An Exact Exponential Theory for the Bending of Simply Supported Laminated Orthotropic Plates*

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A laminated orthotropic plate theory is developed on the assumption that inplane displacements vary exponentially across plate thickness. Analytical solutions are obtained for simply supported, symmetric and anti-symmetric cross-ply rectangular laminates under transverse loading. The numerical results are compared with solutions of the three-dimensional elasticity theory, the third-order shear deformation theory, and the first-order shear deformation theory. It is found that the present exact exponential theory provides very accurate displacements and stresses in comparison to the three-dimensional elasticity theory. In particular, both transverse shear from constitutive equations and transverse normal stresses from equilibrium equations are more accurate than previously developed theories, even for small length-to-thickness ratios.

Key Words: Laminated Plate Theory, Transverse Shear Stress, Transverse Normal Stress, Shear Deformation

1. Introduction

The classical laminated plate theory (CLPT)^{(1),(2)}, which has been widely used for the analysis of composite structures, produces unsatisfactory results in case of the high ratio of elastic modulus to shear modulus and the low length-to-thickness ratio. The inaccuracy of the CLPT results from neglecting transverse shear deformation. To include the shear deformation in the bending of laminated plates, the first-order shear deformation theories⁽³⁾⁻⁽⁵⁾ have been constructed on the basis of the Reissner-Mindlin plate theory. In the analysis of relatively thick laminated plates, the first-order shear deformation theory (FSDT) presents more accurate deflections, natural frequencies and critical buckling loads than the CLPT. The FSDT improved somewhat the accuracies of inplane stresses and transverse shear stresses (calculated from constitutive equations). The

FSDT requires shear correction factors depending on the mechanical properties and stacking sequences of the laminated plates.

Many higher-order plate theories have been developed to evaluate the transverse shear deformation of the laminated plates more accurately. Most of the higher-order plate theories are based on the third-order shear deformation theory (TSDT) that expands inplane displacements up to a cubic term in the thickness coordinate⁽⁶⁾⁻⁽⁸⁾. In a typical TSDT suggested by Reddy⁽⁸⁾, displacement field and associated equilibrium equations have been consistently derived. The displacement field has been selected such that transverse shear stresses are vanished on the top and bottom surfaces of the plate, and consequently it has the same number of unknowns as in the FSDT. However, the transverse shear stresses in the TSDT theories⁽⁶⁾⁻⁽⁸⁾ do not satisfy the continuity conditions at layer interfaces when they are obtained from the constitutive equations. Zigzag varying inplane displacements through thickness⁽⁹⁾⁻⁽¹³⁾ were used to obtain better transverse shear stresses. The zigzag theories provide slightly better transverse shear stresses than the higher-order plate theories⁽⁶⁾⁻⁽⁸⁾ using the constitutive

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equations, instead of the three-dimensional equilibrium equations

The objective of the present paper is to develop a theory for the bending of laminated orthotropic rectangular plates with simply supported boundary conditions by using exact exponential inplane displacements. Most of previous plate theories assumed inplane displacements as functions of polynomials. In this paper, the in-plane displacements are assumed as exponential functions in order to predict more accurately the transverse shear deformation of a plate. Compared to the three dimensional elastic theory, transverse displacement $w(x, y)$ in our theory is assumed as functions of x and y only. Hence, our theory is more simpler than the three dimensional elastic theory. Analytical solutions for the simply supported plates will be obtained and compared with various other solutions such as the three dimensional elasticity solution⁽¹⁴⁾, the third-order shear deformation theory⁽⁸⁾, and the first-order shear deformation theory^{(15),(16)}.

2. Displacement Field

For the bending of a laminated plate composed of N orthotropic layers, it is assumed that the principal axes of the laminate coincide with Cartesian coordinates x, y , and z . The coordinate z is normal to the middle plane defined by x and y as shown in Fig. 1. Equilibrium equations in the k th layer of the plate are expressed as

$$\frac{\partial \sigma_{xx}^{(k)}}{\partial x} + \frac{\partial \sigma_{xy}^{(k)}}{\partial y} + \frac{\partial \sigma_{xz}^{(k)}}{\partial z} = 0 \quad (1.a)$$

$$\frac{\partial \sigma_{xy}^{(k)}}{\partial x} + \frac{\partial \sigma_{yy}^{(k)}}{\partial y} + \frac{\partial \sigma_{yz}^{(k)}}{\partial z} = 0 \quad (1.b)$$

Fig. 1 Geometry of a laminated plate

$$\frac{\partial \sigma_{xz}^{(k)}}{\partial x} + \frac{\partial \sigma_{yz}^{(k)}}{\partial y} + \frac{\partial \sigma_{zz}^{(k)}}{\partial z} = 0 \quad (1.c)$$

