Topology Optimization for the Extruded Three Dimensional Structure with Constant Cross Section

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In this report, the new method of topology optimization technique that is able to deal with the three-dimensional structure with constant cross section is described. In actual design field, the structure with constant cross section is widely adopted because of manufacturing restrictions and efficient design and maintenance. A key technology is to form a longitudinal group by collecting finite elements along the sweeping curve, and the structure is constructed with these groups. Since the design variables are allocated to each group instead of to each finite element, the total number of design variables is drastically reduced. Illustrative examples are shown and the effectiveness of the proposed method is discussed.

Key Words: Finite Element Method, Topology Optimization, Homogenization Method, Frame Based Unit Cell, Convex Linearization Method

1. Introduction

A considerable number of accomplishments have been achieved in the field of concept design of shape of structure, since Bendsoe and Kikuchi(1) proposed the novel method of topology optimization for continua based on the homogenization method. Even in practical or industrial applications, the method has been successfully utilized to decide the optimal shapes of structural members at the very beginning of design processes. In this context, Suzuki and Kikuchi(2) considered a unit cell with a rectangular hole, which is a periodic microstructure of a porous medium, and applied the homogenization method to evaluate the effective material constants under the plane stress condition. Also, we recently have proposed the frame based unit cell model(3), which enables us to obtain optimal topologies without any filtering scheme. With this model, we also have proposed the method to effectively maximize the stiffness with multiple constraints for eigen frequencies and volume of a designed structure(4).

It is recognized that the topology optimization, which is just mentioned in the above paragraph, is very effective, when we appropriately specify the objectives and the conditions in their applications. However, it seems often difficult to utilize them, especially when actual manufacturing processes are to be taken into account. For example, the parts manufactured by extrusion must have the same shape of cross section along an extruding axis, neglecting small amount of extruding gradient. In fact, structural members such as dumpers of trucks and brackets mounting automobile engines are made by extruding, molding or forging and often have constant cross sections. Since the overall structures are arbitrarily shaped and loaded in a three-dimensional manner, the plane strain condition is not acceptable. If we apply the conventional method of topology optimization, to such problems, it is quite few cases that the optimal topology becomes uniform along the axis of extrusion, since the material can be distributed arbitrarily in the design domain.

As an example, applying the conventional method, we present in Fig. 1 the optimal topology of a cantilever beam-like solid structure, which has the maximum stiffness subjected to 36% volume constraint of the design domain. As can be seen from the figure, the structure tapers off to a point and a void is formed near its fixed end; that is, the shape of the cross section cannot be constant along the beam axis. In addition, it appears to be difficult not only to form this kind of geometry in a manufacturing process, but also to conduct the maintenance. Nonetheless, no attempt has been made on the topology optimization method
that can produce optimal structures with a constant cross section, though there hitherto have been numerous discussions about the applications of the optimal structures obtained by the conventional methods to cross sectional design problems.

This paper proposes a new topology optimization method for a solid structure, by which the cross section perpendicular to an arbitrary axial direction can possess the same topology along the axis, even though arbitrary support and loading conditions are imposed in a three-dimensional manner. The method is actually the modification of our recent development for topology optimization, which is based on the frame based unit cell model, and can produce the optimal structure with a constant cross section along a selected axis. In this method, each finite element in a cross section perpendicular to a certain axis is endowed with the same design variables. To demonstrate the effectiveness of the proposed method, we apply the method to the problems of maximizing the stiffness and eigen frequencies of cantilever-like solid structures, which would afford a better understanding of the present contribution. It is also shown that the method is applicable in optimal layout problems of reinforcing ribs in the bottom of a box. Furthermore, concerning with the cross sectional design of a ring-structure called C-arm, which is a part of a diagnostic device for circulators with X-ray, under multiple support and loading boundary conditions, we report that the method is effectively applied to obtain the optimal cross section that is uniform along the circumferential axis of the arm.

2. Formulation for Topology Optimization for a Structure with Constant Cross Section

Suppose that a three dimensional finite element (FE) solid model shown in Fig. 2 be made by extruding a two dimensional section along a selected axis perpendicular to it and subjected to arbitrary support and loading conditions. Here, the axis can be either a straight line or a curve. In the following formulation, we denote the domain of element $i$ in this FE model by $\Omega_{ei}$.

