A Study on the Leakage Error in the Spectrum of Acoustic Intensity*

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Acoustic intensity is usually estimated from the cross-spectrum of acoustic pressures measured by two adjacent microphones. The cross-spectrum, calculated by the digital Fourier transform technique, has unavoidable leakage error because the signal period does not usually coincide with the record length. Therefore, acoustic intensity, estimated by a conventional FFT analyzer, has distorted values. In our research, the expression of Fourier-transformed data of a harmonic signal with a single frequency was formulated when there was leakage error. A method to eliminate the effect of leakage error from contaminated data was also investigated. Some numerical examples show the validation of the proposed method.

Key Words : Acoustic Intensity, Cross Spectrum, Leakage Error, Frequency Resolution Error

1. Introduction

Acoustic intensity is usually estimated by two closely spaced microphones measuring the imaginary part of the cross spectral density of acoustic pressure. Cross spectral density is usually calculated by Fourier transform. Fourier spectrum is continuous when a time signal is acquired over infinite time. In the analyzer, however, a discrete time signal is usually acquired during finite time, and discrete Fourier transform (DFT) is used to get the Fourier spectrum. Leakage error will occur when the data acquisition time is not the integer multiple of the signal period. This error will distort the Fourier spectrum of acoustic pressure measured by two microphones, which leads to the incorrect value for the cross spectrum. Therefore, intensity can not be measured correctly by the analyzer due to leakage error. The window functions, such as hanning window, are used to reduce leakage error in the analyzer. But, these window

functions cannot completely remove leakage error.

There have been some studies to remove leakage error completely in discrete Fourier transform. Dishan⁽¹⁾ showed leakage-induced phase error in discrete Fourier transform is the same regardless of the window function. Thompson⁽²⁾ investigated leakage error of harmonic signal with rectangular window function. Burgess⁽³⁾ formulated the mathematical derivation of the effects of leakage error of single harmonic signal when the number of data is assumed to be infinite. Jeong⁽⁴⁾ proposed a method to remove leakage error of unit impulse response function. However there have been no studies to improve intensity measurement as far as the author's knowledge is concerned.

In this paper, a precise derivation of the effects of leakage error of single harmonic signal is carried out in case of finite number of digital data. And a method to remove the leakage error is also proposed by solving nonlinear equation. The results of Burgess⁽³⁾ is used as the initial value of the roots of nonlinear equation. This method is finally applied to the removal of leakage error of acoustic intensity with single frequency because acoustic intensity is more sensitive to leakage error than acoustic pressure.

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2. Leakage Error in Acoustic Intensity

Instantaneous acoustic intensity is equal to a multiple of instantaneous acoustic pressure and corresponding particle velocity at the same position. Acoustic pressure can be measured directly from two closely spaced microphone signals. The particle velocity can be estimated from the pressure gradient by using the following Euler equation.

$$\rho \frac{\partial \vec{u}}{\partial t} + \nabla p = 0 \quad (1)$$

where $\vec{u}$ represents the particle velocity and p represents the acoustic pressure. ρ is the density of the sound field. In the frequency domain, the averaged acoustic intensity can be expressed by the imaginary part of the cross-spectral density of two pressure signals, which is given by the following equation⁽⁵⁾.

$$I = -\frac{2 \operatorname{Im} S_{AB}}{\omega \Delta r} = -\frac{\operatorname{Im} G_{AB}}{\omega \rho \Delta r} \quad (2)$$

where the cross-spectral density $S_{AB} = E[P_A^* \cdot P_B]$ and $G_{AB} = 2S_{AB}$. Here, P_A and P_B are the Fourier transforms of acoustic pressure measured by two microphones in an intensity probe. $E[\]$ implies the average value and $*$ implies the complex conjugate. Δr is the space between two microphones in the intensity probe. ω is the angular frequency.