In general, the transverse normal stress σ_{zz} is so small as to be negligible in comparison with other normal stresses. The constitutive equations in the k th orthotropic layer can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^{(k)} \quad (2)$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \quad (3)$$

where Q_{ij} are the plane-stress-reduced elastic constants defined by $Q_{ij} = C_{ij} - C_{i3}C_{j3}/C_{33}$ for $i, j = 1, 2, 6$ and $Q_{ij} = C_{ij}$ for $i, j = 4, 5$ where C_{ij} represent three-dimensional elastic constants. The strain-displacement relations take the following form

$$\begin{aligned} \varepsilon_{xx}^{(k)} &= \frac{\partial u^{(k)}}{\partial x}, \quad \varepsilon_{yy}^{(k)} = \frac{\partial v^{(k)}}{\partial y}, \quad \varepsilon_{zz}^{(k)} = \frac{\partial w^{(k)}}{\partial z} \\ \gamma_{xy}^{(k)} &= \frac{\partial v^{(k)}}{\partial x} + \frac{\partial u^{(k)}}{\partial y}, \quad \gamma_{yz}^{(k)} = \frac{\partial w^{(k)}}{\partial y} + \frac{\partial v^{(k)}}{\partial z}, \\ \gamma_{xz}^{(k)} &= \frac{\partial w^{(k)}}{\partial x} + \frac{\partial u^{(k)}}{\partial z} \end{aligned} \quad (4)$$

where $u^{(k)}$, $v^{(k)}$ and $w^{(k)}$ represent the x, y, z components of displacement, respectively.

The displacement fields in the k th layer are assumed to be of the form including exponential terms

$$\begin{aligned} u^{(k)}(x, y, z) &= u^o(x, y) + A^{(k)} e^{s^{(k)}z} \frac{\partial w}{\partial x} \\ v^{(k)}(x, y, z) &= v^o(x, y) + A^{(k)} \lambda^{(k)} e^{s^{(k)}z} \frac{\partial w}{\partial y} \\ w^{(k)}(x, y) &= w(x, y) \end{aligned} \quad (5)$$

where u^o and v^o represent the x, y displacement components of the middle plane, respectively, and $A^{(k)}$, $\lambda^{(k)}$, and $s^{(k)}$ are constants to be determined later.

Substitution of Eqs. (2) - (5) into Eqs. (1.a) and (1.b) leads to

$$\begin{aligned} Q_{11} \frac{\partial^2 u^o}{\partial x^2} + Q_{66} \frac{\partial^2 u^o}{\partial y^2} + (Q_{12} + Q_{66}) \frac{\partial^2 v^o}{\partial x \partial y} \\ + \left[Q_{11} \frac{\partial^3 w}{\partial x^3} + Q_{66} \frac{\partial^3 w}{\partial x \partial y^2} + Q_{55} s^2 \frac{\partial w}{\partial x} \right. \\ \left. + \lambda (Q_{12} + Q_{66}) \frac{\partial^3 w}{\partial x \partial y^2} \right] A e^{sz} = 0 \end{aligned} \quad (6)$$

$$\begin{aligned} (Q_{12} + Q_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + Q_{66} \frac{\partial^2 v^o}{\partial x^2} + Q_{22} \frac{\partial^2 v^o}{\partial y^2} \\ + \left[(Q_{12} + Q_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + \lambda \left(Q_{66} \frac{\partial^3 w}{\partial x^2 \partial y^2} \right. \right. \\ \left. \left. + Q_{22} \frac{\partial^3 w}{\partial y^3} + Q_{44} s^2 \frac{\partial w}{\partial y} \right) \right] A e^{sz} = 0 \end{aligned} \quad (7)$$

where the superscript (k) is omitted for convenience. Since A is arbitrary, the following equations can be derived from Eqs. (6) and (7).

$$Q_{11} \frac{\partial^2 u^o}{\partial x^2} + Q_{66} \frac{\partial^2 u^o}{\partial y^2} + (Q_{12} + Q_{66}) \frac{\partial^2 v^o}{\partial x \partial y} = 0 \quad (8)$$

$$(Q_{12} + Q_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + Q_{66} \frac{\partial^2 v^o}{\partial x^2} + Q_{22} \frac{\partial^2 v^o}{\partial y^2} = 0 \quad (9)$$

$$Q_{11} \frac{\partial^3 w}{\partial x^3} + Q_{66} \frac{\partial^3 w}{\partial x \partial y^2} + Q_{55} s^2 \frac{\partial w}{\partial x} + \lambda (Q_{12} + Q_{66}) \frac{\partial^3 w}{\partial x \partial y^2} = 0 \quad (10)$$

$$(Q_{12} + Q_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} + \lambda \left(Q_{66} \frac{\partial^3 w}{\partial x^2 \partial y} + Q_{22} \frac{\partial^3 w}{\partial y^3} + Q_{44} s^2 \frac{\partial w}{\partial y} \right) = 0 \quad (11)$$

If the transverse displacement w is known, the constants λ and s can be obtained from Eqs. (10) and (11). A double Fourier sine series is assumed for the transverse displacement as follows

