Shape Optimization for Prolonging Fatigue Life

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Shape optimization for two types of specimens in opening mode and a cantilevered beam in mixed mode was accomplished by the linear elastic fracture mechanics and the growth-strain method. Linear elastic fracture mechanics was used to evaluate the stress intensity factors and fatigue lives. And the growth-strain method was used to optimize a shape. From the results, it was found that the optimized shapes of two types of specimens and a cantilevered beam greatly prolong their fatigue lives, and the growth-strain method is an appropriate technique for shape optimization of a structure having a crack.

Key Words: Shape Optimization, Growth-Strain Method, Fatigue Life, Opening Mode, Mixed Mode

1. Introduction

Most failures occurred in machines and structures are caused by fatigue phenomena due to repeated loadings. Such repeated loadings become a direct cause of failure by growing a defect or a crack in materials for vehicles, airplanes, ships and trains, especially. Therefore, design considering fatigue must be accomplished at the beginning design stage in order to secure safety and reliability of machines and structures.

In the last decades, several studies(1)-(3) have shown that various factors such as energy dissipation, early growth of short cracks and friction contribute to fatigue crack growth. But very few studies of shape optimization for prolonging fatigue life have been reported except Gani and Rajan's study(4), which optimized a shape of a cantilevered beam by a mathematical programming technique. It should be noted that a crack growth is not independent of structural geometry.

Therefore, shape optimization for two types of specimens in opening mode and a cantilevered beam in mixed mode was accomplished by linear elastic fracture mechanics(5) and the growth-strain method(6). Linear fracture mechanics was used to evaluate the stress intensity factors and fatigue lives. The growth-strain method was used to optimize the shape. By comparing the fatigue lives of the optimized shape with those of the initial shapes, the applicability of the growth-strain method for shape optimization was verified. It is expected that the result of this study can be applied to shape optimization for the elements and structures of vehicles, airplanes, ships and trains, so that their fatigue lives can be greatly improved.

2. Calculation of the Stress Intensity Factor and Fatigue Life

2.1 Calculation of the stress intensity factor by numerical analysis

In order to calculate the stress intensity factor of a structure including a crack by numerical analysis(7)(8), the singular elements shown in Fig. 1 should be used. In this study the size of the singular elements was established as 10% of the initial crack length(9).
And the transition elements are used for the second elements surrounding a crack tip, and their magnitude was selected as the same as that of the singular elements.

There are several techniques for evaluating the stress intensity factor by numerical analysis; for example, J integral method and QPDT (Quarter Point Displacement Technique), DCT (Displacement Correlation Technique), DEBT (Displacement Extrapolation Technique). Those use the displacements at the quarter point of the singular elements. The QPDT was applied in this study.

The stress intensity factors $K_i$ and $K_{II}$ of opening mode and shearing mode can be obtained by the following Eqs. (1) and (2), respectively.

$$K_i = \sqrt{2\pi} \frac{G}{1 + \nu} \frac{|u_y - u_i|}{\sqrt{r}}$$  \hspace{1cm} (1)

$$K_{II} = \sqrt{2\pi} \frac{G}{1 + \nu} \frac{|u_x - u_i|}{\sqrt{r}}$$  \hspace{1cm} (2)

where, $G$ is the shear modulus, $u_i$ is the displacement in the $u'$ direction and $u_i'$ is the displacement in the $u'$ direction at $i$ node. $r$ is the distance from the crack tip to the $i$ node. $\nu$ is poisson’s ratio.

2.2 Calculation of the stress intensity factor by experimental equations

In order to verify the accuracy of the values of the stress intensity factors obtained by numerical analysis, those obtained by the experimental equations, which are listed in Tada’s handbook, for two typical specimens in opening mode were compared.

The experimental equations for a single edge crack specimen as shown in Fig. 2 are as follows:

$$K_i = \sigma \sqrt{\pi a} F_i(a/b)$$  \hspace{1cm} (3)

$$K_{II} = \tau \sqrt{\pi a} F_{II}(a/b)$$  \hspace{1cm} (4)

where, $\sigma$ is the tensile stress, $\tau$ is the shear stress, $a$ is the crack length, $b$ is the width of a specimen, and $F_i(a/b)$ and $F_{II}(a/b)$ are shape factors. The shape factors $F_i(a/b)$ and $F_{II}(a/b)$ are expressed by the following equations, respectively.