The discretized form of the equilibrium equation of virtual work in terms of displacement is given as follows:

$$a_b(u_h,v_b) = \sum_{e=1}^{N_{EL}} a_e(u_h,v_b),$$

$$a_e(u_h,v_b) = \sum_{i=1}^{IMAX} \int_{\Omega_{ei}} \varepsilon(v_h)^T D(\alpha_e) \varepsilon(u_h) d\Omega_{ei}$$

(1)

where $u_h$ is the FE solution of displacement, $v_h$ is the virtual displacement, $\varepsilon(v_h)$ is the virtual strain, $D$ is the elastic coefficient matrix depending on a set of design variables, $\alpha$. Here, the set consists of design variables, $a$, $b$ and $c$ as shown in Fig. 3, which are the lengths of three edges of a hexagonal void in a periodic structure (unit cell). Also, $N_{EL}$ is the number of elements in a cross section and $IMAX$ is that on the axis perpendicular to the cross section.

Equation (1) indicates that the internal virtual work of the overall structure is the sum of the virtual works done by all the groups identified in a cross section and that the virtual work of each group is stored in a single series of finite elements along the axis. Similarly, the virtual work done by external loading can be provided as

$$f_b(v_h) = \sum_{e=1}^{N_{EL}} f_e(v_h),$$

$$f_e(v_h) = \sum_{i=1}^{IMAX} \left( \int_{\partial \Omega_{ei} \cap \Gamma_i} v_h^T \rho(\alpha_e) \mathbf{b}_i d\Omega_{ei} + \int_{\partial \Omega_{ei} \cap \Gamma_{i}} v_h^T t_i d\Gamma_{i} \right)$$

(2)

where $\rho$ is the mass density, whose value depends on the design variables (that are actually the sizes of voids in Fig. 3), $\mathbf{b}_i$ is the body force and $t_i$ is the distributed load on the boundary $\Gamma_i$. Here, $\partial \Omega_{ei} \cap \Gamma_i$ is the overlapped region of the boundary of element $\Omega_{ei}$ and traction boundary.
\( f_e \) is the nodal force vector for element \( e \).

In terms of these expressions, the weak form corresponding to the principle of virtual work is given as follows:

\[
a_{0h}(u_0, v_0) = f_0(v_0), \quad \forall v_0
\]

The principle of minimum total potential energy equivalent to this weak form reads

\[
\text{Minimize} \Pi(\alpha_e, u_0), \quad \Pi(\alpha_e, u_0) = \frac{1}{2} a_{0h}(u_0, u_0) - f_0(u_0)
\]

The optimization problem for the optimal structure that maximizes the stiffness under volume constraint at \( V_0 \) can be formulated as follows:

\[
\text{Maximize} \Pi(\alpha_e, u_0), \quad \text{s.t.} \quad \sum_{e=1}^{\text{IMAX}} \sum_{i=1}^{\text{NEL}} \rho_e d\Omega_e - V_0 \leq 0 \quad \text{and} \quad 0 \leq \alpha_e \leq 1
\]

Since all the elements contained in each group has to have the same values of design variables, the number of design variables necessary in the topology optimization is only three times of element groups in a cross section.

Next, we consider the maximization problem for eigen frequencies with a volume constraint. Adopting the same grouping scheme as in the stiffness design problem, we employ the mean eigenvalue as an objective function, which was originally introduced by Ma, et al.\(^{(5)}\) and was also employed in our previous report\(^{(4)}\). This objective function is given as

\[
\lambda_{\text{mean}} = \left( \sum_{i=1}^{m} w_i \left( \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \right) \right)
\]

where

- \( \lambda_{ij} \) : \( i \)-th eigenvalue
- \( w_i \) : weight associated with \( i \)-th eigenvalue
- \( \lambda_{\text{mean}} \) : mean eigenvalue
- \( m \) : order of eigenvalues under consideration

Then, the optimization problem of maximizing the eigen frequencies of the structure with a constant cross section can be formulated as follows:

\[
\text{Minimize} (-\lambda_{\text{mean}}), \quad \text{s.t.} \quad \sum_{e=1}^{\text{IMAX}} \sum_{i=1}^{\text{NEL}} \rho_e d\Omega_e - V_0 \leq 0 \quad \text{and} \quad 0 \leq \alpha_e \leq 1
\]

Here, the minus sign of the objective function is used to change the maximization problem to the corresponding minimization problem.

For the formulation of the sensitivity of the mean compliance, that of the mean eigenvalue and that of the constrained volume, one can refer to our previous report \(^{(4)}\). In the next section, we shall present the representative numerical examples, by which we examine the effectiveness of the proposed method.