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (12)$$

where a and b denote plate lengths in x and y coordinates, respectively. Substitution of this equation into Eqs. (10) and (11) leads to

$$Q_{11} p^2 + Q_{66} q^2 - Q_{55} s^2 + \lambda (Q_{12} + Q_{66}) q^2 = 0 \quad (13)$$

$$(Q_{12} + Q_{66}) p^2 + \lambda (Q_{66} p^2 + Q_{22} q^2 - Q_{44} s^2) = 0 \quad (14)$$

where $p = m\pi/a$ and $q = n\pi/b$. Solving these algebraic equations, the following equations are obtained.

$$as^4 - bs^2 + c = 0 \quad (15)$$

$$\lambda = \frac{(Q_{12} + Q_{66}) p^2}{Q_{44} s^2 - Q_{66} p^2 - Q_{22} q^2} \quad (16)$$

The coefficients in Eq. (15) are defined by

$$\begin{aligned} a &= Q_{44} Q_{55} \\ b &= (Q_{11} Q_{44} + Q_{55} Q_{66}) p^2 + (Q_{22} Q_{55} + Q_{44} Q_{66}) q^2 \\ c &= Q_{11} Q_{66} p^4 + (Q_{11} Q_{22} - Q_{12}^2 - 2Q_{12} Q_{66}) p^2 q^2 + Q_{22} Q_{66} q^4 \end{aligned} \quad (17)$$

The four roots of Eq. (15) can be expressed as

$$s = \pm \sqrt{\frac{b + \sqrt{b^2 - 4ac}}{2a}}, \quad \pm \sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}} \quad (18)$$

It is easily proved that the coefficients a and b in Eq. (17) are positive. Since c and $b^2 - 4ac$ are positive in actual composite materials, the roots in Eq. (18) are always real. The value of λ is obtained by substituting Eq. (18) into Eq. (16).

Substituting Eqs. (16) and (18) into Eq. (5), the displacement field in the k th layer of the plate can be written as

$$\begin{aligned} u^{(k)} &= u^o(x, y) + \left(\alpha_1^{(k)} e^{s_1^{(k)} z} + \delta_1^{(k)} e^{-s_1^{(k)} z} + \alpha_2^{(k)} e^{s_2^{(k)} z} + \delta_2^{(k)} e^{-s_2^{(k)} z} \right) \frac{\partial w}{\partial x} \\ \nu^{(k)} &= \nu^o(x, y) + \left(\lambda_1^{(k)} \alpha_1^{(k)} e^{s_1^{(k)} z} + \lambda_1^{(k)} \delta_1^{(k)} e^{-s_1^{(k)} z} + \lambda_2^{(k)} \alpha_2^{(k)} e^{s_2^{(k)} z} + \lambda_2^{(k)} \delta_2^{(k)} e^{-s_2^{(k)} z} \right) \frac{\partial w}{\partial y} \end{aligned} \quad (19)$$

where $s_1^{(k)}$ and $s_2^{(k)}$ are positive roots in Eq. (18) and $\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ are determined from Eq. (16) for $s_1^{(k)}$ and $s_2^{(k)}$, respectively. The coefficients $\alpha_1^{(k)}$, $\alpha_2^{(k)}$, $\delta_1^{(k)}$ and $\delta_2^{(k)}$ in Eq. (19) are determined using continuity conditions that must be satisfied at the interface of layers. The continuity conditions for the displacements and transverse shear stresses are defined between the k th layer and the $(k+1)$ th layer as

$$u^{(k)} = u^{(k+1)}, \quad \nu^{(k)} = \nu^{(k+1)} \\ \sigma_{xz}^{(k)} = \sigma_{xz}^{(k+1)}, \quad \sigma_{yz}^{(k)} = \sigma_{yz}^{(k+1)} \quad k=1, 2, \dots, N-1 \quad (20)$$

The transverse shear stresses are zero at the upper and lower surfaces of the plate

$$\begin{aligned} \sigma_{xz}^{(1)} &= 0, \quad \sigma_{yz}^{(1)} = 0 \quad \text{at } z = -\frac{h}{2} \\ \sigma_{xz}^{(N)} &= 0, \quad \sigma_{yz}^{(N)} = 0 \quad \text{at } z = \frac{h}{2} \end{aligned} \quad (21)$$

where h denotes the plate thickness. Since the laminated plate is composed of N layers, the total number of unknowns for $\alpha_1^{(k)}$, $\alpha_2^{(k)}$, $\delta_1^{(k)}$ and $\delta_2^{(k)}$ in Eq. (19) is $4N$ and the number of equations given by Eqs. (20) and (21) is also $4N$. Therefore, the values of $\alpha_1^{(k)}$, $\alpha_2^{(k)}$, $\delta_1^{(k)}$ and $\delta_2^{(k)}$ are easily determined by solving simultaneous equations constructed by Eqs. (20) and (21).