If acoustic pressures, sensed by two adjacent microphones in the intensity probe have leakage error, the cross spectrum of two acoustic pressures will have leakage errors that are more serious because the effect of errors will be amplified during the product operation, and the acoustic intensity estimated from Eq.(2) will be more severely distorted.

The acoustic pressure $p(r, t)$ at a distance r from the source is given by

$$p(r, t) = A_0 \cos\left(2\pi f t - \frac{2\pi f}{c} r\right) = A_0 \cos(2\pi f t + \Phi_0) \quad (3)$$

where A_0 is the magnitude of the pressure, f is the frequency, and Φ_0 is the phase.

Let us assume the number of period of acoustic signals during record length T is as follows.

$$a = \frac{\text{record length}}{\text{period of signal}} = fT = p(\text{integer part}) + q(\text{decimal part}) \quad (4)$$

Here, a is not an integer quantity but a real quantity in most cases. No leakage error will occur when a is an integer quantity. Substituting frequency resolution $\Delta f = 1/T$ into Eq.(4) leads to the following relation.

$$f = a\Delta f = (p+q)\Delta f \quad (5)$$

where p , the integer part of a and a known quantity, represents a serial number pointing at the maximum peak. q is the decimal part of a and is an unknown quantity. q is called resolution error which we want

Fig. 1 Region of f_{true} in frequency domain

to find out. If the resolution error q can be estimated correctly, the effect of leakage error can be predicted. Then, the true spectrum can be recalculated by removing the effect of resolution error.

As shown in Fig. 1, the range of true frequency f_{true} is

$$f_p - 0.5\Delta f \leq f_{true} \leq f_p + 0.5\Delta f \quad (6)$$

where f_p is the frequency at which a maximum peak occurs and Δf is the frequency resolution.

Substituting $f_p = p\Delta f$ and $f_{true} = a\Delta f$ into Eq.(6) leads to a range of q as follows.

$$-0.5 \leq q \leq 0.5 \quad (7)$$

Provided that n is the total number of sampled data during record length T , the time resolution Δt becomes $\Delta t = T/n$. And the discrete time t_k is given by

$$t_k = k\Delta t = \frac{kT}{n}, \quad k = 0, 1, 2, \dots, n-1 \quad (8)$$

Since true frequency $f = a/T$ and the frequency resolution $\Delta f = 1/T$, the discrete form of Eq.(3) can be written as

$$p_k = A_0 \cos(2\pi a k/n + \Phi_0), \quad k = 0, 1, 2, \dots, n-1 \quad (9)$$

where k is the serial number of time data, and n is the total number of data. Applying DFT to Eq.(9), the Fourier coefficient in the engineering spectrum P_v is given by

$$P_v = \frac{2}{n} \sum_{k=0}^{n-1} p_k e^{-i2\pi f v k T/n} = A_0 \left[\frac{e^{i\Phi_0} [1 - e^{i2\pi(a-v)}]}{n[1 - e^{i2\pi(a-v)/n}]} + \frac{e^{-i\Phi_0} [1 - e^{-i2\pi(a+v)}]}{n[1 - e^{-i2\pi(a+v)/n}]} \right] \quad (10)$$

$$v = 0, 1, 2, \dots, n/2 - 1$$

where v is the serial number of the Fourier coefficient.

3. Improvement of the Leakage Error

Equation (10) can be rewritten as

$$P_v = A_0 [e^{i\Phi_0} V_{v,a} + e^{-i\Phi_0} V_{v,-a}] \quad (11)$$

where

$$V_{v,a} = \frac{1 - e^{i2\pi(a-v)}}{n[1 - e^{i2\pi(a-v)/n}]}, \quad V_{v,-a} = \frac{1 - e^{-i2\pi(a+v)}}{n[1 - e^{-i2\pi(a+v)/n}]}$$

