Theory of Elasticity for Plain-Weave Fabrics*  
(2nd Report, Finite Element Formulation)

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A new breed of finite element is developed to analyze the nonlinear behavior of plain-weave fabrics in the in-plane problems of arbitrary boundary condition. A nondimensional parameter called crimp parameter is introduced as an unknown to represent the crimp condition of warp and weft, and handled as a component of the displacement vector. The plain-weave fabric is homogenized by means of a newly defined strain-displacement relationship including the crimp parameter for the sake of consistent dealing with the three types of thread deformations, that is, skewing, straightening and extension. This homogenized model called pseudo-continuum model induces the geometrical nonlinearity with respect to the finite rotation of the threads, and its finite element is formulated by the principle of virtual work in the total Lagrangian description. The mechanism of nonlinear behavior of the plain-weave fabrics is elucidated through several examples by the proposed finite element.

Key Words: Finite Element Method, Principle of Virtual Work, Finite Deformation Theory, Elasticity, Discontinuity, Plain-Weave Fabric

1. Introduction

The plain-weave fabric has been widely utilized as a light and high-strength membrane material used for airship, large spatial dome and so on, and also as a reinforcement of composite materials in view of formability recently. However, the conventional theory of elasticity is no longer available since the mechanical property of the plain-weave fabric is highly nonlinear. The reason lies on the finite change of fiber orientation with the deformation of woven structure, that is to say, the microscopic geometrical nonlinearity. The material modeling on the constitutive equation such as the homogenization method is being developed to evaluate such nonlinearity by the recent progress but seems impracticable yet from the standpoint of computational efforts

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In the previous paper, we have proposed the pseudo-continuum model which homogenizes the micromechanical property of plain-weave fabrics not on the constitutive law but on the new strain-displacement relationship. The deformation of plain-weave fabric is categorized into the three basic modes, namely, the skewing, straightening and extension. Most of the past-proposed mechanical models cannot take all of these deformation modes into account at once. Meanwhile, the pseudo-continuum model enables the structural analysis in consideration of all the modes and it is versatile to evaluate the microscopic and macroscopic behavior of fabrics in any macroscopic structural condition. The validity of the pseudo-continuum model has already been examined in the biaxial extension analysis, which is a characteristic nonlinear problem of the plain-weave fabric. In this paper, to carry out the structural analysis of fabric in arbitrary problems, the finite element formulation of the pseudo-continuum model is presented.

The pseudo-continuum model contains a geometrically nonlinear problem with infinitesimal strain and finite rotation. The finite element formula-
tion is based on the total Lagrangian form of the principle of virtual work, and consequently, the tangent stiffness of fabric is derived. The obtained tangent stiffness has a numerical instability arising from the lack of shear stiffness, and then we discuss its mathematical singularity and numerical handling. As numerical example, the macroscopic Poisson’s ratio analysis and the uniaxial extension analysis are shown to elucidate the nonlinear deformation mechanism of the plain weave fabrics.

2. Pseudo-Continuum Model for Plain-Weave Fabric

2.1 Problem description

Of interest here is in-plane problem. The warp and weft threads are woven orthogonally, and the weave pattern is uniform and dense enough. Both threads are made of the same material of orthotropic and linear elastic body, and have uniform cross-sections. The flexural rigidity and friction with respect to the threads are supposed to be negligible, while no slip occurs at the center of the crossover region of warp and weft. In these circumstances, the plain-weave fabric possesses the three basic deformation modes in view of thread deformation, that is, the skewing, straightening and extension. The skewing and straightening give rise to the finite rigid-body rotations, meanwhile the strain of thread is regarded as infinitesimal.

The weave pattern is oriented to the direction of angle $\theta$ from the global Cartesian coordinates $x_1-x_2$. We take another Cartesian coordinates $\xi_1-\xi_2$ along the weave pattern where the axis $\xi_1$ indicates the weft direction as shown in Fig. 1. The suffixes 1 and 2 denote the variables concerning the weft and warp, respectively, and the upper tilde means its initial value, hereafter.

2.2 Crimp parameter

Figure 2 shows the detail of the unit weave struc-

\[ h_1 + h_0 = d_1 + d_0 \]  \hspace{5cm} (1)

Assuming that the change of the central thickness is infinitesimal though the change of the crimp height is finite, we consider the thickness to be constant. The crimp height can be represented as Eq.(2) by introducing a nondimensional parameter $\mu$ ($0 \leq \mu \leq 1$).

\[ \begin{cases} h_1 = \mu(d_1 + d_0) \\ h_0 = (1-\mu)(d_1 + d_0) \end{cases} \]  \hspace{5cm} (2)

The parameter $\mu$ is named crimp parameter.