Substituting Eq. (19) into Eq. (4), the strains in the k th layer can be rewritten in terms of the displacements as

$$\begin{aligned} \varepsilon_{xx}^{(k)} &= \varepsilon_{xx}^o + F_1^{(k)} \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy}^{(k)} = \varepsilon_{yy}^o + F_2^{(k)} \frac{\partial^2 w}{\partial y^2}, \\ \varepsilon_{zz}^{(k)} &= 0 \\ \gamma_{xy}^{(k)} &= \gamma_{xy}^o + (F_1^{(k)} + F_2^{(k)}) \frac{\partial^2 w}{\partial x \partial y}, \\ \gamma_{yz}^{(k)} &= F_4^{(k)} \frac{\partial w}{\partial y}, \quad \gamma_{xz}^{(k)} = F_5^{(k)} \frac{\partial w}{\partial x} \end{aligned} \quad (22)$$

where

$$\begin{aligned} \varepsilon_{xx}^o &= \frac{\partial u^o}{\partial x}, \quad \varepsilon_{yy}^o = \frac{\partial \nu^o}{\partial y}, \quad \gamma_{xy}^o = \frac{\partial \nu^o}{\partial x} + \frac{\partial u^o}{\partial y} \\ F_1^{(k)} &= \alpha_1^{(k)} e^{s_1^{(k)} z} + \delta_1^{(k)} e^{-s_1^{(k)} z} + \alpha_2^{(k)} e^{s_2^{(k)} z} + \delta_2^{(k)} e^{-s_2^{(k)} z} \\ F_2^{(k)} &= \lambda_1^{(k)} \alpha_1^{(k)} e^{s_1^{(k)} z} + \lambda_1^{(k)} \delta_1^{(k)} e^{-s_1^{(k)} z} + \lambda_2^{(k)} \alpha_2^{(k)} e^{s_2^{(k)} z} + \lambda_2^{(k)} \delta_2^{(k)} e^{-s_2^{(k)} z} \\ F_4^{(k)} &= \frac{dF_2^{(k)}}{dz} + 1 \\ F_5^{(k)} &= \frac{dF_1^{(k)}}{dz} + 1 \end{aligned} \quad (23)$$

The stresses in the k th layer can be obtained by substituting Eq. (22) into Eqs. (2) and (3).

3. Equilibrium Equations

For a laminated plate under static bending, equilibrium equations and associated boundary conditions are derived from the principle of virtual work defined as

$$\begin{aligned} \iiint_V (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \sigma_{xy} \delta \gamma_{xy} \\ + \sigma_{yz} \delta \gamma_{yz} + \sigma_{xz} \delta \gamma_{xz}) dx dy dz = \iint_{\Omega} q \delta w dx dy \end{aligned} \quad (25)$$

where q is the transversely distributed load applied on the plate surface, and V and Ω represent the volume and area of the plate, respectively. Note that the superscript (k) is omitted again for convenience. Substituting Eqs. (4) and (19) into Eq. (25) and integrating it by parts, the following equation can be obtained.

$$\begin{aligned}
& \iint_{\Omega} \left\{ -\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u^o - \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v^o \right\} dx dy \\
& + \iint_{\Omega} \left\{ \int_{-h/2}^{h/2} \left(F_1 \frac{\partial^2 \sigma_{xx}}{\partial x^2} + (F_1 + F_2) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \right. \right. \\
& + F_2 \frac{\partial^2 \sigma_{yy}}{\partial y^2} - F_4 \frac{\partial \sigma_{yz}}{\partial y} - F_5 \frac{\partial \sigma_{xz}}{\partial x} \left. \right) dz - q \left. \right\} \delta w \, dx dy \\
& + \int_{\Gamma_x} \left\{ N_x \delta u^o + N_{xy} \delta v^o \right. \\
& + \left(\frac{\partial \bar{M}_x}{\partial x} + 2 \frac{\partial \bar{M}_{xy}}{\partial y} + \bar{Q}_x \right) \delta w + \bar{M}_x \frac{\partial \delta w}{\partial x} \left. \right\} dy \\
& + \int_{\Gamma_y} \left\{ N_{xy} \delta u^o + N_y \delta v^o \right. \\
& + \left(2 \frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_y}{\partial y} + \bar{Q}_y \right) \delta w + \bar{M}_y \frac{\partial \delta w}{\partial y} \left. \right\} dx = 0
\end{aligned} \quad (26)$$

where Γ_x and Γ_y are defined along the edges $x=\text{constant}$ and $y=\text{constant}$, respectively, and other variables are defined by

$$(N_x, N_{xy}, N_y) = \int_{-h/2}^{h/2} (\sigma_{xx}, \sigma_{xy}, \sigma_{yy}) dz \quad (27)$$

$$\bar{M}_x = - \int_{-h/2}^{h/2} F_1 \sigma_{xx} dz, \quad \bar{M}_y = - \int_{-h/2}^{h/2} F_2 \sigma_{yy} dz, \quad (28)$$

$$\bar{M}_{xy} = - \int_{-h/2}^{h/2} \frac{1}{2} (F_1 + F_2) \sigma_{xy} dz$$

$$\bar{Q}_x = \int_{-h/2}^{h/2} F_5 \sigma_{xz} dz, \quad \bar{Q}_y = \int_{-h/2}^{h/2} F_4 \sigma_{yz} dz \quad (29)$$