To obtain the unknown quantity q , multiplying $P_p/P_{p\pm 1}$ by $e^{i\Phi_0}$ gives the following

$$\frac{P_p}{P_{p-1}} = \frac{e^{2i\Phi_0} V_{p,a} + V_{p,-a}}{e^{2i\Phi_0} V_{p-1,a} + V_{p-1,-a}} \quad (12. a)$$

$$\frac{P_p}{P_{p+1}} = \frac{e^{2i\Phi_0} V_{p,a} + V_{p,-a}}{e^{2i\Phi_0} V_{p+1,a} + V_{p+1,-a}} \quad (12. b)$$

From Eq.(12. a), $\Phi = e^{i2\Phi_0}$ can be given as

$$e^{i2\Phi_0} = \Phi = \frac{P_p V_{p-1,a} - P_{p-1} V_{p,a}}{P_p V_{p-1,-a} - P_{p-1} V_{p,-a}} \quad (13)$$

Substituting Eq.(13) into Eq.(12. b), the following relation, expressed only in terms of an unknown quantity q , is derived.

$$\left| \frac{P_p}{P_{p+1}} \right| = \left| \frac{\Phi V_{p,a} + V_{p,-a}}{\Phi V_{p+1,a} + V_{p+1,-a}} \right| \quad (14)$$

The unknown quantity q is the root of Eq.(14), and can be obtained by numerical method. From Eq.(13), Φ_0 is given by

$$\begin{aligned} \Phi_0 &= \frac{1}{2i} \ln \Phi \\ &= \frac{1}{2i} \ln \left[-\frac{P_p V_{p-1,a} - P_{p-1} V_{p,a}}{P_p V_{p-1,-a} - P_{p-1} V_{p,-a}} \right] \end{aligned} \quad (15)$$

Finding q and Φ_0 makes it possible to get the true frequency f_{true} as follows.

$$f_{true} = a \Delta f = (p + q) \Delta f \quad (16)$$

Setting $v=p$ in Eq.(11), the amplitude A of acoustic pressure can be obtained as follows.

$$A_0 = \left| \frac{P_p}{e^{i\Phi_0} V_{p,a} + e^{-i\Phi_0} V_{p,-a}} \right| \quad (17)$$

It is necessary to determine the initial value to solve Eq.(14) by using the numerical method. The following will discuss this procedure. When $v=p$ or $v=p \pm 1$, which represents the serial number near the peak, the absolute value of the second term in Eq.(10) is much less than that of the first term. Neglecting the second term in Eq.(10) gives the following approximated equation.

$$P_v \approx A_0 \left[\frac{e^{i\Phi_0} [1 - e^{i2\pi(a-v)}]}{n [1 - e^{i2\pi(a-v)/n}]} \right] \quad (18)$$

and the following relation is obtained if the approximation of n is sufficiently large.

$$P_v \approx A_0 \left[e^{i[\Phi_0 + \pi(a-v)]} \frac{\sin \pi(a-v)}{\pi(a-v)} \right] \quad (19)$$

The initial value of q can be obtained as follows, which will be used in the numerical method for a fast convergence

$$q = \begin{cases} q = -\frac{1}{1+R_-} & P_{p-1} > P_{p+1} \\ q = \frac{1}{1+R_+} & P_{p-1} < P_{p+1} \end{cases} \quad (20)$$

where

$$R_{\pm} = \left| \frac{P_p}{P_{p \pm 1}} \right| = \left| \frac{q \mp 1}{q} \right| > 1$$

Setting $v=p$ in Eq.(19) gives the approximate value of A_0 as follows⁽³⁾.

$$A_0 \approx \frac{P_p \pi q}{e^{i(\Phi_0 + \pi q)} \sin \pi q} \quad (21)$$

And the approximate value of Φ_0 is given by

$$\Phi_0 \approx \angle P_p - \pi q \quad (22)$$

where $\angle P_p$ represents the phase of P_p . There are two values of Φ_0 calculated from Eq.(15). Equation (22) will give the guideline to get the true value of Φ_0 .

