Approach to Analysis of Mechanical Behavior of Textile Composites by Inclusion Element Method*

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The inclusion element method with a simple grid model has been proposed as one of the analytical techniques of the mechanical behavior of textile composites, and the effectiveness of this method has been verified. The inclusion element method is applicable to the analysis of all types of textile composites because the element stiffness obtained by the inclusion method through cooperation with a fabric structure simulator is used. From the result of the analysis by the inclusion element method, it has been confirmed that the peculiar crimp-interchange of woven composites occurs and high tensile stress arises at the elements with fiber bundles oriented in the load direction. A comparison between the analyses using a real model and the inclusion element model has shown relatively good agreement. Although the analytical result is greatly dependent on the grid pattern, the inclusion element method can provide a sufficient accuracy of results even when the number of elements in the model is lower than that in the real model.

Key Words: Reinforced Plastics, Textile Composites, Inclusion Method, Finite Element Method, Computational Mechanics

1. Introduction

Textile composites have good resistance to impact because of the effect of friction between the fiber bundles with a woven structure, and molding to complicated shapes is also easy because of the peculiar flexibility of the fabric. If less expensive base materials, such as plain-woven fabrics, are used, the production cost of textile composites can also be reduced, and further extent of the fields of application can be expected. However, it is very difficult to elucidate the mechanical and fracture behaviors of textile composites because they have complicated characteristics as a result of the interaction between woven fibers.

Concerning an analytical model for use in determining the mechanical behavior of textile composites, various investigations, e.g., the series of studies by Ishikawa and Chou(1) – (3), have been carried out. At present, the RVE method and the homogenization method have been studied, and the usefulness of these techniques has been reported(4). With these techniques, the mechanical properties of textile composites can be obtained through the use of finite element models with several periods or one period (unit cell) assuming material periodicity. However, since in these techniques the microstructure of fabrics is considered, a detailed mesh model of woven structures is indispensable for their application. In addition, problems such as the assumption of perfect periodicity arise.

In the meantime, contrary to the techniques in which the detailed models of textile composites are used, such as the RVE method and the homogenization method, the equivalent inclusion method has been proposed by Eshelby(5) and Mori and Tanaka(6), and several studies in which the fiber bundles of textile composites were treated as equivalent inclusions were carried out(7), (8). This is a very useful technique because its computation speed is rapid and it can be applied to all fabric structures using

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only the fiber orientation angles and volume fractions in textile composites. However, it is unsuitable for local stress estimation.

In this study, to solve the above-mentioned problems, we propose the inclusion element method, in which we can eliminate the labor of making detailed finite element models from the geometric construction of the fiber bundles and which can be used to estimate the local stresses in textile composites. In the inclusion element method, simple grids (subcells) of textile composites are used. Each subcell is a finite element, and the element stiffness of each subcell is calculated by the inclusion method using the fiber orientation and volume fraction obtained from the geometric construction of the fiber bundles in textile composites.

2. Textile Composites

2.1 Geometric construction of textile composites

Textile composites can be largely categorized into two types: two-dimensional fabrics, such as plain, twill and satin, which are produced by interlacing warp and weft yarns, and three-dimensional fabrics produced using multiaxial yarns. As shown in Fig. 1, in plain weave, each warp fiber passes alternately under and over each weft fiber, and in twill weave, two or more warp fibers are alternately woven over and under two or more weft fibers. Satin weave is fundamentally the twill weave modified such that there are fewer intersections of warp and weft yarns. When the total number of warp yarns crossed and passed under weft yarns is \( n \), it is called \( n \)-harness satin weave. Three-dimensional fabrics have not only in-plane yarns but also those through the thickness, as shown in Fig. 2. As above mentioned, there are two types of representative fabrics of textile composites, but there are many kinds of weaves, therefore, it is possible to categorize them in more detail.

2.2 Modeling of woven structure

Although the modeling of the geometric construction of textile composites is very difficult, a fabric structure simulator, WiseTex, has been developed by Lomov et al.\(^{19}\) WiseTex can automatically calculate the geometric construction of textile composites when conditions such as the mechanical properties of fibers and the position of yarns are set. It can also provide a three-dimensional image. This software is able to deal with not only two-dimensional fabrics, such as plain, twill and satin, but also three-dimensional fabrics and knits. Examples of textile composites models prepared by WiseTex are shown in Fig. 3. In this study, we have constructed an analytical system which makes use of the WiseTex software.