$$F_i(a/b) = \frac{2b}{\sin \pi a/2b} \left[0.752 + 2.02(a/b) + 0.371 \left(1 - \sin \left(\frac{\pi a}{2b}\right)\right)\cos \left(\frac{\pi a}{2b}\right)\right]$$

$$F_{II}(a/b) = 1.122 - 0.561(a/b) + 0.085(a/b)^2 + 0.180(a/b)^3$$

\[\sqrt{a - (a/b)}\]

2.3 Calculation of fatigue life

Fatigue life of a structure can be obtained by adding the values of the following two parts.

$$N = N_o + \sum_{i=1}^{n} \Delta N_i, \quad i = 1, 2, 3, \ldots, n$$  \hspace{1cm} (5)

where, the first term $N_o$ denotes the fatigue life to initial crack occurrence, the second term $\sum_{i=1}^{n} \Delta N_i$ means the fatigue life to failure caused by propagating a crack. In the case of a structure with an initial crack, $N_o$ is zero. $\sum_{i=1}^{n} \Delta N_i$ is the smallest value among the fatigue life from initial crack length $a_i$ to the critical crack length $a_x$, the fatigue life until the stress intensity factor $K_i$ at the crack tip reaches the critical stress intensity factor $K_{IC}$ and the fatigue life until the max. von Mises stress reaches the yielding stress $\sigma_y$. In order to estimate $\Delta N_i$, as a crack is growing in mixed mode, the following modified Paris law is used.

$$\Delta K_{eff} = C(a/b)^m\left(\Delta K_i + \sqrt{\Delta K_{Ii} + \Delta K_{II}}\right)$$  \hspace{1cm} (6)

$$K_{Ii} = K_{Ii, max} - K_{Ii, min}$$

$$K_{II} = K_{II, max} - K_{II, min}$$

where, $C$ and $m$ are material constants obtained from experiment, $a$ is a crack length, $da$ is an infinitesimal crack length, and $dN$ is the fatigue life until a crack propagates to $da$. Fatigue life is estimated by integra-
tion of Eq.(6 ) above.

\[ N = \int_{a_0}^{a_1} \frac{1}{C(\Delta K_{eff})^n} da \]  

(8)

The following assumptions were made in order to estimate Eq.(8).

1. The crack increment \( da \) is very small comparing with the magnitude of the structure.

2. The stress intensity factor can be approximated by a linear function of \( K = K(a) \). From the above assumptions, the stress intensity factor can be expressed by \( K = K(a) = pa + b \), and the fatigue life for the crack increment \( da \) can be obtained by the following equation.

\[ \Delta N = \frac{1}{C(1-m)p} \int_{a_0}^{a_1} (pa+b)^{-n} da \]  

(9)

\[ \Delta N = \frac{1}{C(1-m)p} [(pa+b)^{1-n}]_{a_0}^{a_1} \]  

(10)

\[ \Delta N = \frac{1}{C(1-m)p} (\Delta K_{eff}^{a_1} - \Delta K_{eff}^{a_0}) \]  

(11)

where, \( \Delta K_{eff} \) is the value of \( \Delta K_{eff} \) at the beginning of a crack increment. \( \Delta K_{eff}^{a_0} \) is the value of \( \Delta K_{eff} \) at the end of a crack increment. \( p \) is a slope defined by the following equation.

\[ p = \frac{\Delta K_{eff}^{a_1} - \Delta K_{eff}^{a_0}}{\Delta a_i} = \frac{\Delta K_{eff}^{a_1} - \Delta K_{eff}^{a_0}}{\Delta a_i} \]  

(12)

Now, Eq.(8) can be estimated using the above equations.

3. The Growth-Strain Method

In order to improve the fatigue life, the effective stress intensity factor \( \Delta K_{eff} \) in Eq.(8) must be minimized. In other words, since the stress intensity factors \( K_i \) and \( K_{II} \) are calculated by Eqs.(1) and (2), respectively, the displacements on the crack surface must be minimized by minimizing von Mises stresses at the surrounding elements in order to minimize \( \Delta K_{eff} \).

Recently, the growth-strain method\(^{(9)}\), which belongs to the optimality criteria method, has been proposed. The shape of the maximum strength or the maximum stiffness can be designed by the method without boundary parameterization and sensitivity analysis. The method has successfully been applied to shape optimization of two or three dimensional structures\(^{(10)}\). Therefore the growth-strain method, which makes the parametric variables (for example, von Mises stress) uniform, was applied in this study.