Through the method mentioned above, acoustic pressure signals with no leakage error can be obtained in case of single frequency. And the acoustic intensity given in Eq.(2) will also have no leakage error. This method can also be available for multi-frequency signal if peak values are not correlated by the leakage error, for example two peak frequencies are sufficiently separate so that the peak value will not be affected by the leakage error of other peaks.

4. Numerical Examples

4.1 The example of simple source

The theoretical acoustic pressure at distance r from simple source is given by⁽⁶⁾

$$p(r, t) = j\rho c \frac{Qk}{4\pi r} e^{j(\omega t - kr)} \quad (23)$$

And the theoretical intensity of simple source is also given by⁽⁶⁾

$$I(r) = \frac{1}{8} \rho c \left(\frac{Q}{\lambda r} \right)^2 \quad (24)$$

where, ρ is the density of the sound field, c is the phase speed of wave, Q is the source strength, k is the wave number, λ is the wave length, and r is the distance from the source.

Table 1 gives some numerical results when acoustic pressure signal has a single frequency around 251 Hz. Here, $\rho = 1.21 \text{ kg/m}^3$, $c = 343.0 \text{ m/s}$, $Q = 0.0001 \text{ m}^3/\text{s}$, $r = 0.5 \text{ m}$, the total number of data $n = 1024$, $\Delta f = 2 \text{ Hz}$ and $\Delta r = 0.05 \text{ m}$. In Table 1, the true value is a theoretical value calculated from Eq.(23). The conventional method in Table 1 means the value obtained by discrete-Fourier transforming the data

Table 1 The acoustic pressure estimated from the data with leakage error ($\Delta f = 2 \text{ Hz}$, $\Delta r = 0.05 \text{ m}$, simple source)

Microphone 1						Microphone 2					
True Value		Conventional Method		Proposed Method		True Value		Conventional Method		Proposed Method	
$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$	$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$	$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$	$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$	$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$	$f(\text{Hz})$	$P(\times 10^{-2} \text{ Pa})$
250.0	3.184	248.0	0.0	250.0	3.184	250.0	2.881	248.0	0.0	250.0	2.881
		250.0	3.184					250.0	2.881		
		252.0	0.0					252.0	0.0		
250.5	3.191	246.0	0.322	250.5	3.191	250.5	2.887	246.0	0.291	250.5	2.887
		249.0	0.577					248.0	0.522		
		250.0	2.875					250.0	2.601		
		252.0	0.955					252.0	0.864		
		254.0	0.408					254.0	0.369		
		256.0	0.259					256.0	0.234		
251.0	3.197	246.0	0.404	251.0	3.197	251.0	2.892	246.0	0.366	251.0	2.892
		248.0	0.675					248.0	0.611		
		250.0	2.032					250.0	1.839		
		252.0	2.038					252.0	1.844		
		254.0	0.681					254.0	0.616		
		256.0	0.410					256.0	0.371		

contaminated by leakage error. The proposed method in Table 1 means the result of using the suggested method in this paper. Since $\Delta f=2$ Hz, there was no leakage error in the spectrum of acoustic pressure when true frequency of the signal was 250 Hz, while the worst case occurred when true frequency was 251 Hz. The peak value at a certain frequency will split into several frequencies if there is a leakage error, and the peak value looks to be smaller than true value and there seems to be other signals with several frequencies. For example, in case of 251 Hz of true signal in Table 1, the peak value calculated by conventional method (2.038 at 252.0 Hz) becomes smaller than true peak value (3.197 at 251.0 Hz), even though the sum of square values is still preserved such as ;

$$\dots + 0.675^2 + 2.032^2 + 2.038^2 + 0.681^2 + \dots = 3.197^2$$

Table 1 shows that there are frequency split and magnitude distortion in the acoustic pressure estimated by the conventional method, even though the sum of square values will be still preserved. This means that the peak value in the spectrum will vary when resolution error q is changed. However, leakage error can be eliminated by the suggested method, and the true value can be estimated regardless of q value.