2.3 Strain-displacement relationship

A deformation of the unit weave structure is schematically illustrated in Fig. 3. Since the friction between threads is neglected, the shear strain does not take place, and thus the thread is subject to only axial extension and in-plane transverse compression. The strains of axial extension and transverse compression are denoted by $e_{\xi_1}$ and $e_{\xi_2}$, respectively.

The thread skewing gives rise to the in-plane
rotation of \( \xi \) for each of warp and weft as shown in Fig. 3. Denoting the component of displacement vector on \( \xi\tau\xi\zeta \) coordinates by \( u_t \), the relative rotation angle \( \psi \) between warp and weft is approximately represented by
\[
\psi = \xi \cdot \psi + \xi \cdot \psi = \psi_{\alpha} + \psi_{\beta}
\]  
where \((\cdot)\tau \) means the spatial gradient \( \partial (\cdot) \partial \xi \).

We suppose that the change of spacing between neighbor threads of warp or weft is equal to the transverse compressive deformation of the thread since the weave pattern is dense enough. Considering the finite change of crimp wavelength and the in-plane rotation, the transverse compressive strain \( \varepsilon_{\xi} \) is defined as
\[
\varepsilon_{\xi} = 1 - \sqrt{(1 + \psi_{\alpha})^2 + \psi_{\beta}^2} \left[ 1 - \frac{1}{2} (\psi_{\alpha} + \psi_{\beta})^2 \right] \quad (4)
\]
\[
\varepsilon_{\zeta} = 1 - \sqrt{(1 + \psi_{\alpha})^2 + \psi_{\beta}^2} \left[ 1 - \frac{1}{2} (\psi_{\alpha} + \psi_{\beta})^2 \right] \quad (5)
\]
The centerline of thread is supposed to be straight in the unit weave structure before and after the deformation, and its segment length is denoted by \( s_t \). Considering the in-plane and out-of-plane rotation caused by the skewing and straightening, the axial tensile strain \( \varepsilon_t \) is defined as
\[
\varepsilon_{t\alpha} = \phi_{\alpha} (1 + \psi_{\alpha})^2 + \phi_{\beta} \psi_{\beta}^2 + \phi_{\mu} \mu^2 - 1 \quad (6)
\]
\[
\varepsilon_{t\beta} = \phi_{\alpha} (1 + \psi_{\alpha})^2 + \phi_{\beta} \psi_{\beta}^2 + \phi_{\mu} (1 - \mu)^2 - 1 \quad (7)
\]
where \( \phi_{\alpha} \) and \( \phi_{\beta} \) are the geometrical constants obtained by
\[
\phi_{\alpha} = \frac{F_{\alpha}}{s_t}, \quad \phi_{\beta} = \frac{d_1 + d_2}{s_t} \quad (i=1,2).
\]

2.4 Constitutive law

The thread is supposed to be orthotropic and linearly elastic, and Young's modulus and Poisson's ratio are denoted by \( E_{\xi} \) and \( v_{\xi} \) in the axial direction and \( E_{\zeta} \) and \( v_{\zeta} \) in the transverse direction, respectively. On the basis of the plane stress assumption, the axial tensile stress \( \sigma_{t\alpha} \) and transverse compressive stress \( \sigma_{\xi} \) are related with the strains as
\[
\begin{bmatrix} \sigma_{t\alpha} \\ \sigma_{\xi} \end{bmatrix} = [D] \begin{bmatrix} \varepsilon_{t\alpha} \\ \varepsilon_{\xi} \end{bmatrix} \quad (i=1,2) \quad (9)
\]
where \([D]\) is the stress-strain matrix written as
\[
[D] = \begin{bmatrix} E_{\xi} & -v_{\xi} E_{\zeta} \\ -v_{\zeta} E_{\xi} & E_{\zeta} \end{bmatrix}.
\]

