Finite Deformation Analysis Using Natural Strain*

(On Comparison with Other Strain Expressions under Three Different Types of Deformation Paths and the Rationality of Natural Strain)

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The effectiveness of the Natural Strain theory for describing a large deformation is mentioned in this paper. The Natural Strain is obtained by integrating infinitesimal strain increment on an identical line element over the whole process of the deformation path. Consequently, the shearing strain becomes pure angular strain, which is obtained by removing the rigid body rotation from the rotating angle of a line element. Since the expression of the Natural Strain is different from the strain expression of ordinary rate type, the additive law of strain on an identical line element can be satisfied. In this paper, the finite deformation analyses of a pure elastic body concerning the three different types of deformation paths are discussed on the combined deformation of uni-axial tension and simple shear, and the Natural Strain proposed in this paper is compared with other strain expressions and the rationality of this strain expression is confirmed.

Key Words: Finite Deformation Theory, Numerical Analysis, Nonlinear Problem, Natural Strain, Angular Strain, Rate of Deformation Tensor, Plasticity, Deformation History

1. Introduction

In the strain expressions of finite deformation, the A.E.Green and W.Zerna's finite strain theory⁽¹⁾ have been widely applied in the structural analysis. This strain expression is defined by the difference of metric tensor between an original undeformed state and deformed state and is obtained by only the geometrical relation of embedded coordinate axes in the body, and as you know, this method is the total Lagrange style. Therefore, since this strain expression can not exactly represent the deformed intermediate state, this method is not always effective to analyze the relation between stress and strain that depends on the deformation history such as the plastic deformation, even if this strain expression is effective in the analysis of an isotropic elastic body. Hence, the strain expression of rate type, which represents an instantaneous motion in the current condition of a deforming body, i.e., the rate of

deformation tensor, is mainly used in the theoretical studies on an elasto-plastic analysis for large deformation, and this is an update Lagrange method.

However, in the finite deformation analysis, using the rate of deformation tensor as a strain rate and the Jaumann derivative based on spin tensors as a stress rate, unexpected result was derived in the numerical calculation of a simple shear deformation assuming the kinematic hardening materials. Namely, the oscillatory normal and shear stress are induced for monotonically increasing simple shear deformation. Afterwards, assuming that the unexpected result is caused by the spin tensor in a Jaumann type stress rate, many researchers, e.g. E.H.Lee, have been tried to solve the finite deformation problem by introducing the various modified Jaumann derivatives based on the modified spin tensor^{(2),(3)}. As a result, the monotonically increasing stress is obtained from the monotonically increasing simple shear deformation, and unexpected result has been avoided. However, these modified spin tensors in the stress rate are not always clear for a physical meaning. As mentioned above, although many researchers mainly pay attention to the modifications of the spin tensors in the stress rate, the method of strain expression and the modification for the rotation in the strain rate are not espe-

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cially mentioned in those papers. So, it seems that there is an essential problem in the conventional strain expression rather than the stress expression.

As well-known, the ordinary strain expression of rate type can be obtained by resolving the velocity gradient tensor, which is measured from newly defined orthogonal coordinate system at a moving point in the body, into symmetric part $\dot{\epsilon}$, i.e. the rate of deformation tensor, and skew-symmetric part $\dot{\omega}$, i.e. the spin tensor. However, the rate of deformation tensor does not express the strain rate on an identical line element in the body. Namely, the rate of deformation tensor is the strain velocity concerning elements streaming into each orthogonal axis at the moving observation point and replacing one after another. Hence, this is a method that the reference coordinate updates with an increase of deformation. So, it is not able to express the strain on an identical line element in the body, and the physical meaning of this strain that is obtained by integration of the strain rate is not clear. Moreover, the shearing components of this strain are obtained by taking the average of angular velocity of these coordinate axes and, as the result, the spin can be removed from the shearing strain, but the rigid body rotation can not be accurately removed from the total strain which is obtained by integration of this strain rate.

On the other hand, the Natural Strain stated in this paper is a strain expression that is associated with the infinitesimal line element located in the arbitrary direction and is obtained by integration of the infinitesimal strain increment on an identical line element over the whole process of deformation path. Hence, an additive law of strain on an identical line element can be satisfied. Especially, since the shearing component of this strain is obtained by integration of the infinitesimal angular strain, which can remove the rigid body rotation from the current rotating angle of the line element over the whole process of the deformation path, and becomes pure angular strain, this strain expression is different from the above-mentioned rate of deformation tensor. Hence, in the Natural Strain theory, the rigid body rotation caused by deformation can be accurately removed from the shearing strain.