Table 2 shows intensity values calculated from the results in Table 1. In Table 2, the true value is a theoretical intensity given by Eq.(24). The intensity estimated by the conventional method has wide discrepancies as compared with the true value. Comparing Table 1 with Table 2 reveals that leakage error of the intensity becomes more severe than that of acoustic pressure. It is because that the cross-spectral density was calculated by multiplication of two acoustic pressures. However, by using the proposed method in this paper, it was possible to measure intensity with high accuracy in any case because leakage error could be eliminated.

Table 2 The acoustic intensity estimated from the data with leakage error
($\Delta f=2$ Hz, $\Delta r=0.05$ m, simple source)

True Value		Conventional Method		Proposed Method	
$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$	$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$	$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$
250.0	1.102	248.0	0.0	250.0	1.097
		250.0	1.096		
		252.0	0.0		
250.5	1.107	246.0	0.011	250.5	1.100
		248.0	0.036		
		250.0	0.894		
		252.0	0.098		
		254.0	0.018		
		256.0	0.007		
		246.0	0.018		
251.0	1.111	248.0	0.050	251.0	1.104
		250.0	0.449		
		252.0	0.446		
		254.0	0.049		
		256.0	0.018		

4.2 The example of pulsating sphere

The acoustic pressure of pulsating sphere is given by⁽⁶⁾

$$p(r, t) = \rho c U_0 \frac{b}{r} \cos \theta_a e^{j(\omega t - k(r-a) + \theta_a)} \quad (25)$$

And the theoretical intensity of pulsating sphere is also given by⁽⁶⁾

$$I(r) = \frac{1}{2} \rho c U_0^2 \left(\frac{b}{r} \right)^2 \cos^2 \theta_a \quad (26)$$

where, ρ is the density of the sound field, c is the phase speed of the wave, b is the average radius, U_0 is the pulsating speed, k is the wave number, θ_a is equal to ka , λ is the wave length and r is the distance from the center of the sphere source.

Table 3 shows the numerical results of a pulsating sphere when $\rho=1.21$ kg/m³, $c=343.0$ m/s, $a=0.1$ m, $U_0=0.01$ m/s, $r=1.0$ m, the total number of data $n=1024$, $\Delta f=2$ Hz, and $\Delta r=0.05$ m. In Table 3, pulsating sphere has similar results with simple source. That is to say, the intensity could not be estimated correctly by the conventional method. But, the intensity with high accuracy could be obtained in any case by using the proposed method in this paper.

From the above numerical examples, the true frequency and magnitude could be estimated by the proposed method, while those values could not be estimated by the conventional method.

5. Conclusion

(1) The effect of leakage error of Fourier-transformed data was formulated theoretically when the number of data acquired during record length is finite.

(2) A method to remove leakage error of Fourier-transformed data with single frequency was presented.

(3) A true frequency, true magnitude and true phase could be estimated from the data contaminated by leakage error.

Table 3 The acoustic intensity estimated from the data with leakage error
($\Delta f=2$ Hz, $\Delta r=0.05$ m, pulsating sphere)

True Value		Conventional Method		Proposed Method	
$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$	$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$	$f(\text{Hz})$	$I (\times 10^{-4} \text{ W/m}^2)$
300.0	150.90	298.0	0.0	300.0	149.40
		300.0	149.11		
		302.0	0.0		
300.5	150.73	296.0	1.49	300.5	148.93
		298.0	4.83		
		300.0	120.72		
		302.0	13.41		
		304.0	2.46		
		306.0	0.99		
		296.0	2.41		
301.0	150.57	298.0	6.70	301.0	148.76
		300.0	60.29		
		302.0	60.29		
		304.0	6.70		
		306.0	2.41		

(4) Several numerical examples showed that this method could estimate true acoustic intensity from two adjacent acoustic pressures contaminated by leakage error.

(5) The study for the acoustic intensity with multi-frequency coupled by leakage error should be studied further.

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