Note that the signs of \( \sigma_{\xi} \) and \( \sigma_{\zeta} \) are plus when they are compressive\(^{(20)} \). In addition, the transverse compressive stress must have a nonnegative value since it arises from the contact pressure between threads. In the after-mentioned iterative solution, when the transverse stress becomes negative after the calculation by Eq.(10), the stress-strain matrix is converted into that of one-dimensional stress condition:
\[
[D] = \begin{bmatrix} E_{\xi} & 0 \\ 0 & 0 \end{bmatrix}.
\]

At the same moment, the interface between threads is judged to be separated and the analysis proceeds to the next iterative step. In this case, the transverse compressive strain is calculated by Eq.(12) instead of the strain-displacement relationship.
\[
\varepsilon_{\xi} = \nu_{\xi} \varepsilon_{\xi},
\]

3. Finite Element Formulation

3.1 Preparation

Using the above-mentioned strain-displacement relationship and constitutive law, the finite element is formulated in line with the displacement method. We derive a linearized equilibrium equation based on the principle of virtual work in the total Lagrangian description\(^{(9)} \), since the problem is posed in geometrical nonlinearity. The summation convention is used in this chapter, and the stress and strain vectors are defined as
\[
\{\varepsilon\} = \{\sigma_{t\alpha}, \sigma_{\xi}\, \sigma_{t\beta}, \sigma_{\zeta}\}^T \quad (13)
\]
\[
\{\gamma\} = \{\varepsilon_{t\alpha}, \varepsilon_{\xi}, \varepsilon_{t\beta}, \varepsilon_{\zeta}\}^T \quad (14)
\]
where the superscript \( T \) means the matrix transpose. For the simplicity of notation, the stress and strain components are denoted by \( \tau_i \) and \( \gamma_i \) \((i=1,\cdots,4)\), respectively, hereafter. The constitutive equation is rewritten as
\[
\{\varepsilon\} = [D]\{\gamma\} \quad (15)
\]
where \([D]\) is the constitutive matrix expeditiously assembled stress-strain matrices of warp and weft, and its components are determined by Eq.(10) or (11) depending on the stress condition of each thread.

3.2 Principle of virtual work

The reference configuration is fixed to the initial one, and there is no body force acting. The stress and strain of this model are identical with the second Piola-Kirchhoff stress and the Green-Lagrange strain, respectively\(^{(20)} \). The total Lagrangian form of the principle of virtual work is represented by\(^{(40)} \)
\[
\int_{\Omega_t} \delta \varepsilon \cdot \delta \Omega_t = \delta L
\]
where \( L \) is the external work and \( \delta (\cdot) \) means the perturbation. It should be noted that the domain of integration \( \Omega_t \) is the alternative of the initial volume of warp or weft, and the selection is done in consistency with the stress component \( \tau \).

For the sake of simplicity, the displacement gradients and the crimp parameter, which are necessary to calculate the strain, are abridged to one vector as
\[
\{\chi\} = \{\nu_{\alpha}, \nu_{\beta}, \nu_{\mu}, \nu_{\alpha}, \mu\}^T
\]
and its component is denoted by \( \chi_i \) \((i=1,\cdots,5)\). The displacement gradient \( \nu_{\alpha} \) is necessarily transformed from the global coordinate system \( x_1-x_2 \) since it is defined in the weave coordinate system \( \hat{\xi}-\hat{\zeta} \).

To perform the incremental analysis, all the variables are supposed to be given at the time \( t \), and then we analyze the state at the time \( t' \) that is \( dt \) later.
than \( t \). The left superscript \( t \) indicates the occurrence time of the variable, and \( d(\cdot) \) denotes the increment, hereafter. The strain-displacement relationships of Eqs. (4), (5), (6) and (7) are highly nonlinear. We approximate the nonlinear change through the Taylor series expansion at the time \( t \) with the truncation of terms higher than the third-order as represented by

\[
\gamma_t = \gamma + R_{0t} d\Delta t + \frac{1}{2} S_{0t} d\Delta t^2 \tag{18}
\]

where \( R_{0t} \) and \( S_{0t} \) are the first and second-order derivatives of \( \gamma_t \) with respect to \( \Delta t \). The truncated terms are not required because they are eliminated at the after-mentioned linearization of the principle of virtual work in any case. In addition, the stress is also decomposed to the current value and the increments as

\[
\tau_t = \tau + \Delta \tau_t \tag{19}
\]

