The Stress Rate of Hyperelasticity Based on the Updated Lagrangian Formulation*

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A rate-type constitutive equation of compressible isotropic hyperelasticity and a finite element formulation based on the Updated Lagrangian approach are proposed. The constitutive equations of compressible hyperelasticity are simplified by introducing modified tensors in an intermediate configuration, which separates the volumetric deformation from the distortional deformation. Selective integration and a mixed pressure/displacement formulation have been adopted to avoid volumetric locking and the hourglass modes. The deformation of a simple rectangle block was simulated to show the effect of the choice of the finite elements and the integration methods on the stability and the accuracy of proposed method. Moreover, loading tests were carried out on a prototype specimen of a vibration control device made from circumferential rubber parts and reinforcing steel layers in order to evaluate applicability of the proposed formulation to such devices.

Key Words: Finite Element Method, Hyperelasticity, Updated Lagrangian Formulation.

1. Introduction

The Lagrangian formulation is well established and widely used for the analysis of solid and structures⁽¹⁾. In a Lagrangian formulation, stress and strain are referred to a known configuration. Commonly, either the undeformed configuration is chosen as this reference state, the so-called Total Lagrangian formulation, or the most recently calculated configuration, the so-called Updated Lagrangian configuration^{(2),(3)}. The choice for one formulation or the other depends on the constitutive equation and the numerical efficiency.

Hyperelasticity has commonly been used for numerical modeling of rubberlike materials. Rubberlike materials are almost incompressible, which means that the bulk modulus is much larger than the shear modulus. As a consequence, traditional displacement-based low-order finite elements exhibit volumetric locking or hourglass modes, depending on the employed integration rule⁽⁴⁾. A mixed formulation with the nodal mean stress as unknown

was proposed for the analysis of (nearly incompressible) elasto-plastic problems⁽⁵⁾.

In hyperelastic formulations, the constitutive equation is naturally derived in the terms of the Second Piola-Kirchhoff stress⁽⁶⁾. In such a case, the Total Lagrangian formulation can be used efficiently and the performance of displacement/pressure mixed methods and selective integration algorithm has been addressed in the literature⁽⁷⁾. However, there is a tendency to use rubberlike materials in composite structures together with elasto-plastic materials⁽⁸⁾. Then, an Updated Lagrangian formulation may be more appropriate, thus necessitating rewriting the hyperelastic formulations in terms of stress rates.

Moreover, the pressure stress of hyperelasticity has initial value. Following two equations were proposed to ignore the initial stress of hyperelasticity⁽⁹⁾. The first potential is

$$\bar{W} = W(I_1, I_2) - \left\{ \frac{\partial W}{\partial I_1} \Big|_{I_1=3, I_2=3} + 2 \frac{\partial W}{\partial I_2} \Big|_{I_1=3, I_2=3} \right\} \cdot g(I_3), \quad (1)$$

where I_1 and I_2 are the first and second invariants of the right Cauchy-Green deformation tensor, and g is a function of third invariant I_3 . The next equation is

$$\bar{W} = W(\bar{C}) = W(\bar{I}_1, \bar{I}_2), \quad (2)$$

where $\bar{I}_1$ and $\bar{I}_2$ are the first and second invariants preserving volume change.

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In the previous report, we use Eq. (1) for potential energy to ignore the initial stress of hyperelasticity⁽¹⁰⁾. Above equations show the similar stress-strain relation in small strain. However the deviatoric stress part given from Eq. (1) include mean components in finite strain.

In this contribution, the stress rate of isotropic hyperelasticity, either be it incompressible or compressible, is derived in terms of the Truesdell stress rate using by Eq. (2). Subsequently, the governing finite element equations are derived by utilizing a standard mixed displacement/pressure approach and selective integration. The effect of the integration rule and the time stepping algorithm are evaluated by comparison with results obtained from a Total Lagrangian formulation. Finally, loading tests on rubber and steel composite bearings are simulated to assess the applicability of proposed formulation to such structures.

2. Theoretical Developments

The initial and current configurations are denoted as C_0 and C , respectively. When the material particle initially located at some position X moves to a new position x , the deformation gradient tensor F is given by

$$F = \frac{\partial x}{\partial X}. \quad (3)$$

The deformation gradient tensor F is decomposed as:

$$F = \bar{F} \cdot J^{\frac{1}{3}} I \quad (4)$$

The modified deformed gradient tensor $\bar{F}$ is proposed by Froly⁽¹¹⁾

$$\bar{F} = J^{-\frac{1}{3}} F; \quad J := \det(F), \quad (5)$$

where J gives the local volume change and the notation $\bar{\bullet}$ gives modified deformation tensor preserving volume change. Figure 1 shows the relation between the initial configuration C and an intermediate configuration $\bar{C}$ defined by the volumetric deformation. The initial configuration C is moved to the intermediate configuration by $J^{\frac{1}{3}} I$. The intermediate configuration $\bar{C}$ moves to the current configuration C by the modified deformation gradient tensor $\bar{F}$. Accordingly, the intermediate configuration defined by $\bar{C}$ arises from a shape-preserving volume change. Similarly, associated with the intermediate configuration, the modified right Cauchy and left Cauchy deformation tensor are defined as:

Fig. 1 Initial, current and intermediate configurations

$$\bar{C} = \bar{F}^t \cdot \bar{F}, \quad \bar{B} = \bar{F} \cdot \bar{F}^t \quad (6)$$

The third invariants of the above tensors, $\bar{I}_3$, are unity by definition.