In the theoretical studies, not only the modification of spin tensor in the stress rate but also reconsideration of the formulation of strain rate is considered by some researchers other than the Natural Strain theory in this paper, e.g. the stress analysis using the rotationless strain, which is obtained by removing the rigid body rotation from the strain rate, is suggested by Stören-Rice⁽⁴⁾, the strain rate suggested by S.Murakami and N.Ohno⁽⁵⁾, and by M.Gotoh⁽⁶⁾ etc. In these analyses, using the formulation similar to the stress rate, the modification of the rigid body rotation occurring in the body is attempted by introducing the Jaumann type derivative into the ordinary rate of deformation tensor.

Moreover, in the recent study, taking note of the rate type formulation of the Hencky's logarithmic strain, the suggestion about the spin, i.e. logarithmic spin, is attempted by H. Xiao, O.T. Bruhns and A. Meyers⁽⁹⁾. However, since these formulations do not essentially represent the strain associated with an identical line element in the body, the total strains obtained by integration of these strain rates are merely the superposition of the strain rate measured on the updated reference coordinate. Therefore, it is expected that the additive law of strain on an identical line element does not hold as well as the case of the ordinary rate of deformation tensor.

In this paper, the combined deformation of a simple tension, i.e. uni-axial tension, and a simple shear is chosen as the subject of this research, and the numerical results for Natural Strain and results for these strains are compared. Essentially, since the strain expression should be determined from the geometrical relation of deformation, the relation between the given deformation and the strain expression must not involve any inconsistencies regardless of the elastic deformation or the plastic deformation. In general, in the case of an isotropic elastic body, if the final states of each different deformation path are the same, the strain at the final state also must be determined uniquely and must be entirely identical even if those paths are different. When we consider the idealized isotropic elastic body, this is the same as the fact that a final state of stress should be decided uniquely without influence of those loading histories. Hence, three different deformation paths concerning the above-mentioned combined deformation are considered in this paper, and the strains obtained by those deformation paths are compared. As a result, it is revealed that the Natural Strain theory proposed in this paper is more reasonable strain expression in the large deformation analysis compared with other strain expressions.

2. The Rate Type Expressions for Natural Strain in a General Deformation Field

2.1 The decomposition of deformation and the rigid body rotation

Since the Natural Strain theory is the strain expression based on the identical line element in the body, let us consider at first the elongation and the rotation occurring at an infinitesimal line element in the body.

When a body deforms in the orthogonal Cartesian coordinate system (x, y, z) , an infinitesimal line element locating at arbitrary direction ℓ_o changes its length and direction, and turns into the current line element ℓ . The relation between initial and current line element can be decomposed by investigating the principal value of deformation in this state and is represented as

$$\ell = D \ell_o = H A H_o^T \ell_o$$

$$= \mathbf{H} \mathbf{H}_o^T \mathbf{H}_o \mathbf{\Lambda} \mathbf{H}_o^T \ell_o = \mathbf{H} \mathbf{\Lambda} \mathbf{H}^T \mathbf{H} \mathbf{H}_o^T \ell_o \quad (1)$$

where $\mathbf{D}$ is the well-known deformation gradient tensor⁽⁷⁾. $\mathbf{H}$ and $\mathbf{H}_o$ define the coordinate systems coinciding with the directions of current principal axes and its directions of undeformed state, respectively, and $\mathbf{\Lambda}$ is the principal stretch tensor. Equation (1) means that a line element ℓ_o is firstly stretched by $\mathbf{\Lambda}$ without rotation on $\mathbf{H}_o$ and then the line element is rotated rigidly from $\mathbf{H}_o$ to $\mathbf{H}$ and as the results the current line element is obtained. Or, this equation also means that a line element ℓ_o is rotated rigidly from $\mathbf{H}_o$ to $\mathbf{H}$ at first, and is stretched by $\mathbf{\Lambda}$ on $\mathbf{H}$. Hence, it is apparent that the arbitrary deformation can be divided into the rigid body rotation $\mathbf{R} (= \mathbf{H} \mathbf{H}_o^T)$ and the symmetrical stretch. These results shown by Eq. (1) are similar to the well-known polar decomposition of deformation gradient. Namely, $\mathbf{H}_o \mathbf{\Lambda} \mathbf{H}_o^T$ in Eq. (1) corresponds to the right stretch tensor $\mathbf{U}$ and $\mathbf{H} \mathbf{\Lambda} \mathbf{H}^T$ corresponds to the left stretch tensor $\mathbf{V}$.