Substituting Eqs. (18) and (19) into Eq. (16) and taking account of the symmetry of \( S_{0t} \) with respect to the suffixes \( j \) and \( k \), we obtain the incremental form of the principle of virtual work as

\[
\int_{A_t} \left( \frac{\partial \delta}{\partial \tau} \right)^\prime (R_{0t} + S_{0t} \Delta \Delta t) \delta \Delta u dA = \delta^\prime \tau L. \tag{20}
\]

In the process of linearization by taking the limit of \( \Delta t \rightarrow 0 \), the increments \( \Delta \tau_t \) and \( \Delta \Delta t \) are replaced with the rates \( \dot{\tau}_t \) and \( \dot{\Delta} \Delta t \), and consequently we obtain

\[
\int_{A_t} \delta \dot{\Delta} \Delta t \dot{R}_{0t} \delta \Delta u dA + \int_{A_t} \delta \dot{\tau}_t S_{0t} \delta \Delta u dA = \delta^\prime \tau L - \int_{A_t} \delta \dot{\Delta} \Delta t ^{\prime} R_{0t} \delta \Delta u dA. \tag{21}
\]

From the assumption of the linear elastic body, the stress rate \( \dot{\tau}_t \) can be derived as a linear function of \( \dot{\Delta} \Delta t \) by using Eqs. (15) and (18), and then Eq. (21) leads to

\[
\int_{A_t} \delta \dot{\Delta} \Delta t \dot{R}_{0t} \delta \Delta u dA + \int_{A_t} \delta \dot{\tau}_t S_{0t} \delta \Delta u dA = \delta^\prime \tau L - \int_{A_t} \delta \dot{\Delta} \Delta t ^{\prime} R_{0t} \delta \Delta u dA. \tag{22}
\]

The average thickness of each thread, which is the volume of thread included in unit area of fabric in the initial state, is obtained as

\[
C_t = \frac{A_t S_t}{B_t \bar{P}_t} \quad (i=1, 2). \tag{23}
\]

We introduce a new vector \( \{b\} \) in which \( C_i \) is arrayed as Eq. (24), and its component is denoted by \( b_i \) \((i=1, \ldots, 4)\).

\[
\{b\} = \{C_1, C_2, C_3, C_4\}^T \tag{24}
\]

The component \( b_i \) means the average thickness of the thread where \( \tau_i \) acts. Assuming that the stress is uniform along the out-of-plane direction, the volume integral in Eq. (22) can be converted to the area integral of fabric by using \( b_i \) as follows:

\[
\int_{A_t} \delta \dot{\Delta} \Delta t \dot{R}_{0t} \delta \Delta u dA + \int_{A_t} \delta \dot{\tau}_t S_{0t} \delta \Delta u dA = \delta^\prime \tau L - \int_{A_t} \delta \dot{\Delta} \Delta t ^{\prime} R_{0t} \delta \Delta u dA \tag{25}
\]

where \( S \) denotes the initial area of the fabric. Note that the integration in Eq. (25) is carried out without any distinction between warp and weft threads. After the transformation of the displacement gradient from the weave coordinates \( \xi, \eta \) into the global coordinates \( x_1, x_2, x_3, x_4, x_5 \) and the finite element discretization of Eq. (25), the governing equation to be solved is obtained as

\[
[K]\{U\} = \{F\} - \{Q\} \tag{26}
\]

where \( [K] \) is the tangent stiffness matrix, \( \{U\} \) the nodal velocity vector described in the global coordinates, \( \{F\} \) the nodal external force vector and \( \{Q\} \) the nodal internal force vector. In the proposed model, we deal with the crimp parameter as an independent variable along with the displacement. Therefore, the nodal velocity vector contains the rate of the nodal crimp parameter, and consequently the vector consists of three components per node. The component of \( \{F\} \) corresponding to the nodal crimp parameter is equivalent to the out-of-plane force and always become zero.

The tangent stiffness matrix and the internal force vector are detailed as follows:

\[
[K] = \begin{bmatrix} [K_1] & [K_2] \end{bmatrix} \tag{27}
\]

\[
[K_1] = \sum_i \int_{S_v} [\mathcal{W}] [\mathcal{R}] [\mathcal{D}] [\mathcal{D}] [\mathcal{R}] \mathcal{W} dS_v \tag{28}
\]

\[
[K_2] = \sum_v \int_{S_v} [\mathcal{W}] [\mathcal{S}] [\mathcal{W}] \mathcal{D} dS_v \tag{29}
\]