3. Inclusion Element Method

3.1 Preparation of unit cell and subcell partition

In the Inclusion Element Method, the fabric periodicity shown in Fig. 4 is mapped into a unit cell, and the regular grid of subcells shown in Fig. 5 is produced. Using
the geometric construction of the fiber bundles in textile composites models by WiseTex, respective yarn orientations and volume fractions in the subcells can be obtained. Then the homogenized properties of each subcell are calculated by the inclusion method. That is, the inclusion element method is an approach by which to treat the subcells as elements in the finite element method. As the inclusion method in this study, we used Eshelby’s transformation concept combined with the Mori-Tanaka scheme for considering the interaction effects between inclusions.

3.2 Average properties and strain concentration tensor of subcell

The average stiffness of composites is the tensor $C$ that maps a uniform strain to an average stress.

$$\bar{\sigma} = C \bar{\varepsilon}$$  \hspace{1cm} (1)

An important related concept, first introduced by Hill(10), is the idea of strain concentration tensors $A$. These are essentially the following relations between an average inclusion or matrix strain and the corresponding average strain in the composites.

$$\bar{\varepsilon} = A \varepsilon$$ and $$\bar{\sigma} = A \sigma$$  \hspace{1cm} (2)

Superscripts $i$ and $m$ indicate an inclusion and a matrix, respectively.

Here, we implement an alternate strain concentration tensor $\tilde{A}$ that relates the average inclusion strain to the average matrix strain.

$$\varepsilon = \tilde{A} \varepsilon_m$$ \hspace{1cm} (3)

This is related to $A^I$ by

$$A^I = \tilde{A} \left( 1 - \nu' I + \nu' \varepsilon \right)^{-1},$$ \hspace{1cm} (4)

where $\nu'$ indicates the volume fraction of the inclusions. Finally, the effective stiffness tensors of the composites can be expressed as Eq. (5) using the properties of the inclusions and the matrix and using the strain concentration tensors.

$$C = C_m + \nu' \left( C_C + C_m \right) A^I$$ \hspace{1cm} (5)

3.3 Eshelby’s equivalent inclusion method

Eshelby solved the elastic stress field in and around an ellipsoidal particle in an infinite matrix. By assuming the particle to be a prolate ellipsoid, we can use Eshelby’s result to model the stress and strain fields around a cylindrical particle. Eshelby considered an infinite solid body that contains a homogeneous particle with stiffness $C_m$ and that undergoes some transformation. A particular small region of the body is called the inclusion, and the rest of the body is called the matrix. If the inclusion is a separate body, it will acquire a uniform strain $\varepsilon^I$ with no surface traction or stress. $\varepsilon^T$ is called the transformation strain, and this strain might be induced by a phase transformation or a combination of a temperature change and different thermal expansion coefficients in the inclusion and in the matrix. In fact, the inclusion is bonded to the matrix, so when the transformation occurs, the whole body develops a complicated strain field $\varepsilon^C(x)$ related to its shape before the transformation.

$$\sigma^m(x) = C^m \varepsilon^C(x)$$ \hspace{1cm} (6)

Within the inclusion, the transformation does not contribute to the stress, so the inclusion stress is

$$\sigma^I = C^m (\varepsilon^C - \varepsilon^T).$$ \hspace{1cm} (7)

Eshelby showed that within an ellipsoidal inclusion, strain $\varepsilon^C$ is uniform, and is related to the transformation strain by

$$\varepsilon^C = E \varepsilon^T.$$ \hspace{1cm} (8)

$E$ is called the Eshelby tensor, and it depends only on the inclusion aspect ratio and the matrix elastic constants.

The second step in Eshelby’s approach is to demonstrate equivalence between the cause of the homogeneous inclusion and an inhomogeneous inclusion of the same shape. Consider two infinite bodies of a matrix. One has a homogeneous inclusion with some transformation strain $\varepsilon^T$: the other has an inhomogeneous inclusion with a different stiffness $C^I$, but no transformation strain. In subjecting both bodies to a uniform applied strain $\varepsilon^A$ at infinity, two problems arise when we have the same stress and strain distributions.

For the first problem, the inclusion stress is merely Eq. (7) with the applied strain added:

$$\sigma^I = C^m (\varepsilon^A + \varepsilon^C - \varepsilon^T).$$ \hspace{1cm} (9)

The second problem has no $\varepsilon^T$, resulting a stress of

$$\sigma^I = C^I (\varepsilon^A + \varepsilon^C).$$ \hspace{1cm} (10)

Equating these two expressions, using Eq. (8) and carrying out some rearrangement, we get

$$-[C^m + (C^I - C^m) E] \varepsilon^T = (C^I - C^m) \varepsilon^A.$$ \hspace{1cm} (11)

Note that $\varepsilon^T$ is proportional to $\varepsilon^A$, which makes the stress in the equivalent inhomogeneity proportional to the applied strain.