The modified velocity gradient with respect to the intermediate configuration is obtained as:

$$\bar{L} = \dot{\bar{F}} \cdot \bar{F}^{-1}, \quad (\bar{L})_s = \bar{D} = D', \quad (7)$$

where the notation $\bullet'$ gives the deviatoric part of the tensor. The modified spatial rate of deformation tensor $\bar{D}$ is the symmetric part of $\bar{L}$. $\bar{D}$ corresponds to the deviatoric part of the spatial rate of the deformation tensor $\bar{D}'$ and is free from volumetric deformations.

2.1 Stress and stress rate of hyperelasticity

A stored-energy function of hyperelasticity is defined as:

$$W = W(C) \quad (8)$$

For isotropic hyperelasticity, the stored-energy function is a function of the principal invariants of C

$$W(C) = W(I_1, I_2, I_3) \quad (9)$$

which are calculated according to

$$I_1 = C : I, \quad I_2 = \frac{1}{2}(I_1^2 - C^2 : I), \quad I_3 = \det(C). \quad (10)$$

It is assumed that the stored-energy function can be decomposed additively into a volume-preserving and a purely volume-dependent term:

$$W(C) = \bar{W}(\bar{C}) + U^\circ(J) \quad (11)$$

The volume-preserving stored-energy function $\bar{W}$ is a function of the first and the second principal invariants of the modified right Cauchy-Green deformation tensor $\bar{C}$:

$$\bar{W}(\bar{C}) = \bar{W}(\bar{I}_1, \bar{I}_2) \quad (12)$$

where the principal invariants of $\bar{C}$ are calculated according to

$$\bar{I}_1 = \bar{C} : I, \quad \bar{I}_2 = \frac{1}{2}(\bar{I}_1^2 - \bar{C}^2 : I), \quad \bar{I}_3 = 1.0. \quad (13)$$

The derivative of the stored-energy function with respect to the right Cauchy-Green deformation tensor C gives the Second Piola-Kirchhoff stress tensor S^0

$$S^0 = 2 \frac{\partial W(C)}{\partial C} \quad (14)$$

and the Cauchy stress T is related to it as follows:

$$T = \frac{1}{J} F \cdot S^0 \cdot F^t. \quad (15)$$

The Cauchy stress T and 2nd Piola-Kirchhoff stress are decomposed into deviatoric part and mean part in the current configuration as:

$$T = T' + T_m I, \quad S^0 = S'^0 + S_m^0 \quad (16)$$

T' and T_m are deviatoric and mean parts of the Cauchy-stress defined as:

$$T_m \cdot J \cdot C^{-1} = \frac{\partial U^\circ(J)}{\partial C} \quad (17)$$

$$\mathbf{T}' = 2\mathbf{F} \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} \mathbf{F}^t \quad (18)$$

$\mathbf{S}'^0$ and $\mathbf{S}_m^0$ are deviatoric and mean parts of the 2nd Piola-Kirchhoff stress given by

$$\mathbf{S}'^0 = 2 \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} = 2 \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} \frac{\partial \bar{\mathbf{C}}}{\partial \mathbf{C}}, \quad (19)$$

$$\mathbf{S}_m^0 = \frac{\partial U^0(\mathbf{J})}{\partial \mathbf{C}}. \quad (20)$$

The deviatoric part of the 2nd Piola-Kirchhoff stress is calculated as:

$$\mathbf{S}'^0 = 2 \left\{ \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2)}{\partial \bar{\mathbf{I}}_1} \mathbf{J}^{-\frac{2}{3}} \left(\mathbf{I} - \frac{1}{3} \mathbf{I}_1 \mathbf{C}^{-1} \right) + \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2)}{\partial \bar{\mathbf{I}}_2} \mathbf{J}^{-\frac{4}{3}} \left(\mathbf{I}_1 \mathbf{I} - \mathbf{C} - \frac{2}{3} \mathbf{I}_2 \mathbf{C}^{-1} \right) \right\}. \quad (21)$$

The deviatoric part of the Cauchy stress in the current configuration is also calculated as:

$$\mathbf{T}' = \frac{2}{J} \left\{ \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2)}{\partial \bar{\mathbf{I}}_1} \mathbf{J}^{-\frac{2}{3}} \left(\mathbf{B} - \frac{1}{3} \mathbf{I}_1 \mathbf{I} \right) + \frac{\partial \bar{\mathbf{W}}(\bar{\mathbf{I}}_1, \bar{\mathbf{I}}_2)}{\partial \bar{\mathbf{I}}_2} \mathbf{J}^{-\frac{4}{3}} \left(\mathbf{I}_1 \mathbf{B} - \mathbf{B} \cdot \mathbf{B} - \frac{2}{3} \mathbf{I}_2 \mathbf{I} \right) \right\}. \quad (22)$$

These stress have following relationship

$$\mathbf{S}'^0 : \mathbf{C} = 0, \quad \mathbf{T}' : \mathbf{I} = 0. \quad (23)$$

The stress rate of the 2nd Piola-Kirchhoff stress defined by Eq. (21) is calculated using time differentiation of the right Cauchy-Green deformation tensor $\dot{\mathbf{C}}$ according to

$$\dot{\mathbf{S}}'^0 = 2 \left\{ \frac{\partial^2 \bar{\mathbf{W}}}{\partial \mathbf{C} \partial \mathbf{C}} \right\} : \dot{\mathbf{C}}. \quad (24)$$