\[
\{Q\} = \sum_v \int_{S_v} [\mathcal{W}] [\mathcal{D}] [\mathcal{F}] \mathcal{D} dS_v \tag{30}
\]

where \( \sum_v \) means the merge onto the overall matrix of the structure, \( [K_0] \) the initial displacement matrix, \( [K_{\text{int}}] \) the initial stress matrix, \( [\mathcal{W}] \) the matrix to sum up the derivative of shape function and the coordinate transformation, \( [\mathcal{D}] \) consists of \( K_{\theta_{ij}} \), and \( S_v \) denotes the area of a finite element. In addition, \( [\mathcal{F}] \), \( [\mathcal{S}] \) and \( [\mathcal{D}] \) are irregular matrices including the average thickness as follows:

\[
[\mathcal{D}] = \begin{bmatrix} C_1 [D] & [0] \\ [0] & C_4 [D] \end{bmatrix} \tag{31}
\]

\[
[\mathcal{S}] = C_2 \sum_i \tau_i [\mathcal{S}_i] + C_3 \sum_i \mathcal{S}_i \tau_i [\mathcal{S}_i] \tag{32}
\]

\[
[\mathcal{F}] = \left( C_1 \sigma_1, C_1 \sigma_2, C_3 \sigma_3, C_4 \sigma_4 \right)^T \tag{33}
\]

where \( [\mathcal{S}_i] \) consists of \( S_{0t} \) for the \( j \)-th row and \( k \)-th column component, and \( [D] \) is the elastic matrix of thread obtained by Eq. (10) or (11) depending on its stress condition. Through the above-mentioned formulae, the nonlinear finite element analysis can be done. The solution of stress and deformation can be obtained by the incremental analysis using the Newton–Raphson iterative calculations of Eq. (26).

### 3.3 Numerical issues and resolutions

The tangent stiffness matrix has a numerical instability. Particularly at the initial undeformed state, the tangent stiffness has no resistance against the thread skewing since the first-order derivative \( \dot{\theta}_{R_0} \) with respect to \( \theta_{R_0} \) or \( \theta_{R_1} \) is null and also the initial
stress is zero. Consequently, the tangent stiffness matrix of this case becomes singular and then Eq. (26) no longer yields a unique solution. In such case, sufficiently small values of the current $\nu_{1,2}$ and $\nu_{2,3}$ are substituted at the first iteration of the Newton–Raphson procedure to enhance a numerical stability, and they give rise to a small resistance to the thread skewing and enable to obtain a unique solution. This stabilizing operation does not affect the converged solution because the displacement gradient is modified at every iteration to hold the equilibrium. The value of the fake displacement gradient should be determined appropriately in dependency with the practical problem so as to imitate the real situation.

In the developed code of this study, the divergence of solution is prevented by the manner that the displacement vector is updated by adding the incremental solution $\{\Delta U\}$ multiplied by a positive constant less than or equal to 1.0 to keep the crimp parameter within its feasible domain. In addition, discontinuity of the displacement gradient arising from the lack of skewing stiffness has to be taken into account at the finite element discretization. The reason is exemplified by Fig. 4. Figure 4 (a) shows a square finite element in which the warp and weft are diagonally oriented, and the bottom side is fixed. Thick lines indicate the threads. Figure 4 (b) shows the prospective deformation of fabric qualitatively predicted by using the trellis model, in other words, considering only the thread skewing, when the topside of the element is subject to a uniform tensile load. This deformed element can be divided into four regions in each of which the displacement gradient is uniform as shown in Fig. 4 (c). The region 4 cannot yield the thread skewing from the fixation, while the region 1 can be largely deformed because no restriction from the fixed edge affects this region. In the regions 2 and 3, the intermediate skewing deformation between those of the regions 1 and 4 takes place. Therefore, the displacement gradient becomes discontinuous on the boundaries of these four regions. However, in case of the pseudo-continuum model, the rigorous discontinuity does not appear from the effect of the low skewing stiffness. For this reason, the boundary between finite elements should coincides with the discontinuity lines of deformation, or the order of the shape function needs to be high enough to cope with such nonlinearity.