2.2 Relation between deformation increment and rigid body rotation increment on an identical line element

When we consider the state at time $t + dt$ that the deformation is slightly progress from the current configuration at time t , an identical line element vector slightly changes its length and direction (see Fig. 1). In this state, the increment of a line element vector can be written as

$$\dot{\ell} = \dot{\mathbf{D}} \ell_o = \dot{\mathbf{D}} \mathbf{D}^{-1} \ell = \mathbf{L} \ell \quad (2)$$

where $\mathbf{L}$ is the well-known velocity gradient tensor.

As previously stated in the introduction, the tensor $\mathbf{L}$ is decomposed into the symmetric part and the skew-symmetric part, i.e. spin tensor $\boldsymbol{\omega}$. And, the symmetric part is called the rate of deformation tensor and is commonly used in the conventional rate type strain expression. However, although this strain expression can remove the spin tensor from shearing strain component by making the velocity gradient tensor into the symmetrical expression, the increment of rigid body rotation that actually occurs cannot completely remove from the shearing strain component.

Fig. 1 Deformation of a line element

Hence, let us consider the method to completely remove the increment of the rigid body rotation from the deformation increment on a line element vector. As mentioned in the previous paragraph, the deformation of a line element can be decomposed into the rigid body rotation and the pure stretching.

Hence, the line element vector removing the rigid body rotation $\tilde{\ell} (= \mathbf{H} \mathbf{H}_o^T)$ is considered here and is indicated by $\tilde{\ell}$. Taking into account the relation of Eq. (1), it can be written as follows.

$$\tilde{\ell} = \mathbf{R}^T \ell = \mathbf{H}_o \mathbf{H}^T \mathbf{H} \mathbf{\Lambda} \mathbf{H}_o^T \ell_o = \mathbf{H}_o \mathbf{\Lambda} \mathbf{H}_o^T \ell_o \quad (3)$$

The above equation can also be interpreted that the current line element vector is measured from the rotating coordinate system $\mathbf{R}$ which changes its direction every moment depending on the magnitude of the deformation gradient tensor $\mathbf{D}$. Therefore, it can be seen that the rigid body rotation is always removed from the line element vector $\tilde{\ell}$.

Thus, the deformation increment on a line element vector can be written as follows.

$$\begin{aligned} \dot{\tilde{\ell}} &= (\dot{\mathbf{H}}_o \mathbf{\Lambda} \mathbf{H}_o^T + \mathbf{H}_o \dot{\mathbf{\Lambda}} \mathbf{H}_o^T + \mathbf{H}_o \mathbf{\Lambda} \dot{\mathbf{H}}_o^T) \ell_o \\ &= (\dot{\mathbf{H}}_o \mathbf{H}^T + \mathbf{H}_o \dot{\mathbf{\Lambda}} \mathbf{\Lambda}^{-1} \mathbf{H}^T + \mathbf{H}_o \mathbf{\Lambda} \dot{\mathbf{H}}_o^T \mathbf{H}_o \mathbf{\Lambda}^{-1} \mathbf{H}^T) \ell \quad (4) \end{aligned}$$

Since $\tilde{\ell}$ is measured from the rotating coordinate system $\mathbf{R}$, the deformation increment of this line element that is expressed by Eq. (4) should be rewritten from the coordinate system fixed in space again.

Namely, we obtain

$$\begin{aligned} \dot{\tilde{\ell}} &= \mathbf{R}^T \dot{\ell} \\ &= \mathbf{H} (\dot{\mathbf{\Lambda}} \mathbf{\Lambda}^{-1} + \mathbf{H}_o^T \dot{\mathbf{H}}_o + \mathbf{\Lambda} \dot{\mathbf{H}}_o^T \mathbf{H}_o \mathbf{\Lambda}^{-1}) \mathbf{H}^T \ell \\ &= \mathbf{L}^* \ell \quad (5) \end{aligned}$$

where $\mathbf{L}^*$ in Eq. (5) means the velocity gradient tensor that is removed the increment of rigid body rotation. The first term in parenthesis $\dot{\mathbf{\Lambda}} \mathbf{\Lambda}^{-1}$ means the increment of stretch on the coordinate system corresponding to the current principal axis $\mathbf{H}$, and the 2nd and the 3rd term are increment of rotation caused on the line element that becomes a principal axis.