3.4 Eshelby model at low volume fraction

Eshelby’s model can be used to determine the stiffness of composites with ellipsoidal inclusions at dilute concentrations. To determine the stiffness from Eq. (5), we need only to determine the strain concentration tensor $A^I$. To do this, consider that the average strain is identical to the applied strain for dilute composites,

$$\varepsilon = \varepsilon^A.$$ \hspace{1cm} (12)

Also, the inclusion strain is uniform and given by

$$\varepsilon^I = \varepsilon^A + \varepsilon^C.$$ \hspace{1cm} (13)

Now consider the equivalence between the stresses in the homogeneous and inhomogeneous inclusions using Eqs. (9) and (10):
Then, using Eqs. (8), (12) and (13) to eliminate $\varepsilon^T$, $\varepsilon^A$ and $\varepsilon^C$ from Eq. (14), we have

$$[I + ES^n(C' - C^m)]\bar{\varepsilon} = \bar{\varepsilon}. \quad (15)$$

Comparing Eq. (15) with Eq. (2) shows that the strain concentration tensor for Eshelby’s equivalent inclusion is

$$A^i_{\text{Eshelby}} = [I + ES^n(C' - C^m)]^{-1}. \quad (16)$$

We can use Eq. (5) to predict the modulus of macroscopic composites. However, because Eshelby’s solution is applicable only to a single particle surrounded by an infinite matrix, the modulus predictions based on Eqs. (16) and (5) are accurate only at low volume fractions. Therefore, we must obtain the correct results for nondilute composites considering the effects due to their neighbors. The Eshelby tensor $E$ depends only on the material properties of the matrix and on the aspect ratio of the inclusions, i.e., the expressions for the Eshelby tensor of ellipsoidal inclusions are independent of the material properties of the inclusions. They are given as the very simple formulae for continuous fibers with one axis of the ellipsoid being infinity, for spherical inclusions with all axes of the ellipsoid being same length and for thin circular discs with one axis of the ellipsoid being zero.

### 3.5 Mori-Tanaka method

For modeling for nondilute composites, the Mori-Tanaka method has been proposed by Mori and Tanaka\(^{(6)}\). Suppose that composites comprise a single inclusion in an infinite matrix. We know the dilute strain concentration tensor $A^i_{\text{Eshelby}}$ to be

$$\bar{\varepsilon} = A^i_{\text{Eshelby}} \bar{\varepsilon}. \quad (17)$$

On the contrary, the Mori-Tanaka assumption is that when many identical inclusions are introduced in the composites, the average inclusion strain is given by

$$\bar{\varepsilon} = A^i_{\text{Eshelby}} \bar{\varepsilon}. \quad (18)$$

That is, the average inclusion strain is obtained under the assumption of a far-field strain equal to the average strain in the matrix. Using the alternate strain concentrator defined by Eq. (3), the Mori-Tanaka assumption can be restated as

$$\bar{\varepsilon} = A^i_{\text{MT}} \bar{\varepsilon}. \quad (19)$$

Equation (4) then gives the Mori-Tanaka strain concentrator as

$$A^i_{\text{MT}} = A^i_{\text{Eshelby}} \cdot (1 - v' - \frac{1}{3} A^i_{\text{Eshelby}}) \cdot \frac{1}{1 - \nu'}. \quad (20)$$

This is the basic equation for implementing the Mori-Tanaka model. Alternatively, the Mori-Tanaka method can be formulated to directly give the average compliance tensor of composites as

$$S^i_{\text{MT}} = (1 - v') I + v' \langle A^i_{\text{Eshelby}} \rangle \times \langle (1 - v') C^m + v' (C' A^i_{\text{Eshelby}}) \rangle^{-1}. \quad (21)$$

where the notation $\langle x \rangle$ denotes a volume average of tensor $x$.