Since δu^o , δv^o and δw take arbitrary values, the equilibrium equations are obtained from Eq.(26) as

$$\delta u^o: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (30)$$

$$\delta v^o: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (31)$$

$$\begin{aligned}
\delta w: & \int_{-h/2}^{h/2} \left[F_1 \frac{\partial^2 \sigma_{xx}}{\partial x^2} + (F_1 + F_2) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \right. \\
& + F_2 \frac{\partial^2 \sigma_{yy}}{\partial y^2} - F_4 \frac{\partial \sigma_{yz}}{\partial y} - F_5 \frac{\partial \sigma_{xz}}{\partial x} \left. \right] dz = q \quad (32)
\end{aligned}$$

and the boundary conditions along the plate edges are obtained as

$$\begin{aligned}
& u^o \text{ or } N_x \\
& v^o \text{ or } N_{xy} \\
& w \text{ or } \frac{\partial \bar{M}_x}{\partial x} + 2 \frac{\partial \bar{M}_{xy}}{\partial y} + \bar{Q}_x \text{ on } \Gamma_x \quad (33)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial w}{\partial x} \text{ or } \bar{M}_x \\
& u^o \text{ or } N_{xy} \\
& v^o \text{ or } N_y \\
& w \text{ or } 2 \frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_y}{\partial y} + \bar{Q}_y \text{ on } \Gamma_y \quad (34) \\
& \frac{\partial w}{\partial y} \text{ or } \bar{M}_y
\end{aligned}$$

Equations (30)-(32) can be written in terms of the displacements as

$$\begin{aligned}
& \int_{-h/2}^{h/2} \left[Q_{11} \frac{\partial^2 u^o}{\partial x^2} + Q_{66} \frac{\partial^2 u^o}{\partial y^2} + (Q_{12} + Q_{66}) \frac{\partial^2 v^o}{\partial x \partial y} \right] dz \\
& + \int_{-h/2}^{h/2} \left\{ Q_{11} F_1 \frac{\partial^3 w}{\partial x^3} + [Q_{12} F_2 + Q_{66} (F_1 + F_2)] \right. \\
& \times \left. \frac{\partial^3 w}{\partial x \partial y^2} \right\} dz = 0 \quad (35)
\end{aligned}$$

$$\begin{aligned}
& \int_{-h/2}^{h/2} \left[(Q_{12} + Q_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + Q_{66} \frac{\partial^2 v^o}{\partial x^2} + Q_{22} \frac{\partial^2 v^o}{\partial y^2} \right] dz \\
& + \int_{-h/2}^{h/2} \left\{ [Q_{12} F_1 + Q_{66} (F_1 + F_2)] \frac{\partial^3 w}{\partial x^2 \partial y} \right. \\
& + Q_{22} F_2 \frac{\partial^3 w}{\partial y^3} \left. \right\} dz = 0 \quad (36)
\end{aligned}$$

$$\begin{aligned}
& \int_{-h/2}^{h/2} F_1 \frac{\partial}{\partial x} \left[Q_{11} \frac{\partial^2 u^o}{\partial x^2} + Q_{66} \frac{\partial^2 u^o}{\partial y^2} + (Q_{12} + Q_{66}) \right. \\
& \times \left. \frac{\partial^2 v^o}{\partial x \partial y} \right] dz + \int_{-h/2}^{h/2} F_2 \frac{\partial}{\partial y} \left[(Q_{12} + Q_{66}) \frac{\partial^2 u^o}{\partial x \partial y} \right. \\
& + Q_{66} \frac{\partial^2 v^o}{\partial x^2} + Q_{22} \frac{\partial^2 v^o}{\partial y^2} \left. \right] dz + \int_{-h/2}^{h/2} Q_{11} F_1^2 \frac{\partial^4 w}{\partial x^4} \\
& + \int_{-h/2}^{h/2} [2 Q_{12} F_1 F_2 + Q_{66} (F_1 + F_2)^2] \frac{\partial^4 w}{\partial x^2 \partial y^2} \\
& + \int_{-h/2}^{h/2} Q_{22} F_2^2 \frac{\partial^4 w}{\partial y^4} - \int_{-h/2}^{h/2} Q_{55} F_5^2 \frac{\partial^2 w}{\partial x^2} \\
& - \int_{-h/2}^{h/2} Q_{44} F_4^2 \frac{\partial^2 w}{\partial y^2} = q \quad (37)
\end{aligned}$$

In these equations, the middle plane displacements u^o , v^o and the transverse displacement w are uncoupled each other. By applying Eqs.(8), (9) to Eqs.(35), (36), the following conditions can be obtained.