The growth-strain method optimizes a shape of a structure by volume deformation occurring in the process of making a parametric variable uniform. It consists of two steps. The first is a stress analysis step for calculating a parametric variable under the given boundary conditions. The second is a growth analysis (volume deformation) step for changing a shape based upon the calculated result.

In order to create a shape where a parametric variable is made uniform, volume deformation is developed by the growth law of Eq.(13). Reduction of the volume deformation occurs for the elements where the stress \( \sigma \) is less than the average stress \( \bar{\sigma} \). Whereas expansion of the volume deformation occurs for the elements where the stress \( \sigma \) is greater than the average stress \( \bar{\sigma} \). This procedure can be explained by Fig.3.

\[ \varepsilon_i^U = \frac{\sigma - \bar{\sigma}}{\bar{\sigma}} \delta_k h \]  

(13)

In Eq.(13), \( \varepsilon_i^U \) is volume strain, \( \sigma \) is a parametric variable (for example, von Mises stress), \( \bar{\sigma} \) is a standard value of the parametric variable (for example, average stress), \( \delta_k \) is the Kronecker delta, and \( h \) is a growth ratio, which is an arbitrary constant less than 1 and controls the magnitude of volume deformation.

If a thermal stress analysis is considered, thermal deformation for thermal isotropic materials causes only volume deformation without shear components by Eq.(14). Therefore, Eq.(13) of the growth law is very similar to Eq.(14) of thermal deformation.

\[ \varepsilon_i^U = a \Delta T \delta_i \]  

(14)

In order to control volume to the objective volume value effectively, Eq.(13) has been modified by Eq.(15) using the PIP (proportional-integral-derivative) control theory. By using Eq.(15), the stresses in a structure can be made uniform under the condition that volume is constrained. It is called volume control. The average stress in Eq.(16) is used as a standard value.

\[ \varepsilon_i^{(n)} = \frac{\sigma_i^{(n-1)} - \sigma_i^{(n-1)}}{\bar{\sigma}^{(n-1)}}, \delta_k h - \left[ R_p \frac{V^{(n-1)} - V_{obj}}{V_{obj}} \right] \]  

\[ + R_i \sum_{k=1}^{n-1} \frac{V^{(k)} - V_{obj}}{V_{obj}} + R_p \frac{V^{(n-1)} - V_{obj}}{V_{obj}} \]  

(15)

\[ \bar{\sigma}^{(n)} = \frac{1}{n} \sum_{k=1}^{n} \varepsilon_i^{(k)} V^{(k)} \]  

(16)

where, \( (n) \) denotes the \( n \)th iteration, \( R_p, R_i, R_p \) are proportional constants, \( V_{obj} \) the volume of each element, \( \sigma_i \) the representative stress of each element, \( V_{obj} \) the total objective volume, and \( V^{(n)} \) is the total volume at
the \( n \)th iteration, \( \sigma_{\text{max}} \) is the objective stress value and \( \sigma_{\text{max}}^{\text{0}} \) is the max. stress at the \( n \)th iteration.

The second term with \( R_e \) in Eq. (15) responds to the proportional behavior in PID control. The third term with \( R_i \) responds to the integral behavior and the fourth term with \( R_d \) to the derivative behavior. But since the fourth term makes the growth strain oscillating, \( R_d \) was set to be zero in this study.

4. Examples of Shape Optimization

4.1 A single edge crack specimen

The material of a single edge crack specimen was selected as AL7075-T6. The repeated tension stress \( \sigma \) having the stress width between the max. tensile stress 28.821 MPa and the min. tensile stress 0 MPa was given at the ends. As a shape constraint, only the top and bottom surfaces were allowed for changing shape. The crack length was kept in optimization process. Data for a single edge crack specimen are denoted in Table 1.

A shape optimization was accomplished by volume control for a single edge crack specimen. The objective value of volume was established by 100% of the initial volume. The growth ratio \( h \) was 0.05. The values of \( R_e, R_i, R_d \) were established as 2.7, 2.7, 0., respectively. The values of \( R_e, R_i, R_d \) were obtained empirically.

In the shape optimization process, remeshing is automatically accomplished for the surrounding elements of a crack tip at each iteration in order to prevent the premature overexpansion of them. The initial shape and given boundary conditions, and the optimized shape are shown in Figs. 4 and 5, respectively.