### 3.6 Stiffness of fiber bundle

The local axes of the fiber bundles corresponding to the inclusions are denoted by the coordinate system (1,2,3), as shown in Fig. 6. Axis-3 indicates the direction of fiber bundles. Hooke’s Law for an orthotropic material is given by

$$
\begin{bmatrix}
\sigma_{11}^r \\
\sigma_{22}^r \\
\sigma_{33}^r \\
\tau_{12}^r \\
\tau_{13}^r \\
\tau_{23}^r
\end{bmatrix} =
\begin{bmatrix}
C_{11}^r & C_{12}^r & C_{13}^r & 0 & 0 & 0 \\
C_{12}^r & C_{22}^r & C_{13}^r & 0 & 0 & 0 \\
C_{13}^r & C_{23}^r & C_{33}^r & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^r & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55}^r & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^r
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{13} \\
\gamma_{12}
\end{bmatrix}.
$$

Since the fiber bundles are transversely isotropic, the properties of the axis-1 and -2 are equal. Therefore the stiffness constants on Eq. (22) can be found to be

$$
\begin{bmatrix}
C'_{11} = C'_{22} = E_2 \left(1 - \nu_{12}^2 \frac{E_2}{E_1} \right) / \text{det} \\
C'_{33} = E_1 (1 - \nu_{23}^2) / \text{det} \\
C'_{12} = E_2 \left(\nu_{12} + \nu_{23} \frac{E_2}{E_1} \right) / \text{det} \\
C'_{13} = C'_{23} = E_2 (\nu_{12} + \nu_{13} \nu_{23}) / \text{det} \\
C'_{44} = C'_{55} = G_{12} \\
C'_{66} = G_{23}
\end{bmatrix},
$$

where $E_1$, $E_2$, $G_{12}$, $G_{23}$, $\nu_{12}$ and $\nu_{23}$ indicate the mechanical properties of the fiber bundles, which can be calculated using Chamis’s equation\(^{(11)}\) or by the Mori-Tanaka method with the properties of the fibers and matrix which consist of the fiber bundles.

Because the fiber bundles in textile composites are arbitrarily distributed in a three-dimensional space, their local axes denoted by the local coordinate system (1,2,3) must be transformed to the global coordinate system $(x,y,z)$. The transformation matrices are defined as...
with

\[
M = \begin{bmatrix}
  l_1 & m_1 & n_1 \\
  l_2 & m_2 & n_2 \\
  l_3 & m_3 & n_3
\end{bmatrix}.
\]  

(26)

Using these equations, we can obtain Eqs. (27) and (28) for the transformation of the strain and the strain concentration tensor from the local to the global coordinate system.

\[
C' = T_c C'' T_c^{-1}
\]  

(27)

\[
A' = T_c A'' T_c^{-1}
\]  

(28)

\[
M' = T_c M T_c^{-1}
\]  

(29)

If only the components of direction cosines \(l_1, m_1\) and \(n_1\) for the \(z\)-axis of a yarn direction are given, the direction cosines for the \(x\)- and \(y\)-axes are obtained from the following equations.

\[
l_1 = \frac{-m_3}{\sqrt{l_3^2 + m_3^2}}, \quad m_1 = \frac{l_3}{\sqrt{l_3^2 + m_3^2}}, \quad n_1 = 0
\]  

(30)

\[
l_2 = \frac{-m_2}{\sqrt{l_2^2 + m_2^2}}, \quad m_2 = \frac{-m_1}{\sqrt{l_2^2 + m_2^2}}, \quad n_2 = \sqrt{l_2^2 + m_2^2}
\]  

(31)

3.7 Cooperation with finite element method

The finite element method based on the displacement method is widely used to analyze mechanical behavior. In this method, the mechanical characteristics of an analytical object are replaced by the relational expression between loaded external forces and displacements of the object by the principle of virtual work, one of the mechanical principles. The elastic stress-strain relation for general anisotropic materials is expressed below, using 21 components of symmetry.

\[
\begin{bmatrix}
  \sigma_x \\
  \sigma_y \\
  \sigma_z \\
  \tau_{xy} \\
  \tau_{xz} \\
  \tau_{yz}
\end{bmatrix} = \begin{bmatrix}
  C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
  C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
  C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
  C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
  C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
  C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix} \begin{bmatrix}
  \varepsilon_x \\
  \varepsilon_y \\
  \varepsilon_z \\
  \gamma_{xy} \\
  \gamma_{xz} \\
  \gamma_{yz}
\end{bmatrix}
\]  

(31)

In the inclusion element method, the stiffness of each subcell obtained from Eq. (5) corresponds to each element stiffness in the general finite element method, and whole stiffness equations of a structure can be calculated. Therefore, the inclusion element method needs a solver of the finite element method that can set all the components shown in Eq. (31) as element stiffness. In this study, the finite element software developed by the authors was used, but it is also possible to use the general-purpose finite element analysis systems that are compatible with the input method of the element stiffness.