$$\begin{aligned}
& \int_{-h/2}^{h/2} \left\{ Q_{11} F_1 \frac{\partial^3 w}{\partial x^3} + [Q_{12} F_2 + Q_{66} (F_1 + F_2)] \frac{\partial^3 w}{\partial x \partial y^2} \right\} dz = 0 \\
& \int_{-h/2}^{h/2} \left\{ [Q_{12} F_1 + Q_{66} (F_1 + F_2)] \frac{\partial^3 w}{\partial x^2 \partial y} + Q_{22} F_2 \frac{\partial^3 w}{\partial y^3} \right\} dz = 0 \quad (38)
\end{aligned}$$

By substituting Eq.(38) into Eqs.(35), (36) and then Eqs.(8), (9) into Eq.(37), the equilibrium equations for the laminated orthotropic plate subjected to static bending take the simpler form

$$A_{11} \frac{\partial^2 u^o}{\partial x^2} + A_{66} \frac{\partial^2 u^o}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 v^o}{\partial x \partial y} = 0 \quad (39)$$

$$(A_{12} + A_{66}) \frac{\partial^2 u^o}{\partial x \partial y} + A_{66} \frac{\partial^2 v^o}{\partial x^2} + A_{22} \frac{\partial^2 v^o}{\partial y^2} = 0 \quad (40)$$

$$\begin{aligned}
& \bar{D}_{11} \frac{\partial^4 w}{\partial x^4} + 2(\bar{D}_{12} + 2\bar{D}_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \bar{D}_{22} \frac{\partial^4 w}{\partial y^4} \\
& - H_{55} \frac{\partial^2 w}{\partial x^2} - H_{44} \frac{\partial^2 w}{\partial y^2} = q \quad (41)
\end{aligned}$$

where A_{ij} are the inplane moduli

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ij} dz \quad (42)$$

and $\bar{D}_{ij}$ and H_{ij} are defined by

$$\begin{aligned}
& \bar{D}_{11} = \int_{-h/2}^{h/2} Q_{11} F_1^2 dz, \quad \bar{D}_{22} = \int_{-h/2}^{h/2} Q_{22} F_2^2 dz, \\
& \bar{D}_{12} = \int_{-h/2}^{h/2} Q_{12} F_1 F_2 dz \\
& \bar{D}_{66} = \frac{1}{2} \int_{-h/2}^{h/2} Q_{66} (F_1 + F_2)^2 dz, \\
& H_{44} = \int_{-h/2}^{h/2} Q_{44} F_4^2 dz, \quad H_{55} = \int_{-h/2}^{h/2} Q_{55} F_5^2 dz \quad (43)
\end{aligned}$$

Equations (39) - (41) are similar to the equilibrium equations derived from the classical laminated plate theory except $H_{55} \frac{\partial^2 w}{\partial x^2}$ and $H_{44} \frac{\partial^2 w}{\partial y^2}$ in Eq.(41), which are related to the transverse shear deformation.

4. Solution Procedure

Here analytical solutions of Eqs.(39) - (41) are obtained for simply supported laminated rectangular plates. The boundary conditions are obtained from Eqs.(33) and (34) as follows.

$$w = N_x = N_{xy} = \bar{M}_x = 0 \quad \text{at } x=0 \text{ and } a \text{ (on } \Gamma_x) \quad (44)$$

$$w = N_y = N_{xy} = \bar{M}_y = 0 \quad \text{at } y=0 \text{ and } b \text{ (on } \Gamma_y) \quad (45)$$

The applied transverse load q is expanded in the double Fourier sine series as

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (46)$$

The displacements are assumed as

$$\begin{aligned} u^o &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ \nu^o &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \quad (47)$$

These displacements result in zero values for σ_{xx} and σ_{xy} along $x=0$ and a , and for σ_{yy} and σ_{xy} along $y=0$ and b . Hence, Eq.(47) satisfies all the boundary conditions given by Eqs.(44) and (45). Then, substitution of Eqs.(46) and (47) into Eqs.(39) - (41) leads to

$$\begin{aligned} U_{mn} &= V_{mn} = 0 \\ \left[\bar{D}_{11} \left(\frac{m\pi}{a} \right)^4 + 2(\bar{D}_{12} + 2\bar{D}_{66}) \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 \right. \\ &\quad \left. + \bar{D}_{22} \left(\frac{n\pi}{b} \right)^4 + H_{55} \left(\frac{m\pi}{a} \right)^2 + H_{44} \left(\frac{n\pi}{b} \right)^2 \right] W_{mn} \\ &= q_{mn} \end{aligned} \quad (48)$$

Hence the deflection w is expressed as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{q_{mn}}{D} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (50)$$

where D denotes the bracket in the left-hand side of Eq.(49).