### 3. Numerical Examples

In order to demonstrate the effectiveness of the proposed method, we here present five representative numerical examples. As a lucid example, we first consider two optimization problems of maximizing the stiffness for cantilever-like solid structures and then illustrate the applicability to the maximization problem of eigen frequency. Next, the optimal layout problem for reinforcing ribs on a thin plate affords scope for our study. Finally, we report the success when we applied the method to the design of an actual medical device.

#### 3.1 Topology for a cantilever-like solid structure with maximum stiffness (a single loading)

We examine the capability of the proposed method by using the model shown in Fig. 1. The whole size of the design domain is \( 9 \times 10^{-3} \text{m} \times 9 \times 10^{-3} \text{m} \times 4 \times 10^{-2} \text{m} \), and the material constants are Young’s modulus 70 300 Mpa and Poisson’s ratio 0.3. Also, \( 9 \times 9 \times 40 = 3240 \) cubic finite elements are used for the spatial discretization. The one end surface is fixed and the load of 40 N is applied to the center of the other side. In this study, the possibility of buckling is not considered, when the state variables are evaluated.

Figure 4 shows the topology obtained by minimizing the mean compliance with volume constraint of less than or equal to 36%. A beam-like structure of I-shaped cross section is obtained as an optimal one. The material left at the connection between the web and flange indicates the need for the curvature there. For comparison, Fig. 5 shows the fixed-end-side of one-half of the optimal structure given in Fig. 1, which was obtained by the conventional method.

![Fig. 4 Resulted topology and its cross section](image)

![Fig. 5 Cross section by using the conventional method](image)
The history of the mean compliance calculated by the conventional method (labeled by A) and that of the proposed method (labeled B) are shown in Fig. 6. Here, the horizontal axis is the number of iterations and the vertical one is the mean compliance normalized by the initial value (in the 0th iteration). The calculation spent 286 seconds for 20 iterations by using a notebook PC with a Mobile Pentium III of 600 MHz CPU.

For both optimal structures, the maximum values of displacement ($\delta_{\text{max}}$) and von-Mises stress ($\sigma_{\text{max}}$) are found at the loaded points and near the surface of the fixed end, respectively. However, the optimal structure with a constant cross section exhibits smaller values of displacement and stress in this particular example than that of the conventional method. That is $\delta_{\text{max}} = 4.00 \times 10^{-5}$ m and $\sigma_{\text{max}} = 21.1$ MPa for the latter, whereas $\delta_{\text{max}} = 3.89 \times 10^{-5}$ m and $\sigma_{\text{max}} = 17.0$ MPa for the proposed method. Although the optimal topologies of these structures are quite different, the distinction between their performances cannot be recognized clearly. Nonetheless, the superiority of the proposed method in the manufacturability is evident.

### 3.2 Topology for a cantilever-like solid structure with maximum stiffness (multiple loading)

We consider the same design domain as in the previous section and apply two separate loading patterns of 40 N to the center of the other side; one is the vertical direction and the other the horizontal one. Constraining the volume of the structure into less than or equal to 40% of that of the design domain, we try to obtain the optimal topology that maximizes the stiffness. To deal with these two separate loading patterns simultaneously, we simply follow the method introduced in Ref. (3); that is, we minimized the mean compliance that is calculated by accumulating the largest values among the strain energies of all the elements, each of which is separately calculated by one of these loading patterns. The calculation to obtain the optimal topology spent 291 seconds for 20 iterations. Also, in solving for the state variables, we do not take into account the possibility of buckling. The optimal structure obtained in this calculation is shown in Fig. 8 and the one by the conventional method is in Fig. 9.

The conventional method provides smaller values of maximum displacement and stress than the proposed method; that is, the maximum values of displacement ($\delta_{\text{max}}$) and von-Mises stress ($\sigma_{\text{max}}$) are $\delta_{\text{max}} = 3.61 \times 10^{-5}$ m and $\sigma_{\text{max}} = 14.5$ MPa for the conventional method, and $\delta_{\text{max}} = 6.55 \times 10^{-5}$ m and $\sigma_{\text{max}} = 22.2$ MPa for the proposed method. As can be seen from Fig. 9, the topology obtained by the conventional method has a large hole and tapers off to the loading point. On the other hand, the proposed method provides the topology shown in Fig. 8. This optimal structure has a cross-web in its cross section, since the loading is applied at the center of the cross section of the tip. However, this web seems play no role at the fixed end and, therefore, its volume is unnecessary. This presumably causes the aforementioned differences between the maximum displacements. Also, the reason why the center of the cross-web is void of the size of a single finite element is that the forces can be transmitted only through the surrounding nodes. It is to be noted that we have employed the displacement-assumed formulation for the finite element approximation for solving the state equations.