4. Example of Analysis and Comparison with Real Model

4.1 Analytical model of woven composites

The woven composites used in an example analysis is a plain-weave fabric model 12 mm wide by 12 mm long by 1.002 mm thick, as shown in Fig. 7, which consists of 36.8% carbon fiber and epoxy resin. We performed the uniaxial tensile analysis in which the boundary conditions were one end fixed and a tensile strain of 1% on the other side. The mechanical properties of the fiber and resin used in the analysis are shown in Table 1. In addition, the mechanical properties of the fiber bundles, which was assumed to make up the fiber volume fraction of 60%, was calculated using Chamis’s equation(11).

4.2 Result of analysis using inclusion element model

The inclusion element model is based on the plain-
weave fabric produced by WiseTex shown in Fig. 8 and has 48 subcells each in the x- and y-directions and 8 subcells in the z-direction, respectively, as shown in Fig. 9. The total number of subcells is 18,432 and the total number of nodes is 21,609. This analytical model is composed of $6 \times 6$ unit cells.

Figure 10 shows the state of deformation and the Von Mises stress distribution (MPa) obtained from the analytical result. Since the deformation magnification in the visualization is 5 times, a wavelike deformation under tensile load can be observed in the $z$-direction, because, when the waves become small upon pulling the fiber bundles of the load direction, the deformations increase the wave undulation of the fiber bundles in the direction perpendicular to the load direction. This is a characteristic of woven composites and is called crimp-interchange under tensile loads. We can observe high tensile stresses at the subcells with fiber bundles oriented in the load direction.

### Table 1 Mechanical properties of the fiber and matrix

<table>
<thead>
<tr>
<th></th>
<th>Fiber</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young's modulus : $E_L$</td>
<td>220,483</td>
<td>3,000</td>
</tr>
<tr>
<td>Young's modulus : $E_T$</td>
<td>137,800</td>
<td>3,000</td>
</tr>
<tr>
<td>Poisson's ratio : $v_{LT}$</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Poisson's ratio : $v_{TZ}$</td>
<td>0.0125</td>
<td>0.3</td>
</tr>
<tr>
<td>Shearing modulus : $G_{LT}$</td>
<td>8,957</td>
<td>1,154</td>
</tr>
<tr>
<td>Shearing modulus : $G_{TZ}$</td>
<td>6,805</td>
<td>1,154</td>
</tr>
</tbody>
</table>

4.3 Comparison with result of analysis using real model

In order to verify the inclusion element method, we compared the results obtained using the inclusion element...
Fig. 14 Results of displacement $z$ along the $y$-axis at $x = 6$ mm

Fig. 15 Relationship between max. Von Mises stress and number of nodes

The fiber bundles oriented can be highly stressed in the load direction. The maximum Von Mises stress in the real model is 974 MPa, but in the inclusion element model, it is 899 MPa, which is about 8% lower. This is because the element stiffness of the inclusion element model is calculated as an average value between the fiber bundles and resin, and greatly depends on the approach to the division into subcells. However, the total number of elements and nodes of the inclusion element model is about 1/2 that of the real model in this analysis. Then the results of varying the element number are shown in Fig. 15. As seen from this result of the analysis, the greater the number of nodes, the greater is the associated maximum stress. The result strongly depends on the approach to the division into subcells. However, it is proven that the result is constant over about 10,000 nodes. When the number of element divisions is similar between these two models, the maximum stress obtained using the inclusion element model has the higher value. The calculation times using the two models are nearly equal when the finite element model has the same degrees of freedom. Therefore, it is an advantage of the inclusion element method that analytical results do not differ greatly from those obtained using the real model even if a model with a small number of elements and a simple grid mesh is used.

5. Conclusions

As one of the analytical techniques of the mechanical behavior of textile composites, the inclusion element method with a simple grid model has been proposed, and the effectiveness of this method has been verified. The inclusion element method is applicable to the analysis of all types of textile composites using the element stiffness obtained by the inclusion method along with the fabric structure simulator WiseTex. From the results of the analysis by the inclusion element method, we have confirmed that a peculiar crimp-interchange of woven composites occurs and high tensile stress arises at the elements with the fiber bundles oriented in the load direction. Although the analytical result obtained using the inclusion element model is greatly dependent on the grid pattern, even when the number of elements in the inclusion element model is lower than that in the real model, sufficiently accurate of results can be obtained. In addition the inclusion element method being applicable to all types of textile structures, because of the use of a very simple finite-element-model-like grid, the application of parallel calculation methods with unit division, such as the substructure method, is also easy. Therefore, the application of the inclusion element method to actual structures can also be expected. It is considered that the inclusion element method will be applicable also to a nonlinear analysis of damage because the stress and strain arising in the fiber bundles and matrix can be evaluated using the results of stress and strain obtained by the inclusion element method.
References


