and for the case of $t/w=0.1$, the plastic region is developed all over the strip model. Nevertheless, as seen in Fig. 7, the unique relation between $K_e$ and $K_p$ for $t/w=0.1$ is still satisfied.

Figure 9 shows distributions of stress $\sigma_o$ on direction of $\theta=0^\circ$ for different notch depths.

3.3 Discussion of $K_{e\theta}^{-\lambda_0}/K_{p\theta}^{-\lambda_0}$

In order to discuss the relation between $K_e$ and...
$K_e$, first we consider two types of notch problems as shown in Fig. 10 (a) and (b). The models of Fig. 10 (a) and Fig. 10 (b) are similar in shape, and are subjected to the same load. Because of similarity in shape, the parameter $K_0^{(1)} / K_0^{(2)}$ should be constant regardless of the scale of notch model. Next, we consider the notch models of Fig. 10 (b) and Fig. 10 (c) which are not similar in shape, but only have the same notch angle $\gamma$. From the result in Fig. 7, one can say if both models have the same value of $K_e$ under linear elastic deformation, the values of $K_e$ under nonlinear plastic deformation are also the same with each other. Therefore, we can imagine whether the parameter $K_0^{(2)} / K_0^{(3)}$ may be also constant regardless of the scale of notch model or loading levels for V-notch problems which are dissimilar in shapes. In our previous paper, we have confirmed that the parameter $K_0^{(2)} / K_0^{(3)}$ under plane stress condition is constant regardless of loading level. Figure 11 shows the relation of normalized parameter $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ and $K_e$ under plane stress condition by using the circular model (see Fig. 1(a)). Here, Fig. 11 (a) shows the result for $\gamma=30^\circ$, and Fig. 11 (b) shows the result for $\gamma=90^\circ$. As seen from these figures, it is found that the parameter $K_0^{(2)} / K_0^{(3)}$ is also constant for notch of every $n$ and $\gamma$ under plane strain condition.

Table 2 shows the constant values of $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ obtained from our numerical results for some cases of $\gamma (0^\circ \leq \gamma \leq 150^\circ)$. As seen from Table 2, it is found that the parameter $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ is a function of $n$ and $\gamma$. Moreover, the values of $(K_0^{(2)})$ under plane strain condition are larger and vary more rapidly with $\gamma$ than the values under plane stress condition.

For V-notch problems with various notch depth $t$, the relations of $K_e$ and $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ are shown in Figs. 12 to 15. In Figs. 12 and 13, we show the relations of $\gamma=30^\circ$ and $90^\circ$ under plane stress condition, and in Figs. 14 and 15, the relations under plane strain condition. In these figures, the results obtained by circular notch model (Fig. 1(a)) are shown as dotted lines. As seen from these figures, the values of $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ fall in a certain range for notch of every $n$ and $\gamma$, and the difference from the values as shown in Table 2 is within 6 percent.

3.4 Application to another V-notch problem

From the above results, we can confirm that for V-notch problems, the relation between $K_e$ and $K_0$ is expressed by a unique curve for notch of every $n$ and $\gamma$, and the parameter $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ seems to be constant. Therefore, we can say that the plastic stress intensity factor $K_e$ for a V-notch problem can be obtained approximately by using the elastic stress intensity factor $K_e$ and the constant parameter $(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}$ for notch of every $n$ and $\gamma$. That is, $K_e$ can be calculated from $K_e$ by using the following equation:

$$K_e = [(K_0^{(2)} / K_0^{(3)}) \cdot \sigma_0^{(2-\gamma)}] \cdot K_0^{(1)}$$

(7)
In Eq. (7), the term in \( K_{p}^{-1}/K_{v}^{-1} \) is a constant value which is obtained previously for notch of every \( n \) and \( \gamma \). So as to confirm validity of such approximate evaluation, a V-notch problem of \( \gamma = 30^\circ \) and \( n = 3 \) subjected to 3-point bending load is studied as shown in Fig. 16, and the plastic stress intensity factor \( K_p \) around the notch tip is calculated. In Fig. 17 the variations of \( K_p \) for various load \( P \) is shown as \( \circ \) and \( \bullet \). Also in Fig. 17, the approximate results obtained from Eq. (7) are shown as solid lines. As seen from Fig. 17, the calculated \( K_p \) obtained from \( K_e \) by using Eq. (7) can give a satisfactory approximation for both plane conditions.
4. Conclusion

In our study, the plastic singular stress occurring in a V-notch strip is studied by using the finite element analysis, and the relation between the plastic stress intensity factor $K_p$ and the elastic stress intensity factor $K_e$ obtained from linear elastic analysis is discussed. Based on our numerical results, some remarkable conclusions are obtained as follows:

1. For V-notch problems with the same notch opening angle $\gamma$ and the same hardening exponent $n$, the relation between $K_p$ and $K_e$ can be shown by a unique curve regardless of notch depth. This feature can be seen for a problem over a small scale yielding condition.

2. The parameter $K_p^{1/n} / K_e^{1/n}$ is independent of notch depth and loading levels, where $1 - \lambda_e$ and $1 - \lambda_p$ are the orders of the elastic and the plastic singularity, respectively.

3. By using the parameter $K_p^{1/n} / K_e^{1/n}$ which depends on $\gamma$ and $n$, the plastic stress intensity factor $K_p$ can be evaluated approximately from the elastic stress intensity factor $K_e$.

References