### 3.3 Topology for a cantilever-like solid structure with maximum eigen-frequency

In this section, considering the same cantilever-like
model, we try to obtain an optimal topology that maximizes several eigen-frequencies. In addition to the material constants adopted in the above, the mass density $\rho = 2.68 \times 10^3 \text{ kg/m}^3$ is used for the eigenvalue problem. With the volume constraint set at 40\% of the design domain, the first, second and third eigen frequencies are maximized by means of the mean eigenvalue defined by Eq. (6).

First, we present the topology obtained by the conventional method in Fig. 10. In the figure, the transparent portion of the design domain is the domain where the optimization process assigns no material. The optimal structure has the longitudinal length of 40\% of the design domain, which reveals very stiff response. The first and the second eigen frequencies are coincident and their corresponding modes are similar bending ones about the two orthogonal axes in the cross section. The third eigen mode is a twisting one about the longitudinal axis. The histories of convergence of these eigen frequencies are shown in Fig. 11.

Next, the topology obtained by the proposed method is shown in Fig. 12, and the first and the third modes are depicted in Fig. 13. As in the above, the first and second eigen frequencies are all the same and the corresponding eigen modes are similar, while the third mode is a twisting mode about the longitudinal axis. The convergence histories of these three eigen frequencies are given in Fig. 14. The calculation required 604 seconds for 20 iterations.

For reference, we present the optimal topology and its cross section in Fig. 15, when maximizing only the third eigen frequency whose corresponding eigen mode is a twisting one about the longitudinal axis. It is recognized that this maximization is equivalent to that of the stiffness with respect to the torsion. The circular cross section of a cylindrical structure can be obtained by the proposed method, though the spatial resolution is rather low in this present calculation.

3.4 Rib-reinforcement on the bottom surface of a box structure

In the previous sections, we have considered the constant cross section over the whole domain of the structure. In this section, we demonstrate that the proposed method can be applied to optimal layout problems of rib-reinforcements by dividing of a design domain. As an
example, we consider an optimal layout problem of rib-reinforcements on the bottom of the box shown in Fig. 16, which is subjected to out of plane shear loading.

Figure 16 shows the box structure under consideration, whose bottom can be reinforced, and the illustration of the support and loading conditions. The outer size of the box is $40 \times 10^{-3} \text{ m} \times 20 \times 10^{-3} \text{ m} \times 10 \times 10^{-3} \text{ m}$. The thickness of the lateral walls and the bottom of the box are $1 \times 10^{-3} \text{ m}$ and $4 \times 10^{-3} \text{ m}$, respectively. The material constants are Young’s modulus of 70 300 MPa and Poisson’s ratio of 0.3. The forces indicated by white arrows in Fig. 16 are supposed to cause out of plane shear of the bottom and each of them is set at 100 N. The total numbers of nodes and elements are 9 471 and 3 896, respectively. The bottom part of this box structure is composed of four layers of finite elements. Three layers out of the four layers are used for the design domain and all the remaining regions of the structure are not designed. Thus, by designing the constant cross section with a certain volume constraint in the design domain, we are expecting that reinforcements be placed on the bottom modeled by a single layer of elements. Figure 17 shows the optimal distribution of materials, i.e., the optimal layout of reinforcement ribs, when we maximize the stiffness of the whole structure with volume constraint of less than 10% of the design domain. Figure 18 shows the deformed configuration of the optimal structure and Fig. 19 presents the convergence history of the mean compliance. Probably due to the support condition that actually causes instability, the mean compliance increases up to 12th iteration, but finally tends to convergence. The calculation time required for 40 iterations was 618 seconds.

If the same amount of material is uniformly distributed in the design domain, the structure exhibits the maximum displacement of $\delta = 4.08 \times 10^{-3} \text{ m}$. On the other hand, the optimal structure obtained here has the maximum displacement of $\delta = 1.50 \times 10^{-3} \text{ m}$. The effect of the placed rib-reinforcements is evident.

As can be seen in this example, we are able to design a constant cross section only in a part of the structure, by relevantly setting the design domain. Thus, we conclude that this kind of optimal layout problems of rib-reinforcements can be applied to various design problems in practice.

### 3.5 Cross section design of a structure with a circumferential axis

As a final example, we here try to obtain the optimal topology of a ring-like structure called C-arm, as shown in Fig. 20, which is a part of a diagnostic device for circulators with X-ray. The optimal structure should be of a constant cross section along the circumferential axis.