If we indicate this relation by using the right stretch tensor $\mathbf{U}$ and the left stretch tensor $\mathbf{V}$ which correspond to the well-known polar decomposition of deformation gradient tensor, it can be rewritten as follows.

$$\text{Firstly, Eq. (3) is rewritten as} \quad \tilde{\ell} = \mathbf{R}^T \mathbf{D} \ell_o = \mathbf{U} \ell_o = \mathbf{R}^T \mathbf{V} \mathbf{R} \ell_o \quad (6)$$

and secondly, Eq. (5) is

$$\begin{aligned} \dot{\tilde{\ell}} &= \mathbf{R}^T \dot{\ell} = (\dot{\mathbf{D}} \mathbf{D}^{-1} - \dot{\mathbf{R}} \mathbf{R}^T) \ell = (\mathbf{L} - \dot{\mathbf{R}} \mathbf{R}^T) \ell \\ &= \mathbf{R} \dot{\mathbf{U}} \mathbf{U}^{-1} \mathbf{R}^T \ell = (\dot{\mathbf{V}} \mathbf{V}^{-1} + \mathbf{V} \dot{\mathbf{R}} \mathbf{R}^T \mathbf{V}^{-1} - \dot{\mathbf{R}} \mathbf{R}^T) \ell \\ &= \dot{\mathbf{V}} \mathbf{V}^{-1} \ell \quad (7) \end{aligned}$$

Thus, we can notice from the above equation that the increment of the rigid body rotation $\dot{\mathbf{R}} \mathbf{R}^T$ is removed from the velocity gradient tensor $\mathbf{L}$.

In the Natural Strain theory, the strain increment on an identical line element can be obtained by using the deformation increment of a line element vector that the rigid body rotation is always removed. Then, in the following paragraph, let us explain about the formulation of the strain rate for the Natural Strain theory.

2.3 The rate type expression for the extensional strain

As mentioned in the previous paper⁽⁷⁾, the extensional strain component in the Natural Strain theory is quite different from its component of Hencky's logarithmic strain which is obtained by performing the coordinate transformation. Namely, the extensional strain component in the Natural Strain theory can be obtained by using an identical line element, which changes its length and direction, and it is represented by logarithms of the ratio of current deformed length to original length.

Taking account the relation of Eq. (3), it is expressed as follows.

$$\begin{aligned} e &= \ln \frac{\|\ell\|}{\|\ell_o\|} = \frac{1}{2} \ln \left(\frac{\ell^T \ell}{\ell_o^T \ell_o} \right) = \frac{1}{2} \ln \left(\frac{\tilde{\ell}^T \tilde{\ell}}{\ell_o^T \ell_o} \right) \\ &= \frac{1}{2} \ln \left(\frac{\ell_o^T \mathbf{H}_o \Lambda^2 \mathbf{H}_o^T \ell_o}{\ell_o^T \ell_o} \right) \end{aligned} \quad (8)$$

Moreover, the rate type expression for extensional strain component is obtained by differentiating the above equation. Namely, we have

$$\begin{aligned} \dot{e} &= \frac{1}{2} \frac{\dot{\tilde{\ell}}^T \tilde{\ell} + \tilde{\ell}^T \dot{\tilde{\ell}}}{\tilde{\ell}^T \tilde{\ell}} = \frac{1}{2} \frac{\dot{\tilde{\ell}}^T \mathbf{R}^T \mathbf{R} \tilde{\ell} + \tilde{\ell}^T \mathbf{R}^T \mathbf{R} \dot{\tilde{\ell}}}{\tilde{\ell}^T \mathbf{R}^T \mathbf{R} \tilde{\ell}} \\ &= \frac{1}{2} \frac{\dot{\ell}^T \ell + \ell^T \dot{\ell}}{\ell^T \ell} = \left(\frac{\ell^T}{\sqrt{\ell^T \ell}} \dot{\ell} \right) \frac{1}{\sqrt{\ell^T \ell}} \end{aligned} \quad (9)$$

where $\ell / \sqrt{\ell^T \ell}$ is a unit vector along the deformed current line element, and it is indicated as i_{β_x} in this paper. Therefore, the notation enclosed in the parenthesis of Eq. (9) corresponds to the projection of increment of a line element vector to a current line element vector. And, it means the infinitesimal elongation of a line element (see Fig. 2). If we represent it by using the unit vector i_{β_x} , it can be written as

$$i_{\beta_x}^T \dot{\ell} = \dot{\ell}_n \quad (10)$$

Fig. 2 Deformation increment of a line element

Thus, the rate type expression for extensional strain component of Natural Strain theory can be obtained by dividing the infinitesimal elongation of a line element by the length of a current line element. Substituting the relation of Eq. (5), or Eq. (7), into the Eq. (10), it can be formulated as follows.

$$\begin{aligned} \dot{e} &= \frac{\dot{\ell}_n}{\|\ell\|} = i_{\beta_x}^T \mathbf{L}^* i_{\beta_x} \\ &= i_{\beta_x}^T \mathbf{H} (\dot{\Lambda} \Lambda^{-1} + \Lambda \dot{\mathbf{H}}_o^T \mathbf{H}_o \Lambda^{-1} - \dot{\mathbf{H}}_o^T \mathbf{H}_o) \mathbf{H}^T i_{\beta_x} \\ &= i_{\beta_x}^T \dot{\mathbf{V}} \mathbf{V}^{-1} i_{\beta_x} \end{aligned} \quad (11)$$