Deviatoric part of the second Piola-Kirchhoff stress tensor is translated to the stress rate $\dot{\mathbf{S}}'$ in the intermediate configuration $\bar{\mathbf{C}}$ using $\mathbf{J}^{1/3} \mathbf{I}$

$$\dot{\mathbf{S}}' = \mathbf{J}^{2/3} \mathbf{I} \cdot \dot{\mathbf{S}}' \cdot \mathbf{I}. \quad (25)$$

The Truesdell stress rate has following relationship

$$\begin{aligned} \overset{\circ}{\mathbf{T}}_T &= \frac{1}{J} \mathbf{F} \cdot \dot{\mathbf{S}}' \cdot \mathbf{F}^t \\ &= \dot{\mathbf{T}} - \mathbf{L} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{L}^t + (\text{tr } \mathbf{D}) \mathbf{T}. \end{aligned} \quad (26)$$

The deviatoric part of the Truesdell stress rate in the intermediate configuration is calculated as:

$$\begin{aligned} \overset{\circ}{\mathbf{T}}'_T &= \frac{1}{J} \bar{\mathbf{F}} \cdot \dot{\mathbf{S}}' \cdot \bar{\mathbf{F}}^t \\ &= \dot{\mathbf{T}}' - \bar{\mathbf{L}} \cdot \mathbf{T}' - \mathbf{T}' \cdot \bar{\mathbf{L}}^t + (\text{tr } \mathbf{D}) \mathbf{T}'. \end{aligned} \quad (27)$$

The 2nd Piola-Kirchhoff stress rate is calculated as:

$$\begin{aligned} \dot{\mathbf{S}}' &= 2 \left\{ \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1 \partial \bar{\mathbf{I}}_1} \left(\mathbf{I} - \frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\mathbf{I} - \frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \right) \right. \\ &\quad + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1 \partial \bar{\mathbf{I}}_2} \left(\mathbf{I} - \frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\bar{\mathbf{I}}_1 \mathbf{I} - \bar{\mathbf{C}} - \frac{2}{3} \bar{\mathbf{I}}_2 \bar{\mathbf{C}}^{-1} \right) \\ &\quad + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2 \partial \bar{\mathbf{I}}_1} \left(\bar{\mathbf{I}}_1 \mathbf{I} - \bar{\mathbf{C}} - \frac{2}{3} \bar{\mathbf{I}}_2 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\mathbf{I} - \frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \right) \\ &\quad \left. + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2 \partial \bar{\mathbf{I}}_2} \left(\bar{\mathbf{I}}_1 \mathbf{I} - \bar{\mathbf{C}} - \frac{2}{3} \bar{\mathbf{I}}_2 \bar{\mathbf{C}}^{-1} \right) \otimes \left(\bar{\mathbf{I}}_1 \mathbf{I} - \bar{\mathbf{C}} - \frac{2}{3} \bar{\mathbf{I}}_2 \bar{\mathbf{C}}^{-1} \right) \right\} \end{aligned}$$

$$\begin{aligned} &+ \frac{\partial \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1} \frac{1}{3} \left(\frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} - \mathbf{I} \otimes \bar{\mathbf{C}} - \bar{\mathbf{C}}^{-1} \otimes \mathbf{I} - \bar{\mathbf{I}}_1 \frac{\partial (\bar{\mathbf{C}}^{-1})}{\partial \bar{\mathbf{C}}} \right) \\ &+ \frac{\partial \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2} \left(\mathbf{I} \otimes \mathbf{I} - \frac{\partial \bar{\mathbf{C}}}{\partial \bar{\mathbf{C}}} - \frac{2}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{C}}^{-1} \otimes \mathbf{I} - \frac{2}{3} \bar{\mathbf{I}}_1 \mathbf{I} \otimes \bar{\mathbf{C}}^{-1} \right. \\ &\quad + \frac{2}{3} \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}} + \frac{2}{3} \bar{\mathbf{C}} \otimes \bar{\mathbf{C}}^{-1} + \frac{4}{9} \bar{\mathbf{I}}_2 \bar{\mathbf{C}}^{-1} \otimes \bar{\mathbf{C}}^{-1} \\ &\quad \left. - \frac{2}{3} \bar{\mathbf{I}}_2 \frac{\partial (\bar{\mathbf{C}}^{-1})}{\partial \bar{\mathbf{C}}} \right) \left\} : \dot{\mathbf{C}} = \overset{(t)}{\mathbf{C}}_{ijkl} : \dot{\mathbf{C}}_{kl}, \end{aligned} \quad (28)$$

where $\overset{(t)}{\mathbf{C}}_{ijkl}$ is the consistent stiffness matrix for the total Lagrangian method. From Eqs. (24) and (27), the Truesdell stress rate in the intermediate configuration is calculated using the relation $\dot{\mathbf{C}} = 2\bar{\mathbf{F}} \cdot \mathbf{D}' \cdot \bar{\mathbf{F}}^t$