5. Numerical Results and Discussion

Symmetric and anti-symmetric cross-ply laminates are suggested to estimate the accuracy of the present theory for laminated plates with simply supported boundary conditions. The laminated plates are graphite/epoxy composites and composed of layers with equal thickness. Mechanical properties of the layer^{(4),(5)} are

$$\begin{aligned} E_{11} &= 174.6 \text{ GPa } (25 \times 10^6 \text{ psi}), \\ E_{22} &= E_{33} = 7 \text{ GPa } (1 \times 10^6 \text{ psi}), \\ G_{12} &= G_{13} = 3.5 \text{ GPa } (0.5 \times 10^6 \text{ psi}), \\ G_{23} &= 1.4 \text{ GPa } (0.2 \times 10^6 \text{ psi}), \\ \nu_{12} &= \nu_{13} = \nu_{23} = 0.25 \end{aligned}$$

The transverse load q applied on the upper surface of the plate is assumed to be sinusoidal.

$$q = q_o \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (51)$$

For $[0/90]_s$ symmetric cross-ply square laminate, the variations of deflections and stresses for length-to-thickness ratios of 4, 10, 20 and 100 are presented in Table 1. The deflections and stresses from the present theory are compared with those of the three dimensional elasticity theory⁽¹⁴⁾, the third-order shear deformation theory⁽⁸⁾ and the first-order shear deformation theory⁽¹⁵⁾. The deflections and stresses shown in Table 1 are nondimensionalized as

$$\begin{aligned} \bar{w} &= w(a/2, b/2) \frac{100 E_{22} h^3}{q_o a^4} \\ \bar{\sigma}_{xx} &= \sigma_{xx}(a/2, b/2, h/2) \frac{h^2}{q_o a^2}, \\ \bar{\sigma}_{yy} &= \sigma_{yy}(a/2, b/2, h/4) \frac{h^2}{q_o a^2}, \\ \bar{\sigma}_{zz} &= \sigma_{zz}(a/2, b/2, 0) \frac{1}{q_o}, \\ \bar{\sigma}_{yz} &= \sigma_{yz}(a/2, 0, 0) \frac{h}{q_o a}, \\ \bar{\sigma}_{xz} &= \sigma_{xz}(0, b/2, 0) \frac{h}{q_o a}, \\ \bar{\sigma}_{xy} &= \sigma_{xy}(0, 0, -h/2) \frac{h}{q_o a^2} \end{aligned} \quad (52)$$

The three-dimensional deflection w of the TSDT⁽¹⁴⁾ in Table 1 is calculated at $x=a/2, y=b/2$ and $z=0$. The transverse normal stress σ_{zz} in the present exact exponential theory is obtained by integrating the equilibrium equation in Eq.(1.c) with respect to the thickness coordinate as follows.

$$\sigma_{zz}(x, y, z) = - \int_{-h/2}^z \left(\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) dz \quad (53)$$

In particular, the transverse shear stresses σ_{xz} and σ_{yz} in the present exponential theory are obtained using the constitutive equations given in Eq.(3) instead of the equilibrium equations. As shown in Table 1, the present exponential theory predicts the deflection and

Table 1 Comparison of deflections and stresses for $[0/90/90/0]$ laminated square plates ($a/b=1$) with simply supported boundary condition

a/h	Source	$\bar{w}$	$\bar{\sigma}_{xx}$	$\bar{\sigma}_{yy}$	$\bar{\sigma}_{zz}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{xz}$
4	Pagano ^a	1.9367	0.7203	0.6626	0.4957	0.2915	0.2193
	Present	1.9691	0.7045	0.6762	0.5000	0.2924	0.2201
	TSDT ^b	1.8937	0.6651	0.6322	-	0.2389	0.2064
	FSDT ^c	1.7100	0.4059	0.5765	-	(0.298) ^d	(0.231)
10	Pagano	0.7370	0.5586	0.4010	0.4989	0.1960	0.3014
	Present	0.7395	0.5605	0.4025	0.5000	0.1955	0.3017
	TSDT	0.7147	0.5456	0.3888	-	0.1531	0.2640
	FSDT	0.6628	0.4989	0.3615	-	(0.192)	(0.307)
20	Pagano	0.5130	0.5428	0.3084	0.5000	0.1556	0.3282
	Present	0.5134	0.5434	0.3087	0.5000	0.1555	0.3283
	TSDT	0.5060	0.5393	0.3043	-	0.1234	0.2825
	FSDT	0.4912	0.5273	0.2957	-	(0.154)	(0.330)
100	Pagano	0.4346	0.5389	0.2710	0.5000	0.1389	0.3388
	Present	0.4346	0.5389	0.2710	0.5000	0.1389	0.3388
	TSDT	0.4343	0.5387	0.2708	-	0.1117	0.2897
	FSDT	0.4337	0.5382	0.2705	-	(0.139)	(0.339)

The superscripts a, b, c and d denote the numerical results obtained from the 3-D elasticity solution⁽¹⁴⁾, the third order shear deformation theory⁽⁸⁾, and the first order shear deformation theory⁽¹⁵⁾, respectively.

stresses more accurately than the TSDT and the FSDT. When compared to the three dimensional elasticity solutions⁽¹²⁾, the errors in deflections and stresses for the length-to-thickness ratio a/h equal to or greater than 10 are less than one percent. Moreover, both the transverse shear stresses and the transverse normal stresses are predicted very accurately even for $a/h=4$.