2.4 The rate type expression for the shearing strain

The shearing strain component in the Natural Strain theory is expressed by using the difference of angle between undeformed state and deformed current state of an identical line element, which is measured from the principal coordinate system $\mathbf{H}_o$ and $\mathbf{H}^{(7)}$. Namely, it is a pure angular strain, which can be obtained by removing the rigid body rotation from the rotating angle of a line element vector over the whole process of deformation. Taking into consideration of the relation of Eq. (3), it can be formulated in the general deformation field as

$$\gamma = \cos^{-1} \left(\frac{\ell_o^T \mathbf{H}_o \mathbf{H}^T \ell}{\sqrt{\ell_o^T \ell_o} \sqrt{\ell^T \ell}} \right) = \cos^{-1} \left(\frac{\ell_o^T \tilde{\ell}}{\sqrt{\ell_o^T \ell_o} \sqrt{\tilde{\ell}^T \tilde{\ell}}} \right) \quad (12)$$

The rate type expression for shearing strain component is obtained by differentiating Eq. (12) in the same way as the extensional strain component. Namely, we have

$$\dot{\gamma} = \frac{-1}{\sin \gamma} \frac{\ell_o^T \dot{\tilde{\ell}} - \ell_o^T \tilde{\ell} \frac{1}{2} \frac{(\dot{\tilde{\ell}}^T \tilde{\ell} + \tilde{\ell}^T \dot{\tilde{\ell}})}{\tilde{\ell}^T \tilde{\ell}}}{\sqrt{\ell_o^T \ell_o} \sqrt{\tilde{\ell}^T \tilde{\ell}}}$$

and, the above equation can be rewrite as

$$\dot{\gamma} = - \frac{\ell_o^T}{\sqrt{\ell_o^T \ell_o}} \left(\dot{\tilde{\ell}} - \frac{\tilde{\ell}^T \dot{\tilde{\ell}}}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} \frac{\tilde{\ell}}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} \right) \frac{1}{\sin \gamma} \frac{1}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} \quad (13)$$

Moreover, a term enclosed in the parenthesis of Eq. (13) can be rewritten as

$$\dot{\tilde{\ell}} - \left(\frac{\tilde{\ell}^T \dot{\tilde{\ell}}}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} \right) \frac{\tilde{\ell}}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} = \dot{\tilde{\ell}} - \dot{\tilde{\ell}}_n = \dot{\tilde{\ell}}_t \quad (14)$$

where from the geometrical relation indicated in Fig. 3, $\dot{\tilde{\ell}}_t$ means the remaining deformation increment that can be obtained by removing the extensional component $\dot{\tilde{\ell}}_n$ from the deformation increment of a line element $\dot{\tilde{\ell}}$ measured from the co-rotational coordinate system $\mathbf{R}$. Hence, Eq. (13) can be expressed as

$$\dot{\gamma} = - \frac{\ell_o^T}{\sqrt{\ell_o^T \ell_o}} \dot{\tilde{\ell}}_t \frac{1}{\sin \gamma} \frac{1}{\sqrt{\tilde{\ell}^T \tilde{\ell}}} \quad (15)$$

Fig. 3 Geometrical relation of a line element

Moreover, since the inner product of a unit vector defining the direction of an undeformed line element $\ell_o / \sqrt{\ell_o^T \ell_o}$ and deformation increment $\dot{\ell}_t$ is represented as

$$\frac{\ell_o^T}{\sqrt{\ell_o^T \ell_o}} \dot{\ell}_t = \sqrt{\dot{\ell}_t^T \dot{\ell}_t} \cos\left(\frac{\pi}{2} + \gamma\right) = -\sqrt{\dot{\ell}_t^T \dot{\ell}_t} \sin \gamma,$$

and Eq. (15) is rewritten as follows.

$$\begin{aligned} \dot{\gamma} &= \frac{\|\dot{\ell}_t\|}{\|\dot{\ell}\|} = \frac{\sqrt{\dot{\ell}_t^T \dot{\ell}_t}}{\sqrt{\dot{\ell}^T \dot{\ell}}} = \frac{\sqrt{\dot{\ell}_t^T \mathbf{R}^T \mathbf{R} \dot{\ell}_t}}{\sqrt{\dot{\ell}^T \mathbf{R}^T \mathbf{R} \dot{\ell}}} \\ &= \frac{\sqrt{\dot{\ell}_t^T \dot{\ell}_t}}{\sqrt{\dot{\ell}^T \dot{\ell}}} = \frac{\|\dot{\ell}_t\|}{\|\dot{\ell}\|} \end{aligned} \quad (16)$$