$$\begin{aligned} (\overset{\circ}{\mathbf{T}}'_T)_{ij} &= \frac{4}{J} \left\{ \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1 \partial \bar{\mathbf{I}}_1} \left(\bar{\mathbf{B}} - \frac{1}{3} \bar{\mathbf{I}}_1 \mathbf{I} \right)_{ij} \left(\bar{\mathbf{B}} - \frac{1}{3} \bar{\mathbf{I}}_1 \mathbf{I} \right)_{kl} \right. \\ &\quad + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1 \partial \bar{\mathbf{I}}_2} \left(\bar{\mathbf{B}} - \frac{1}{3} \bar{\mathbf{I}}_1 \mathbf{I} \right)_{ij} \left(\bar{\mathbf{I}}_1 \bar{\mathbf{B}} - \bar{\mathbf{B}} \cdot \bar{\mathbf{B}} - \frac{2}{3} \bar{\mathbf{I}}_2 \mathbf{I} \right)_{kl} \\ &\quad + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2 \partial \bar{\mathbf{I}}_1} \left(\bar{\mathbf{I}}_1 \bar{\mathbf{B}} - \bar{\mathbf{B}} \cdot \bar{\mathbf{B}} - \frac{2}{3} \bar{\mathbf{I}}_2 \mathbf{I} \right)_{ij} \left(\bar{\mathbf{B}} - \frac{1}{3} \bar{\mathbf{I}}_1 \mathbf{I} \right)_{kl} \\ &\quad + \frac{\partial^2 \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2 \partial \bar{\mathbf{I}}_2} \left(\bar{\mathbf{I}}_1 \bar{\mathbf{B}} - \bar{\mathbf{B}} \cdot \bar{\mathbf{B}} - \frac{2}{3} \bar{\mathbf{I}}_2 \mathbf{I} \right)_{ij} \left(\bar{\mathbf{I}}_1 \bar{\mathbf{B}} - \bar{\mathbf{B}} \cdot \bar{\mathbf{B}} - \frac{2}{3} \bar{\mathbf{I}}_2 \mathbf{I} \right)_{kl} \\ &\quad + \frac{\partial \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_1} \frac{1}{3} \left(\frac{1}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{I}}_{ij} \bar{\mathbf{I}}_{kl} - \bar{\mathbf{B}}_{ij} \bar{\mathbf{I}}_{kl} - \bar{\mathbf{I}}_{ij} \bar{\mathbf{B}}_{kl} + \bar{\mathbf{I}}_1 \bar{\mathbf{I}}_{ik} \bar{\mathbf{I}}_{lj} \right) \\ &\quad + \frac{\partial \bar{\mathbf{W}}}{\partial \bar{\mathbf{I}}_2} \left(\bar{\mathbf{B}}_{ij} \bar{\mathbf{B}}_{kl} - \bar{\mathbf{B}}_{ik} \bar{\mathbf{B}}_{lj} + \frac{2}{3} (\bar{\mathbf{B}} \cdot \bar{\mathbf{B}})_{ij} \bar{\mathbf{I}}_{kl} + \frac{2}{3} \bar{\mathbf{I}}_{ij} (\bar{\mathbf{B}} \cdot \bar{\mathbf{B}})_{kl} \right. \\ &\quad \left. - \frac{2}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{I}}_{ij} \bar{\mathbf{B}}_{kl} - \frac{2}{3} \bar{\mathbf{I}}_1 \bar{\mathbf{B}}_{ij} \bar{\mathbf{I}}_{kl} + \frac{4}{9} \bar{\mathbf{I}}_2 \bar{\mathbf{I}}_{ij} \bar{\mathbf{I}}_{kl} + \frac{2}{3} \bar{\mathbf{I}}_2 \bar{\mathbf{I}}_{ik} \bar{\mathbf{I}}_{lj} \right) \left\} \mathbf{D}'_{kl} \\ &= \overset{(u)}{\mathbf{C}}_{ijkl} \mathbf{D}'_{kl}, \end{aligned} \quad (29)$$

where $\overset{(u)}{\mathbf{C}}$ is the tangent stiffness matrix to be employed in the finite element formulation.

We note that the left Cauchy-Green tensor and principal invariants in Eq. (29) are volume-preserving. For an incompressible problem, the intermediate configuration corresponds to the initial configuration.

2.2 Updated Lagrangian formulation

We start from the following principle of stationary potential energy for the updated Lagrangian formulation

$$\begin{aligned} \Pi^U(\mathbf{v}) &= \int \left[U_T + \frac{1}{2} \mathbf{T} : (\nabla \mathbf{v} \cdot \mathbf{v} \nabla) \right] dV \\ &\quad - \int \dot{\mathbf{t}} \cdot \mathbf{v} dS - \int \rho \dot{\mathbf{g}} \cdot \mathbf{v} dV, \end{aligned} \quad (30)$$

where V is the volume occupied by the body in the current configuration and dS is the surface bounding this volume. ρ is the density of material in the current configuration, $\dot{\mathbf{t}}$ is the rate of surface traction vector and $\dot{\mathbf{g}}$ is the rate of body force vector. U_T is the internal work rate related to the Truesdell stress rate $\overset{\circ}{\mathbf{T}}_T$ and its variation is given by

$$\delta U_T = \overset{\circ}{\mathbf{T}}_T : \delta \mathbf{D}. \quad (31)$$

From Eq. (30), the updated Lagrangian formulation of the principle of stationary potential energy using the Truesdell stress rate in the intermediate configuration is given as:

$$\begin{aligned} \Pi^P(\mathbf{v}) = & \int \left[U'_G(\mathbf{D}') + U_{Gm}(\mathbf{D}_m) - 2D_m(\mathbf{T}' : \mathbf{D}') \right. \\ & \left. - T_m(\mathbf{D}' : \mathbf{D}') + \frac{3}{2}T_m D_m^2 + \frac{1}{2}\mathbf{T} : (\nabla \mathbf{v} \cdot \mathbf{v} \nabla) \right] dV \\ & - \int \rho \dot{\mathbf{g}} \cdot \mathbf{v} dV - \int \dot{\mathbf{t}} \cdot \mathbf{v} dS, \end{aligned} \quad (32)$$

where U_G and U_{Gm} have following relationships

$$\delta U'_G = \dot{\mathbf{T}}'_T : \delta \mathbf{D}', \quad \delta U_{Gm} = 3\dot{T}_m \cdot \delta D_m. \quad (33)$$

When a linear volumetric stress-strain relationships is assumed, U_{Gm} is calculated as $U_{Gm} = 9\kappa_0 J D_m^2$, where κ_0 is the bulk modulus⁽¹²⁾. The virtual work expression for the mixed method is obtained using Lagrangian multipliers as:

$$\begin{aligned} & \int \left[\left\{ \frac{\partial U'_G}{\partial \mathbf{D}'} - 2(\mathbf{v} \nabla)_m \mathbf{T}' - 2T_m \mathbf{D}' \right\} : \delta \mathbf{D}' \right. \\ & \left. + 3 \left\{ \dot{T}_m + T_m(\mathbf{v} \nabla)_m - \frac{2}{3}(\mathbf{T}' : \mathbf{D}') \right\} \delta(\mathbf{v} \nabla)_m \right. \\ & \left. + \mathbf{T} : (\nabla \mathbf{v} \cdot \delta \mathbf{v} \nabla) - 3 \left\{ \frac{1}{3J\kappa_0} \dot{T}_m - (\mathbf{v} \nabla)_m \right\} \delta \dot{T}_m \right] dV \\ & = \int \rho \dot{\mathbf{g}} \cdot \delta \mathbf{v} dV + \int \dot{\mathbf{t}} \cdot \delta \mathbf{v} dS, \end{aligned} \quad (34)$$

where $\dot{T}_m$ is the time derivative of mean part of the Cauchy stress and it correspond to $(\dot{\mathbf{T}} : \mathbf{I})/3$.

To apply selective integration, the stress and strain in Eq. (32) are decomposed into the mean and deviatoric parts and different integration rules are applied according to:

$$\begin{aligned} & \int_{F.I.} \left[\dot{\mathbf{T}}'_T : \delta \mathbf{D}' - 2\delta D_m(\mathbf{T}' : \mathbf{D}') \right. \\ & \left. - 2D_m(\mathbf{T}' : \delta \mathbf{D}') + \mathbf{T}' : (\nabla \delta \mathbf{v} \cdot \mathbf{v} \nabla) \right] dV \\ & + \int_{R.I.} [9\kappa_0 D_m \delta D_m - 2T_m(\mathbf{D}' : \delta \mathbf{D}')] \\ & + 3T_m D_m \delta D_m + T_m \text{tr}(\nabla \delta \mathbf{v} \cdot \mathbf{v} \nabla)] dV \\ & = \int \rho \dot{\mathbf{g}} \cdot \delta \mathbf{v} dV + \int \dot{\mathbf{t}} \cdot \delta \mathbf{v} dS, \end{aligned} \quad (35)$$

where the notation *F.I.* and *R.I.* are the full and reduced integration rules, respectively.

Here, the velocity vector $\mathbf{v}$, the rate of deformation tensor $\mathbf{D}$ and pressure stress rate $\dot{T}_m$ are expressed using node velocity $\psi^{(e)}$ and node pressure $\dot{T}_m^{(q)}$ discretely as:

$$\begin{aligned} v_i &= N_i^{(m)} \cdot \psi^{(m)}, \quad (\nabla v)_{ij} = A_{ij}^{(m)} \cdot \psi^{(m)}, \quad D'_{ij} = H'_{ij} \cdot \psi^{(m)}, \\ D_m &= H_m^{(m)} \cdot \psi^{(m)}, \quad \dot{T}_m = \dot{M}^{(q)} \cdot \dot{T}_m^{(q)}, \end{aligned} \quad (36)$$

where $N_i^{(m)}$, $A_{ij}^{(m)}$ and H'_{ij} are the shape functions for velocity and deformation. $\dot{M}^{(q)}$ is the shape function for

pressure stress. m and q are number of node in an element. The stiffness matrix of the mixed method is then given by substituting the discrete deformation and stress tensors in Eq. (34):

$$\begin{aligned} & \int \left[\left\{ \mathbf{C}_{ijkl} H'_{ij} H'_{kl} - 2T'_{ij} H'_{ij} H_m^{(m)} - 2T'_{ij} H_m^{(n)} H'_{ij} \right. \right. \\ & \left. \left. - 2T_m H'_{ij} H'_{ij} + 3T_m H_m^{(n)} H_m^{(m)} + T_{ij} A_{ih}^{(n)} A_{jh}^{(m)} \right\} \psi^{(n)} \right. \\ & \left. + 3H_m^{(m)} \dot{M}^{(r)} \dot{T}_m^{(r)} \right] dV_e \\ & = \int \rho \dot{\mathbf{g}}_i \cdot N_i^{(m)} dV_e + \int \dot{\mathbf{t}}_i \cdot N_i^{(m)} dS_e, \end{aligned} \quad (37)$$

$$\int \left[3H_m^{(m)} \dot{M}^{(q)} \psi^{(n)} - \frac{1}{J\kappa_0} \dot{M}^{(q)} \dot{M}^{(r)} \dot{T}_m^{(r)} \right] dV_e = 0, \quad (38)$$

where n and r are also number of the velocity node and the pressure stress. V_e and S_e are volume and surface of the element, respectively.