4. Numerical Example

4.1 Macroscopic Poisson's ratio analysis

A uniform deformation of fabric under uniaxial extension is analyzed by one finite element in this section. The finite element is a square of 100 mm on a side as shown in Fig. 5, and we employ the four-node element from the viewpoint of uniform deformation analysis. The lower left node is completely fixed and the lower right one is fixed in vertical direction. The upper nodes are subject to a prescribed uniform displacement of $U$ in vertical direction and free in horizontal direction. The mechanical property of the thread and the geometrical parameters of the weave are listed in Table 1. All the property and parameters are identical for the warp and weft in initial state. We analyze four cases of the weave angle $\theta$ equal to 0, $\pi/12$, $\pi/6$, $\pi/4$, and calculate the macroscopic Poisson's ratio as the ratio of the macroscopic horizontal strain $\varepsilon_{1}$ to the vertical strain $\varepsilon_{2}$, where $u_{1}$ and $u_{2}$ are the components of displacement vector described in the global coordinate system. The numerical integration is implemented by the full inte-

Fig. 4  Discontinuity of displacement gradient

Fig. 5  Uniform uniaxial extension
Table 1 Mechanical and geometrical properties

<table>
<thead>
<tr>
<th></th>
<th>$E_t$</th>
<th>5.00 GPa</th>
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<tbody>
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<td>Young's modulus</td>
<td>$E_C$</td>
<td>0.50 GPa</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu_t$</td>
<td>0.30</td>
</tr>
<tr>
<td></td>
<td>$\nu_C$</td>
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</tr>
<tr>
<td>Cross-sectional area</td>
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</tr>
<tr>
<td>Central thickness</td>
<td>$d_t, d_C$</td>
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</tr>
<tr>
<td>Initial wavelength</td>
<td>$2\beta_1, 2\beta_2$</td>
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</tr>
<tr>
<td>Initial wave height</td>
<td>$h_t, h_C$</td>
<td>0.15 mm</td>
</tr>
</tbody>
</table>

Fig. 6 Variation of macroscopic Poisson's ratio

The prescribed displacement $U$ is set to 20 mm, which corresponds to 20% macroscopic strain, and the displacement is applied by equally divided 50 incremental steps. Figure 6 shows the variation of Poisson's ratio obtained from the finite element analysis. The variation is complicatedly nonlinear with respect to the tensile strain and the Poisson's ratio exceeds 1.0 in some cases. The value over 1.0 is unrealistically high for continuum, but is realistic for fabric. As described in Section 3.3, the fabric has no skewing stiffness at initial state and the skewing is dominant at the initial deformation. Consequently, the Poisson's ratio of 1.0 is possible to occur since the planar area is invariant in initial skewing. Furthermore, the macroscopic Poisson's ratio exceeds 1.0 by the effect of thread straightening. Figure 7 shows the variation of the crimp parameter, the gradient of which means the magnitude of thread straightening. In case of $\theta=0$, the thread straightening is dominant at initial deformation until 5% strain and this yields high Poisson's ratio as shown in Fig. 6. But as the crimp parameter approaches to 1.0, which means complete straightening, the horizontal strain reaches to stable and the macroscopic Poisson's ratio drastically decreases. This complicated nonlinear behavior qualitatively coincides with the experimental result of coated fabrics. In case of $\theta=\pi/12$, though the thread skewing is dominant at initial deformation, the large thread straightening appears after the warp rotates and becomes parallel to the tensile direction over 2% strain as observed in initial deformation of $\theta=0$. In case of $\theta=\pi/6$, the skewing is dominant and the effect of straightening is relatively small. In case of $\theta=\pi/4$, the straightening never appears because the tensions on the warp and weft are equal, and then the macroscopic Poisson's ratio remains high by the effect of skewing.

4.2 Analysis of uniaxial tensile test

Suppose that the rectangle specimen of width $B$ and length $H$ is subject to uniaxial extension as shown in Fig. 8. The lower edge is completely fixed and the upper edge is clamped and extended upward by the uniform displacement $U$, and the side edges are free. The mechanical properties of the threads and the geometrical parameters of the weave are listed in Table 2. We calculate the three cases of the aspect ratio $H:B$ equal to 2:1, 1:1 and 1:2, where the length $H$ is fixed at 100 mm in all the cases. In addition, the weave angle $\theta$ is set to four cases of 0,
Table 2 Mechanical and geometrical properties