In the process of derivation of above equation, since $\dot{\ell}_t$ is measured from the coordinate system $\mathbf{R}$, $\dot{\ell}_t$ is re-measured from the coordinate system fixed in the space. Furthermore, $\dot{\ell}_t$ can also be rewritten as

$$\dot{\ell}_t = \mathbf{R} \dot{\ell}_t = \mathbf{R} (\dot{\ell} - \dot{\ell}_n) = \dot{\ell} - \dot{\ell}_n \quad (17)$$

where $\dot{\ell}_n$ is indicated by

$$\dot{\ell}_n = \left(\frac{\ell^T}{\sqrt{\ell^T \ell}} \dot{\ell} \right) \frac{\ell}{\sqrt{\ell^T \ell}}$$

As mentioned above, $\dot{\ell}_n$ means the extensional component for the increment of a line element vector and is obtained by projecting the increment of a line element vector to the direction of a current line element vector. Hence, this vector is the same direction as the current line element i_{β_x} . Therefore, since $\dot{\ell}_t$ means the remaining deformation increment which is obtained by removing this extensional component $\dot{\ell}_n$ from the increment of a line element vector $\dot{\ell}$ as indicated in Eq. (17), $\dot{\ell}_t$ is perpendicular to the current line element vector. Hence, the magnitude of this deformation increment, i.e. $\|\dot{\ell}_t\|$ is represented by using the

orthogonal unit vector to the current line element vector. Namely, we have

$$\|\dot{\ell}_t\| = i_{\beta_y}^T \dot{\ell} \quad (18)$$

Moreover, substituting the relation of Eq. (5), or Eq. (7), into Eq. (18), the rate type expression for shearing strain component can be formulated as follows.

$$\begin{aligned} \dot{\gamma} &= \frac{\|\dot{\ell}_t\|}{\|\dot{\ell}\|} = i_{\beta_y}^T \mathbf{L}^* i_{\beta_x} \\ &= i_{\beta_y}^T \mathbf{H} (\dot{\Lambda} \Lambda^{-1} + \Lambda \dot{\mathbf{H}}_o^T \mathbf{H}_o \Lambda^{-1} - \dot{\mathbf{H}}_o^T \mathbf{H}_o) \mathbf{H}^T i_{\beta_x} \\ &= i_{\beta_y}^T \dot{\mathbf{V}} \mathbf{V}^{-1} i_{\beta_x} \end{aligned} \quad (19)$$

Thus, since Eq. (19) is constituted by not only the velocity gradient tensor removing the rigid body rotation $\mathbf{L}^*$ but also the vector $(i_{\beta_x}, i_{\beta_y})$ defining the direction of line element rotating with deformation, it is obvious that this strain expression is associated with an identical line element in the body during whole process of deformation path. Consequently, the shearing component in the Natural Strain theory is expressed by a pure angular strain that the increment of rigid body rotation can be completely removed from the current rotating angle of an identical line element.

3. The Strain Expression by Stören-Rice and Other Strain Expression

As stated in the introduction, taking note of the necessity for the re-examination of strain expression, the strain expression that reflects the effect of the rigid body rotation to the strain rate is suggested by Stören and Rice⁽⁴⁾. This strain expression is derived by using the symmetric tensor, i.e. the right stretch tensor $\mathbf{U}$, which is removed the rigid body rotation $\mathbf{R}$, and it is defined as

$$\mathbf{E}_R = \int_0^t \frac{1}{2} \left\{ \dot{\mathbf{U}} \mathbf{U}^{-1} + (\dot{\mathbf{U}} \mathbf{U}^{-1})^T \right\} dt \quad (20)$$

Taking into account that the rotationless strain $\mathbf{E}_R$ is measured from the coordinate system rotating with $\mathbf{R}$ in order to remove the rigid body rotation, if we re-measure this strain from the coordinate system fixed in the space, it is expressed as

$$\mathbf{e}_R = \mathbf{R} \mathbf{E}_R \mathbf{R}^T \quad (21)$$

Here, by differentiating Eq. (21) with respect to time, the strain rate of the rotationless strain is represented as follows.

$$\dot{\mathbf{e}}_R = \dot{\mathbf{e}} + \dot{\mathbf{R}} \mathbf{R}^T \mathbf{e}_R - \mathbf{e}_R \dot{\mathbf{R}} \mathbf{R}^T \quad (22)$$

Therefore, it can be interpreted that the strain rate $\dot{\mathbf{e}}_R$ can be obtained by removing the increment of the rigid body rotation $\dot{\mathbf{R}} \mathbf{R}^T$ from the ordinary rate of deformation tensor $\dot{\mathbf{e}} (= (\mathbf{L} + \mathbf{L}^T)/2)$.