$$\begin{aligned} & \int_{F.I.} \left[\left\{ \mathbf{C}_{ijkl} H'_{ij} H'_{kl} + T'_{ij} A_{ih}^{(m)} A_{jh}^{(n)} - 2T'_{ij} H'_{ij} H_m^{(n)} \right. \right. \\ & \left. \left. - 2T'_{ij} H_m^{(n)} H'_{ij} \right\} \psi^{(n)} \right] dV_e \\ & + \int_{R.I.} \left[\left\{ 9\kappa_0 J H_m^{(m)} H_m^{(n)} - 2T_m H'_{ij} H'_{ij} \right. \right. \\ & \left. \left. + 3T_m H_m^{(m)} H_m^{(n)} + T_m A_{ij}^{(m)} A_{ij}^{(n)} \right\} \psi^{(n)} \right] dV_e \\ & = \int \rho \dot{\mathbf{g}}_i \cdot N_i^{(m)} dV_e + \int \dot{\mathbf{t}}_i \cdot N_i^{(m)} dS_e \end{aligned} \quad (39)$$

2.3 Total Lagrangian formulation

The principle of stationary potential energy for the total Lagrangian formulation is given by:

$$\Pi^T(\mathbf{u}) = \int W dV_0 - \int \mathbf{t} \cdot \mathbf{u} dS_0 - \int \rho_0 \mathbf{g} \cdot \mathbf{u} dV_0, \quad (40)$$

where V_0 is the volume occupied by the body in the initial configuration and S_0 is the surface in the initial configuration. $\mathbf{t}$ and $\mathbf{g}$ are the traction and body force vector, respectively, and ρ_0 is the density of material in initial configuration. For a Total Lagrangian formulation, a mixed displacement-pressure method reads:

$$\begin{aligned} & \int \left[\left\{ \frac{\partial \bar{W}}{\partial \mathbf{C}} + p \frac{\partial J}{\partial \mathbf{C}} \right\} : \delta \mathbf{C} + \left\{ J - 1 - \frac{\partial U_C^o(p)}{\partial p} \right\} \delta p \right] dV \\ & = \int \mathbf{t} \cdot \delta \mathbf{u} dS + \int \rho_0 \mathbf{g} \cdot \delta \mathbf{u} dV, \end{aligned} \quad (41)$$

where the Lagrangian multiplier p corresponds to the pressure stress and is related to the updated formulation via $p = T_m$.

$$\begin{aligned} & \int \left[\left\{ \mathbf{C}_{ijkl} \left\{ B_{ij}^{(m)} + 2B_{ij}^{nl(mo)} \right\} \left\{ B_{kl}^{(n)} + 2B_{kl}^{nl(np)} \right\} \right. \right. \\ & \left. \left. + S_{ij} B_{ij}^{nl(mm)} \right\} \Delta u^{(e)} \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{J}{2} (C^{-1})_{ij} \left\{ B_{ij}^{l(m)} + 2B_{ij}^{nl(mn)} u_{(n)}^{(e)} \right\} M^{(r)} \Delta p_{(r)}^{(e)} \Big] dV_e \\
& = \int (t_i + \Delta t_i) N_i^{(m)} dS_e + \int \rho_0 (g_i + \Delta g_i) \cdot N_i^{(m)} dV_e \\
& - \int \left[\frac{1}{2} S_{ij}^{(m)} \left\{ B_{ij}^{l(m)} + 2B_{ij}^{nl(mn)} u_{(n)}^{(e)} \right\} \right] dV_e \quad (42)
\end{aligned}$$

$$\begin{aligned}
& \int \left\{ \frac{J}{2} (C^{-1})_{ij} \left\{ B_{ij}^{l(m)} + 2B_{ij}^{nl(mn)} u_{(n)}^{(e)} \right\} \Delta u_{(m)}^{(e)} M^{(r)} \right. \\
& \left. - \left(\frac{1}{\kappa_0} M^{(q)} M^{(r)} \Delta p_{(q)}^{(e)} \right) \right\} dV_e \\
& = - \int \left[\left\{ (J-1) - \frac{p}{\kappa_0} \right\} M^{(r)} \right] dV_e \quad (43)
\end{aligned}$$

$$\begin{aligned}
u_i &= N_i^{(m)} \cdot u_{(m)}^{(e)}, \quad p = M^{(q)} \cdot p_{(q)}^{(e)}, \\
C_{ij} &= (B^{l(m)} + B^{nl(mn)} u_{(n)}^{(e)}) \cdot u_{(m)}^{(e)} + \delta_{ij}, \quad (44)
\end{aligned}$$

The commercial FEM code ABAQUS is based on the Updated Lagrangian formulation using the virtual work principle⁽¹³⁾

$$\int \mathbf{T} : \delta \mathbf{D} dV - \int \bar{\mathbf{t}} \cdot \delta \mathbf{v} dS - \int \rho \mathbf{g} \cdot \delta \mathbf{v} dV. \quad (45)$$

3. Numerical Analysis

The objective of this section is to assess the formulation and numerical implementation of the formulations presented in the preceding sections.