The change of the ratio of the volume to the initial volume, the max. von Mises stress to the initial max. von Mises stress and the stress intensity factor to the initial stress intensity factor at each iteration are shown in Fig. 6.

From the results, it was found that the volume of the optimized shape was converged to the initial

Table 1 Data for a single edge crack specimen

<table>
<thead>
<tr>
<th>Item</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E )</td>
<td>Modulus of Elasticity</td>
</tr>
<tr>
<td>( \sigma_{\text{max}} )</td>
<td>Maximum load</td>
</tr>
<tr>
<td>( \sigma_{\text{min}} )</td>
<td>Minimum load</td>
</tr>
<tr>
<td>( \nu )</td>
<td>Poisson ratio</td>
</tr>
<tr>
<td>( t )</td>
<td>Thickness</td>
</tr>
<tr>
<td>( k_{C} )</td>
<td>Material toughness</td>
</tr>
<tr>
<td>( \sigma_y )</td>
<td>Yielding stress</td>
</tr>
<tr>
<td>( a )</td>
<td>Initial crack length</td>
</tr>
</tbody>
</table>

volume as established, the max. von Mises stress was greatly reduced to 48.2% of the initial max. von Mises stress, and the stress intensity factor was also greatly reduced to 46.8% of the initial stress intensity factor.

The fatigue life was calculated by substituting, \( N_0=0, C=2.71 \times 10^{-8} \text{ mm/cycle} \), \( \gamma=3.70 \) in Eq. (13). The fatigue life of the initial shape of the specimen was evaluated as 6 cycles, whereas the fatigue life of the optimized shape was greatly improved to 105 cycles. The max. error of the stress intensity factor by numerical analysis was less than 2% for the initial shape.

4.2 A compact tension specimen

The material of a compact tension specimen was selected as AL7075-T6. The repeated load \( P \) having the load width between the max. load 4.88 kN and the min. load 0 kN was applied at the ends. As shape constraints, the top and bottom surfaces were allowed for moving horizontally and the right surface was allowed for changing shape freely. Data for a compact tension specimen are given in Table 2.

A shape optimization was accomplished by volume control for a compact tension specimen. The objective value of volume was established by 100% of the initial volume. The growth ratio \( h \) was 0.05. The

Fig. 4 Initial shape of a single edge crack specimen

Fig. 5 Optimized shape of a single edge crack specimen

Fig. 6 History of iteration for the change of volume, max. von Mises stress, and \( K_i \) of a single edge crack specimen
values of $R_r$, $R_t$, $R_o$ were established as 15, 2.0, 0., respectively. The initial shape and given boundary conditions, and the optimized shape are shown in Figs. 7 and 8, respectively.

The change of the ratio of the volume to the initial volume, the max. von Mises stress to the initial max. von Mises stress and the stress intensity factor to the initial stress intensity factor at each iteration are shown in Fig. 9.

From the results, it was found that the volume of the optimized shape was converged to the initial volume as established, the max. von Mises stress was greatly reduced to 82.8% of the initial max. von Mises stress, and the stress intensity factor was also greatly reduced to 73.6% of the initial stress intensity factor.

The fatigue life was calculated by substituting $N_a=0, \ C=2.71 \times 10^{-8}\frac{\text{mm}}{\text{cycle}}\frac{\text{MPa}^m}{\sqrt{m}}, \ m=3.70$ into Eq.(13). The fatigue life of the initial shape of the specimen was evaluated as 66 cycles, whereas the fatigue life of the optimized shape was greatly improved to 245 cycles. The max. error of the stress intensity factor by numerical analysis was less than 4% for the initial shape.

### 4.3 A cantilevered beam

Data for a shape optimization of a cantilevered beam were taken from Gani and Rajan's paper(1). One end of a cantilevered beam was fixed, and the repeated shear stress $r$ between the max. shear stress 5.171 MPa and the min. shear stress $-5.171$ MPa was applied. As shape constraints, the right surface was allowed for moving vertically and the top and bottom surfaces were allowed for changing shape freely. Data for a cantilevered beam is given in Table 3.

A shape optimization was accomplished by volume control for a cantilevered beam. The objective value of volume was established by 100% of the initial volume.