Similarly, the strain expressions of the analogous form are suggested by Murakami, Ohno⁽⁵⁾ and Goto⁽⁶⁾. It

is discussed in those papers that the strain rate form similar to the Jaumann type stress rate is necessary in order to satisfy the objectivity for the coordinate transformation.

Hence, this strain rate is expressed by similar form to the Jaumann type stress rate as

$$\dot{\mathbf{e}}_\omega = \dot{\mathbf{e}} + \boldsymbol{\omega} \mathbf{e}_\omega - \mathbf{e}_\omega \boldsymbol{\omega} \quad (23)$$

Although $\boldsymbol{\omega}$ indicated in Eq. (23) is ordinary spin tensor, it is mentioned in those papers that the ordinary spin tensor is not always adopted but the different type spin tensor should be properly used depending on each conditions. For example, if the rotation increment for the current principal axis of deformation is adopted as the spin tensor, this formulation is represented as follows.

$$\dot{\mathbf{e}}_H = \dot{\mathbf{e}} + \dot{\mathbf{H}} \mathbf{H}^T \mathbf{e}_H - \mathbf{e}_H \dot{\mathbf{H}} \mathbf{H}^T \quad (24)$$

On the other hand, as previously stated, there is a strain expression suggested by Hencky^{(8),(9)}, which is widely known as a logarithm strain. This strain expression is derived by introducing the concept of the logarithm strain only in the principal strain component, and the strain components in the arbitrary direction are obtained by the transformation of coordinate system with respect to the principal strain component as indicated in Eq. (25).

$$\mathbf{e}_H = \mathbf{H} \ln \boldsymbol{\Lambda} \mathbf{H}^T \quad (25)$$

As is evident from Eq. (25), this strain expression is not associated with an identical line element in the body. So, this strain is essentially different from the Natural Strain theory suggested in this paper. By differentiating Eq. (25) with respect to time, the strain rate of Hencky's logarithm strain is rewritten as follows.

$$\dot{\mathbf{e}}_H = \mathbf{H} \dot{\boldsymbol{\Lambda}} \boldsymbol{\Lambda}^{-1} \mathbf{H}^T + \dot{\mathbf{H}} \mathbf{H}^T \mathbf{e}_H - \mathbf{e}_H \dot{\mathbf{H}} \mathbf{H}^T \quad (26)$$

Although this formulation is not quite same as Eq. (24), it can be seen that the first term of this equation is merely different form.

Therefore, since these strains are not expressions on an identical line element in the body during whole process of deformation path, these strain expressions are essentially different from Natural Strain theory suggested in this paper.

4. Combined Deformation of Uni-Axial Tension and Simple Shear

In this chapter, in order to compare the Natural Strain with other strain expressions, the combined deformations of uni-axial tension and simple shear are considered, and each strain is examined changing these deformation paths (see Fig. 4) by performing the numerical calculation.

Namely, following three types of deformation paths are considered here.

- (1) Simple shear after uni-axial tension
- (2) Uni-axial tension after simple shear
- (3) Combined deformation of uni-axial tension and simple shear with the constant ratio

$$\mathbf{D}_s \mathbf{D}_T = \begin{bmatrix} 1 & k_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{m} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1/\sqrt{m} \end{bmatrix} \\ = \begin{bmatrix} 1/\sqrt{m} & k_1 m & 0 \\ 0 & m & 0 \\ 0 & 0 & 1/\sqrt{m} \end{bmatrix}$$

(1) Simple shear after uni-axial tension ($A_0 \rightarrow A_1 \rightarrow A_2$)

$$\mathbf{D}_T \mathbf{D}_s = \begin{bmatrix} 1/\sqrt{m} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1/\sqrt{m} \end{bmatrix} \begin{bmatrix} 1 & k_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1/\sqrt{m} & k_2/\sqrt{m} & 0 \\ 0 & m & 0 \\ 0 & 0 & 1/\sqrt{m} \end{bmatrix}$$

(2) Uni-axial tension after simple shear ($C_0 \rightarrow C_1 \rightarrow C_2$)

$$\mathbf{D}_c = \begin{bmatrix} D_{11} & D_{12} & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{m} & \eta/l_{0y} & 0 \\ 0 & m & 0 \\ 0 & 0 & 1/\sqrt{m} \end{bmatrix}$$

(3) Combined uni-axial tension and simple shear with a constant ratio ($B_0 \rightarrow B_1$)

Fig. 4 Various deformation paths and deformation gradient tensors

These deformation paths and deformation gradient tensors corresponding to each deformation path are prescribed as indicated in Fig. 4, where k_1 and k_2 are slip, i.e. the values

of shear deformation, and m is the value of the deformation for uni-axial tension in the y axis and $m = l_y/l_{oy}$. In these calculations, it is assumed that the deformation gradient tensors at the final state for these deformation paths are the same value and these are non-volumetric deformations in order to simplify the numerical calculations. By using these deformation gradient tensors, each strain rate can be calculated from Eqs. (11), (19), (22), (23), and (24).