3.1 Time stepping algorithm

The Newton-Raphson and the forward-Euler algorithm were applied for the time stepping algorithm of the updated Lagrangian formulation. For the total Lagrangian formulation, only the Newton-Raphson algorithm was applied. In the Newton-Raphson algorithm, the iteration calculation would be repeated until the un-constrained node force would be small.

In the forward-Euler algorithm, $\mathbf{u}_n$ and $\mathbf{u}_{n+1}$ are defined as displacement in n and $n+1$ steps, where $\mathbf{u}_n$ is already calculated and $\mathbf{u}_{n+1}$ is unknown. Then, $\mathbf{u}_{n+1}$ is updated according to

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \cdot \mathbf{v}. \quad (46)$$

The deformation gradient tensor $\mathbf{F}$ and the Cauchy-stress are updated according to

$$\mathbf{F}_{n+1} = \mathbf{F}_n + \Delta t \cdot \mathbf{F}_n \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}_n}, \quad (47)$$

$$\begin{aligned}
\mathbf{T}_{n+1} &= \mathbf{T}_n \\
&+ \Delta t \cdot \{ \overset{\circ}{\mathbf{T}}_T + \mathbf{L} \cdot \mathbf{T}_n + \mathbf{T}_n \cdot \mathbf{L}^T - (\text{tr } \mathbf{D}) \mathbf{T}_n \}. \quad (48)
\end{aligned}$$

Node force is given by integration of Eq. (30) for an element.

$$\mathbf{f}_{n+1}^{(e)} = \int \mathbf{T}_{n+1} : \mathbf{H} dV_e \quad (49)$$

3.2 Numerical example

3.2.1 Plane strain deformation of square block

Numerical calculations of plane strain tensile deformation of square block were conducted to assess proposed formulation. The square block is 20 mm of width and 20 mm height. One plane-strain 4-node element is used for modeling. Following Mooney-Rivlin model is introduced for material model^{(14), (15)}

$$\bar{W}(\bar{\mathbf{C}}) = C_1 (\bar{I}_1 - 3.0) + C_2 (\bar{I}_2 - 3.0) \quad (50)$$

Material parameters are assumed as:

$$C_1 = 1.0 \text{ MPa}, \quad C_2 = 0.5 \text{ MPa} \quad (51)$$

Volumetric stress-strain relation assumed incompressible. Then the selective integration method is not able to apply, because it can be applied only for compressible condition.

Both of the forward-Euler and Newton-Raphson algorithms were applied for the updated Lagrangian formulation. For 1 incremental calculation of the Newton-Raphson algorithm, 50% of tensile strain $\Delta \varepsilon_v$ in vertical direction and 2 times iteration calculations were applied. 1% of tensile strain ($\Delta \varepsilon_v$) and 2 times iteration calculations were applied for forward-Euler algorithm. FEM mesh, stress-strain relation and volume-deformation relation are shown in Fig. 2.

Stress-strain relations computed by the Newton-Raphson algorithm are correspond to theoretical results. The total and updated Lagrangian formulation shows the same results, if the Newton-Raphson algorithm was applied. The updated Lagrangian formulation applied the forward-Euler algorithm show higher tensile load and smaller volume compared with the Newton-Raphson algorithm.

It would be hard for the forward-Euler algorithm to keep incompressible condition completely even if the incremental deformation would be very small.

3.2.2 Deformation of a rectangle block

Deformation of a simple rectangle block was considered. The rectangle block has 200 mm of width and depth and 400 mm of height. Linear volumetric stress-strain relationships is considered and the bulk modulus κ_0 is assumed as 1 000 MPa. 8-node solid element was applied and the boundary conditions include constrained top and bottom surface.

For time stepping algorithm, the Newton-Raphson algorithm was applied for mixed method by total and updated Lagrangian formulations and the forward-Euler algorithm was applied for selected integration method by updated Lagrangian formulation.

First, a horizontal deformation (u_h) of 400 mm was applied at the top surface. For the Newton-Raphson algorithm, 8 mm of incremental horizontal deformation and one iteration was applied per step. For the forward-Euler algorithm, 4 mm of deformation was applied for one step. Plots of the resulting load-displacement curves are shown

Fig. 2 Relationships above vertical load, area of one element and vertical displacement in plane strain and incompressible conditions ($C_1 = 1.0$, $C_2 = 0.5$ [MPa], $\kappa_0 = \infty$)

in Fig. 3, where the horizontal strain has been calculated as $e_h = u_h/200.0$. All curves look similar. However, a numerical error was observed in the results of forward Euler method, because it was calculated without iterations.

Fig. 3 Relationships between vertical load and vertical strain of 3-dimensional analysis ($\kappa_0 = 1000$ MPa)

Fig. 4 Shear deformation of rectangle block

Finite element meshes with a mean stress contour corresponding to 100 mm, 200 mm, 300 mm, 400 mm of horizontal displacement are shown in Fig. 4. Neither lock-

Fig. 5 Relationships between vertical load and vertical strain of 3-dimensional analysis ($\kappa_0 = 1000$ MPa)

Fig. 6 Deformation of compressed rectangle block under 400 mm shear deformation

ing, nor hourglass modes were observed and the deformation patterns calculated from the various formulations are rather similar.