### Table 2 Data for compact tension specimen

<table>
<thead>
<tr>
<th>Item</th>
<th>Value</th>
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<tr>
<td>$E$</td>
<td>Modulus of Elasticity</td>
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<tr>
<td>$P_{max}$</td>
<td>Maximum force</td>
</tr>
<tr>
<td>$P_{min}$</td>
<td>Minimum force</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson ratio</td>
</tr>
<tr>
<td>$t$</td>
<td>Thickness</td>
</tr>
<tr>
<td>$K_c$</td>
<td>Material toughness</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Yielding stress</td>
</tr>
<tr>
<td>$a$</td>
<td>Initial crack length</td>
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</tbody>
</table>

### Table 3 Data for a cantilevered beam

<table>
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<th>Value</th>
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</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Modulus of Elasticity</td>
</tr>
<tr>
<td>$r_{max}$</td>
<td>Maximum shear load</td>
</tr>
<tr>
<td>$r_{min}$</td>
<td>Minimum shear load</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Poisson ratio</td>
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<tr>
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</tr>
<tr>
<td>$a$</td>
<td>Initial crack length</td>
</tr>
</tbody>
</table>

Fig. 7 Initial Shape of a compact tension specimen

Fig. 8 Optimized shape of a compact tension specimen

Fig. 9 History of iteration for the change of volume, max. von Mises stress, and $K_c$ of a compact tension specimen

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The growth ratio $h$ was 0.1. The values of $R_r$, $R_i$, $R_o$ were established as 6, 20, 0, respectively. The initial shape and given boundary conditions, and the optimized shape are shown in Figs. 10 and 11, respectively.

The change of the ratio of the volume to the initial volume, the max. von Mises stress to the initial max. von Mises stress and the stress intensity factor $K_r$ to the initial stress intensity factor $K_i$ at each iteration are shown in Fig. 12. Also the changes of the stress intensity factors $K_i$ and $K_r$ normalized by the initial stress intensity factor $K_i$ at each iteration are shown in Fig. 13.

From the results, it was found that the volume of the optimized shape was converged to the initial volume as established, the max. von Mises stress was greatly reduced to 66.7% of the initial max. von Mises stress. The stress intensity factor $K_i$ was also greatly reduced to 67.1% of the initial stress intensity factor $K_i$, whereas the stress intensity factor $K_r$ is slightly increased by 0.03% of the initial intensity factor $K_i$.

The max. principle stress theory was used for determining the propagating direction of the crack in mixed mode. The fatigue life was calculated by substituting $N_0=0$, $C=2.8 \times 10^{-12}$ mm/cycle, $m=4.9$ into Eq.(13). The fatigue life of the initial shape of a cantilevered beam was evaluated as 18 cycles, whereas the fatigue life of the optimized shape was greatly improved to 148 cycles.

It is difficult to compare this result with Gani and Rajan's directly even though the same cantilevered beam was optimized under the same conditions. The shape optimization of Gani and Rajan was accomplished by the mathematical programming method. After a half shape of the cantilevered beam was optimized by use of Bezier curve with four control points on the top surface, the entire shape was determined by symmetry. Whereas the shape optimization of this study was accomplished only by the growth law of the growth-strain method. Although the two optimized shapes are very similar to each other, the optimized shape in this study is not symmetric as was Gani and Rajan’s result.

With regard to the fatigue life point, the fatigue life of Gani and Rajan’s shape was improved from 14 cycles to 430 cycles, but that of this study from 18 cycles to 148 cycles. The difference between the two results seems to be caused by the different optimized
shapes and numerical techniques. In evaluating the fatigue life, Gani and Rajan used a special rosette element consisting of six singular elements. The magnitude of the rosette element was established as 50% of the initial crack length, whereas the magnitude of the singular element in this study was established as 10% of the initial crack length. Since the numerical error is generally caused by the magnitude of the singular and transition elements, such difference made a big difference in the fatigue life.

Therefore, it seems that the shape optimization by the growth-strain method gives more practical and better results since the magnitude of the singular and transition elements are smaller than Gani and Rajan’s.

5. Conclusions

The following conclusions were obtained by accomplishing a shape optimization using the growth-strain method for prolonging the fatigue life of a structure in this study.

① The fatigue lives of the structures having a crack were greatly improved by shape optimizations using the growth-strain method.

② The stress intensity factor $K_{\text{eff}}$ in mixed mode was influenced by the stress intensity factor $K_I$ in shearing mode at the beginning of propagating a crack. However, as the crack grew, the stress intensity factor $K_{\text{eff}}$ in mixed mode was close to the intensity factor $K_I$ in opening mode.

③ It was verified that the growth-strain method is an appropriate technique for shape optimization of a structure having a crack.

References