5. Numerical Calculations and Results

Calculated results obtained by using the Natural Strain theory described in chapter 2, which is performed in above-mentioned three kinds of different deformation paths, are indicated in Fig. 5. Figure 5 (a) shows the extensional strain component and the shearing strain component for an identical line element, and Fig. 5 (b) shows calculated results of the principal strain components. On the other hand, the calculated results of the principal strain for other strain expressions discussed in chapter 3 are indicated in Fig. 6 (a)–(c), where parameter ξ in these figures means the ratio of the shearing deformation to the elongation, and in these figures calculations are performed under the same condition ($\xi = 6$).

If the final deformation of each deformation path (A_2, B_1, C_2) is the same state, the strain at the final state must also be the same value and uniquely determined regardless of the different deformation paths. It is evident from an isotropic elastic body that the strain at the final

(a) Natural Strain ($\beta_o = 120$ [deg.])

(b) Natural Strain (Principal strain)

Fig. 5 Strains for combined deformation of uni-axial tension and simple shear (Natural Strain)

state is the same value regardless of different deformation path and the directions of the principal axis of stress and strain coincide with each other. The strains at the final state for three types of different deformation path agree precisely in Fig. 5 (a) and (b). Thus, reasonable result is obtained in case of the Natural Strain theory. On the other hand, the calculated results for the rotationless strain suggested by Stören and Rice are shown in Fig. 6 (a). Since the strains at the final state for three types of different deformation path do not coincide with completely, unreasonable result is obtained in this strain calculation. Moreover, similar results are obtained for other strain rates, see Fig. 6 (b) and (c).

Next, the relation between the direction of the prin-

(a) The rotationless strain suggested by Stören-Rice ($\dot{R}R^T$)

(b) The modified rate of deformation tensor
(Based on spin ω)

(c) The modified rate of deformation tensor (Based on $\dot{H}H^T$)

Fig. 6 Strains for combined deformation of uni-axial tension and simple shear (The modified strain rate)

(a) The rotationless strain suggested by Stören-Rice ($\dot{R}R^T$)

(b) The modified rate of deformation tensor
(Based on spin ω)

Fig. 7 The direction of the principal axis

pical axis of strain and the principal axis of deformation for three types of different deformation path are indicated in Fig. 7 (a) and (b). The calculated results for rotationless strain suggested by Stören and Rice and the results for Natural Strain theory are compared in Fig. 7 (a). On the other hand, the calculated result using the strain rate similar to the Jaumann type stress rate (Eq. (23)) is compared with in Fig. 7 (b). In these figures, the direction of the principal axis of the Natural Strain and the principal axis of deformation agree precisely over the whole region without influencing the deformation paths. Contrary to this, the direction of the principal axis of other strain expressions does not coincide with the direction of the principal axis of deformation over the whole region. As the results, the direction of principal axis at the final state also does not coincide.

6. Concluding Remarks

By comparing the Natural Strain theory proposed in this paper with other strain expressions, which is variously modified the effect of rigid body rotation, the difference in the formulation was revealed. Moreover, in order to verify the rationality of the Natural Strain theory, the three types of different deformation path concerning the combined deformation of uni-axial tension and simple shear are considered, and the calculated strains for these deformation

paths were compared. As the results, since the strain at the final deformation state in the Natural Strain theory coincided with each other without influencing the deformation path, the reasonable results was obtained. On the other hand, unreasonable results were obtained in other strain expressions in spite of modifying the rotation. Furthermore, from the results obtained by comparing the direction of principal axis of strain with the direction of principal axis of deformation, it was revealed that the directions of both principal axes completely coincide in the Natural Strain theory. On the other hand, since the directions of both principal axes are different with an increase of deformation, unreasonable results were obtained in other strain theories. Therefore, it is clear that there is a contradiction in other strain rate because the rigid body rotation cannot be completely removed from the integrated total strain. This suggests that not only the modification of the spin tensors in the strain rate but also reconsideration of strain expression which is associated with an identical line element in the body during the deformation path is necessary in the formulation of strain.

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