Next, for $[0/90/0/90]$ anti-symmetric square laminate, the deflection for varying length-to-thickness ratio is presented in Fig. 2. Also, the variations of inplane displacements and stresses through the thickness for the length-to-thickness ratios $a/h=4$ and 10 are presented in Figs. 3 - 10. In these figures, the dashed line represents the three dimensional elasticity solutions while the results of the present theory are

Fig. 2 Nondimensionalized deflection $\bar{w}$ versus length-to-thickness ratio a/h for antisymmetric cross-ply square plates

Fig. 3 Nondimensionalized inplane displacement $\bar{u}$ versus thickness z/h for antisymmetric cross-ply square plates

shown by solid line. The displacements and stresses are nondimensionalized as

$$\bar{u} = u(0, b/2, z) \frac{E_{22} h^2}{q_0 a^3},$$

$$\bar{v} = v(a/2, 0, z) \frac{E_{22} h^2}{q_0 a^3},$$

Fig. 4 Nondimensionalized inplane displacement $\bar{v}$ versus thickness z/h for antisymmetric cross-ply square plates

Fig. 5 Nondimensionalized inplane normal stress $\bar{\sigma}_{xx}$ versus thickness z/h for antisymmetric cross-ply square plates

Fig. 6 Nondimensionalized inplane normal stress $\bar{\sigma}_{yy}$ versus thickness z/h for antisymmetric cross-ply square plates

Fig. 7 Nondimensionalized transverse normal stress $\bar{\sigma}_{zz}$ versus thickness z/h for antisymmetric cross-ply square plates

Fig. 8 Nondimensionalized transverse shear stress $\bar{\sigma}_{yz}$ versus thickness z/h for antisymmetric cross-ply square plates

Fig. 9 Nondimensionalized transverse shear stress $\bar{\sigma}_{xz}$ versus thickness z/h for antisymmetric cross-ply square plates

$$\bar{w} = w(a/2, b/2) \frac{100 E_{zz} h^3}{q_0 a^4},$$

$$\bar{\sigma}_{xx} = \sigma_{xx}(a/2, b/2, z) \frac{h^2}{q_0 a^2},$$

Fig. 10 Nondimensionalized inplane shear stress $\bar{\sigma}_{xy}$ versus thickness z/h for antisymmetric cross-ply square plates

$$\begin{aligned} \bar{\sigma}_{yy} &= \sigma_{yy}(a/2, b/2, z) \frac{h^2}{q_0 a^2}, \\ \bar{\sigma}_{zz} &= \sigma_{zz}(a/2, b/2, z) \frac{h^2}{q_0 a^2}, \\ \bar{\sigma}_{yz} &= \sigma_{yz}(a/2, 0, z) \frac{h}{q_0 a}, \\ \bar{\sigma}_{xz} &= \sigma_{xz}(0, b/2, z) \frac{h}{q_0 a}, \\ \bar{\sigma}_{xy} &= \sigma_{xy}(0, 0, z) \frac{h^2}{q_0 a^2} \end{aligned} \quad (54)$$

As shown in Figs. 2–10, the results of the present exponential theory and the three dimensional elasticity solutions agree very well for $a/h=4$ and are too close to distinguish for $a/h=10$. It is shown that the results of $a/h=10$ agrees more accurately with the theory because the transverse displacement $w(x, y)$ is assumed as functions of x and y only. As the aspect ratio a/h of a plate increases, the difference in between the present theory and the three dimensional elastic theory decreases. Thus, it can be concluded that the present theory predicts very accurately the responses of symmetric and anti-symmetric cross-ply laminated plates with simply supported boundary conditions.

6. Conclusions

A new laminated orthotropic plate theory has been developed on the assumption that the inplane displacements vary exponentially across the plate thickness. The equilibrium equations and boundary conditions of the laminated plate are derived using the principle of virtual work, and analytical solutions are obtained for symmetric and anti-symmetric cross-ply rectangular laminates with simply supported boundary conditions under sinusoidal loading.

In order to investigate the accuracy of the present exact exponential theory, the numerical results are

compared with those obtained by the three-dimensional elasticity theory, the TSDT and the FSDT. The present exact exponential theory predicts the displacements and stresses of the laminated plates more accurately than the three-dimensional elasticity solutions, the TSDT and the FSDT. The transverse shear stresses that are obtained from the constitutive equations are predicted very accurately. In addition, the transverse normal stress obtained from the integration of the equilibrium equation across the plate thickness is also very accurate. Hence, the present exponential theory can describe very accurately both inplane and transverse responses of the laminated orthotropic plates with simply supported boundary conditions. The present exponential theory can be extended to the finite element analysis for performing better laminated plate analysis.

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