Let the C-arm of internal radius 0.890 m shown
in Fig. 21 be subjected to multiple support and loading boundary conditions. We impose seven sets of constraints, each of which is shifted from adjacent one by 20 degree. For each set of constraint, adopting cylindrical coordinate system \((R, \theta, Z)\), we fix eight points in the \(R\)-direction (radius direction), 4 points in the \(Z\)-direction (longitudinal direction) and 2 points in the \(\theta\)-direction. As can be shown in the left of Fig. 22, we consider both the dead loadings of the X-ray generator (A: 97 kg mass) and of the detector (B: 77 kg mass), each of which is connected by rigid elements to one of the two ends of the arm. Rotating five times the arm by 45 degree each, we have \(7 \times 5 = 35\) combinations of the support and loading conditions. The design domain is illustrated at the right of Fig. 22. The material constants used in this example are Young’s modulus \(6.8941 \times 10^4\) MPa, Poisson’s ratio 0.33 and the mass density \(2.75 \times 10^3\) kg/m\(^3\) with the gravity \(9.80665\) m/s\(^2\).

Under these conditions, we have obtained the optimal structure that maximizes the stiffness with volume constraint of less than 50% of the design domain. Figure 23 presents the topology of the designed cross section. Also, for reference, the optimal structure obtained by the conventional method is shown in Fig. 24 and the topologies of two representative cross sections of this structure are given in Fig. 25. As can be seen from Fig. 25, the conventional topology optimization is no longer applicable to this kind of design problem. Table 1 presents the maximum values of displacements and von-Mises stresses among all the results with 35 separate support-loading conditions. Although both values by the conventional method are smaller than those of the proposed method, it should be noted that we could not manufacture the designed structure with the cross sections shown in Fig. 25.

Figure 26 illustrates the convergence histories of the mean compliances calculated by both the present and conventional methods. Here, label A indicates the conver-
gence history by the proposed method, whereas label B is for that by the conventional one.

The total numbers of nodes and elements of the FE model in this example are 30,332 and 24,208 (hexagonal elements), respectively, and the number of constraint and loading cases is 35. Since the computational cost for this model appears to be relatively high, the iteration process is terminated at the 10th iteration in this particular computation. Although, according to Fig. 26, a few more iterations are needed to attain the final convergence state, the proposed method provides a stable convergence trend. This trend is probably due to the fact that the number of the design variables in the proposed method is much smaller than that of the conventional one.

The calculation spent 33,853 seconds with a desktop PC with a Pentium III 1.0 GHz CPU. As a final comment, the obtained optimal topology is in good agreement with that of the actual cross section that has already been designed independently.

4. Conclusions

We have developed a new topology optimization method for a class of structures with constant cross section, which includes members manufactured by extruding and molding, even though they are subjected to three dimensionally arbitrary support and loading conditions.

Architectural members such as I-shaped steel ones and structural members such as bumpers of trucks have high bending rigidity and easily manufactured by molding and extruding. Such topologies cannot be obtained by the direct application of the conventional method for topology optimization, but the proposed one makes it possible.

In the proposed method, for a three dimensional FE solid model that is generated by extruding a plane FE model, each finite element in its cross section perpendicular to a certain axis is set to belong to a group of finite elements along the axis, and each group is endowed with the same design variables.

When using, for instance, the mean compliance as an objective function, we assign the sum of them for all the finite elements in a group to each finite element located in a single cross section as a representative value. Then, the optimization process can be conducted on this virtual cross section, in which each finite element has the assigned representative mean compliance. Similarly, the sum of the sensitivities of objective or constraint functions of all the elements in this group represents the ones for each finite element on this cross section. Thus, this type of grouping has been recognized as a key technology to realize the topology optimization for structures with a constant cross section.

As were shown in the numerical examples, the proposed method enables us to design not only the I-shaped, rectangular and circular cross sections of the cantilever beams subjected to bending loading, but also the cross section of a ring-structure with a curved axis. In addition, we have demonstrated that the optimal layout problems of rib-reinforcements can be resolved by carefully contriving the usage of the proposed method.

The conventional topology optimization method tends to distribute materials move the most outer region of the design domain so that the designed structure can most effectively resist bending and torsion, and, as a result, often generates a void in the structure. This design is itself meaningful, but carries burdens in manufacturing processes and therefore may not be acceptable in practice. This paper has brought about a satisfactory solution.

In all the example problems, we have utilized the frame based unit cell model, which was developed in our previous report. Also, we employed the optimization algorithm called CONLIN⁶, which is a combination of the convex linearization and the dual method.

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