<table>
<thead>
<tr>
<th>Property</th>
<th>$E_r$</th>
<th>$E_c$</th>
<th>$v_r$</th>
<th>$v_c$</th>
<th>$A_0$, $A_0$</th>
<th>$d_1$, $d_2$</th>
<th>$2\rho_1$, $2\rho_2$</th>
<th>$h_1$, $h_2$</th>
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<tr>
<td>Poisson's ratio</td>
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<td>0.30</td>
<td>0.03</td>
<td></td>
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<tr>
<td>Cross-sectional area</td>
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<td>Central thickness</td>
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<td></td>
</tr>
<tr>
<td>Initial wave height</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.10 mm</td>
<td></td>
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</tr>
</tbody>
</table>

Fig. 11 Load-displacement curves for $H:B=1:2$

in the aspect ratios, and there appears a weak non-linearity caused by the thread straightening. In case of $\theta=\pi/4$, the stiffness of fabric becomes lowest and the initial tangent stiffness is almost null from the reason mentioned in Section 3.3. On the one hand, the fabric is extremely flexible in the bias direction in case of $H:B=2:1$ because no thread connects the upper and lower edges and there appears a large thread skewing. On the other hand, in case of $H:B=1:2$, the stiffness in bias direction becomes very high and the load-displacement curve just shows a slight nonlinearity from the thread straightening since the threads connecting the upper and lower edges prevent the thread skewing geometrically. These results imply that the nonlinear behavior of the plain weave fabric is governed by the boundary condition, and then the conventional material modeling methodology based on the constitutive law is not acceptable for the fabric.

Figure 12 shows the deformations and the crimp parameter distributions in case of $\theta=\pi/4$ for each width of specimen when the applied force is approximately equal to 10 N/mm. Figure 12 (a) is for $H:B=2:1$, (b) 1:1 and (c) 1:2. The bright region has relatively high crimp parameter and the dark region low. The dots indicate the point of nodes. The ranges of crimp parameters are indicated under the distribution figures, and the maximum or minimum values happen at the corner of specimens and exceed the domain bounds of 0 and 1 for all the cases. The reason is that the interpolated crimp parameter seems to locally take an abnormal value unable to hold natural weave configuration at the corners since the out-of-plane equilibrium is considered by the weak form in the principle of virtual work. However, the crimp parameters of all the integral points stay in the feasible domain.

Figures 13, 14 and 15 show the stress distribution on center and bottom cross-sections at the time corre-
Fig. 12 Deformation and distribution of $\mu$

(a) $H:B=2:1$

(b) $H:B=1:2$

(c) $-0.05 \leq \mu \leq 1.05$

Fig. 13 Stress distribution for $H:B=2:1$

(a) Cross-section at $x_2=51.1$ mm

(b) Cross-section at $x_2=1.1$ mm

Fig. 14 Stress distribution for $H:B=1:1$

(a) Cross-section at $x_2=51.1$ mm

(b) Cross-section at $x_2=1.1$ mm

Fig. 15 Stress distribution for $H:B=1:2$

(a) Cross-section at $x_2=51.1$ mm

(b) Cross-section at $x_2=1.1$ mm
sponding to the deformations in Fig. 12. The plots indicate the stress of integral points. In all the cases, an extremely high stress concentration appears at the corners of specimen in the axial tensile stress. In case that there is a thread connecting the upper and bottom edges, that thread incurs most of the tensile load applied to the specimen. The stress distributions on both the center and bottom cross-sections of Fig. 15 represent the situation of connecting thread. In case of $H:B=2:1$, such thread does not exist so that the stress distribution on the center cross-section becomes almost uniform and the transverse compressive stress rises because of the large skewing deformation. A stress undulation is recognized near the region of stress concentration. This arises from the reason that the finite element discretization used in this work is too coarse to evaluate such high stress concentration.

5. Concluding Remarks

For the purpose of stress and deformation analyses, the finite element formulation of the pseudo-continuum model was presented for practical evaluation of the complicated nonlinear behavior in plain-weave fabric. Since the proposed model can be mathematically handled in the framework of the continuum, the finite element is formulated in line with the conventional total Lagrangian formulation. We investigated the instability of the obtained tangent stiffness matrix and the discontinuity of the displacement gradient. In addition, we described the method to overcome these difficulties.

As numerical example, the macroscopic Poisson’s ratio analysis and the uniaxial extension analysis were performed and we made it clear in the nonlinear behavior of plain-weave fabric that the mechanical anisotropy was strong and it greatly depended on boundary condition. The thread skewing is particularly governed by the geometrical boundary condition, and this complicates the mechanical behavior of fabric. The proposed model was proven to be able to evaluate the deformation of fabric and thread comprehensively and easily.

References