Next, a vertical deformation (u_v) was applied after 400 mm of horizontal deformation until divergence of the calculation occurred. 8 mm of incremental vertical deformation and one iteration per step were applied in case of the Newton-Raphson algorithm and increments of 4 mm were applied for the forward Euler algorithm. Plots of the resulting load-deformation curves are shown in Fig. 5, where the vertical strain has been calculated as $\epsilon_v = u_v/400.0$. FEM meshes with the mean stress contour corresponding to 100 mm, 200 mm, 300 mm, 400 mm of vertical displacement are shown in Fig. 6.

The load-displacement curves for the Newton-Raphson algorithm show convergence characteristics similar to those of ABAQUS. The forward-Euler algorithm

Table 1 Geometrical data of the thick natural rubber bearing

Diameter of rubber bearing	252mm
Thickness of one rubber layer	31.5mm
Number of rubber layers	6
Total thickness of rubber layers	189mm
Thickness of one steel layer	2mm
Number of steel layers	5
Loaded area	50200mm ²

Table 2 Coefficients of the Eq. (52)

A	C_{A1}	C_{A2}	C_{A3}	C_{A4}	C_{A5}
1	2.05×10^{-1}	1.32×10^{-2}	2.35×10^{-4}	1.71×10^{-1}	-2.12
2	1.35×10^{-2}	-1.61×10^{-3}	6.32×10^{-5}	-6.87×10^{-2}	-6.44

shows slightly better convergence behavior, but some elements in the corner exhibit suspicious deformation patterns.

3.2.3 Simulation of isolated rubber Loading tests on a prototype rubber bearing were conducted to assess the applicability of the computer code. The bearing consists six rubber layers and five reinforcing steel plates. The diameter is 252 mm. The thicknesses of the rubber layers and reinforcing steel platens are 31.5 mm and 2 mm, respectively. These measures have been chosen such that the rubber can undergo large deformations and steel platens can experience bending, but are outside the normal design of rubber bearings used for isolation of buildings. The geometrical data are summarized in Table 1. The rubber used for the specimen is a natural rubber with a very small material damping. The potential energy function of the natural rubber material has been evaluated by coupon tests⁽⁸⁾:

$$\frac{\partial \bar{W}^o}{\partial \bar{I}_A} = C_{A1} + C_{A2} \cdot (\bar{I}_A - 3) + C_{A3} \cdot (\bar{I}_A - 3)^2 + C_{A4} \cdot \exp\{C_{A5} \cdot (\bar{I}_A - 3)\} \quad (A = 1, 2) \quad (52)$$

Material parameters in Eq. (52) are shown in Table 2. The volumetric stress-strain relation is assumed linear and the Poisson's ratio is defined as 0.4995. The boundary conditions include a constrained bottom surface, a vertical force of 0 MPa, 1.22 MPa and 2.45 MPa and shear deformation applied to the top surface simultaneously. The finite element mesh is divided into 4 elements in the vertical and radial direction and into 12 elements in the circumferential direction.

Selective integration and the forward-Euler algorithm are applied because of less number of degree of freedom and the increased numerical stability.

For the constitutive equation of reinforcing steel platens, the Jaumann rate of Cauchy stress and an isotropic hardening law are used. The parameters of the hardening law were determined from a tensile loading test on a mild

Fig. 7 Comparison loading tests with finite element analysis ($P = 1.22 \text{ MPa}$)

(a) Relationships between horizontal load and shear strain

(b) Relationships between vertical displacement and horizontal displacement

Fig. 8 Comparison of simulated results with experiments

steel. The relationship between tensile stress and strain is expressed by:

$$\sigma_t = 179.0 + 345.0[1.0 - \exp(-3.40\zeta)] \quad [\text{MPa}] \quad (53)$$

where σ_t is tensile stress and ζ is the equivalent plastic strain. For the elastic properties of the steel, the Young's modulus is taken as 200 GPa and the Poisson's ratio is taken as 0.3.

The pictures of loading experiment and of the finite element mesh with mean stress contours after the application of 1.25 MPa of vertical load are shown in Fig. 7. The

numerical calculation well predicts the large deformation of the rubber layers and the bending behavior of the reinforcing steel platen.

Relationships between horizontal deformation and horizontal load and relationships between horizontal and vertical deformation are shown in Fig. 8. In the case of application of 2.42 MPa vertical load, the vertical displacement could not be measured because it was beyond the capacity of displacement sensor.

In the other cases, the numerical calculation well predicts the vertical displacement due to horizontal displacement. These agreements show the applicability of the formulation to predict coupled behavior of large deformation of rubber material and elasto-plastic deformation of mild steel.

4. Conclusions

Finite element equations based on the Truesdell stress rate of isotropic compressible hyperelasticity are presented. By introducing an intermediate volume-preserving configuration, the stress rate and the consistent stiffness matrix were described using only the deformation in the intermediate configuration. This stress rate refers the same configuration as that of the elasto-plastic constitutive equation.

By comparing with Total Lagrangian formulation and a commercial FEM code, similar accuracy and response characteristics were observed, provided of course that the same algorithm was applied.

Finally, loading test of a prototype of rubber bearing was simulated to assess the applicability of our formulation to this class of problems. Deformation and load-deformation responses are well predicted even if rubber and steel material exhibit large deformations.

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