Two Collinear Interface Cracks between Two Dissimilar Functionally Graded Piezoelectric/Piezomagnetic Material Layers under Anti-Plane Shear Loading∗

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In this paper, the behavior of two collinear interface cracks between two dissimilar functionally graded piezoelectric/piezomagnetic material layers subjected to an anti-plane shear loading is investigated. To make the analysis tractable, it is assumed that the material properties vary exponentially with coordinate vertical to the crack. By using the Fourier transform, the problem can be solved with the help of a pair of triple integral equations in which the unknown variable is the jump of the displacement across the crack surfaces. These equations are solved using the Schmidt method. The normalized stress, the electrical displacement and the magnetic flux intensity factors are determined for different geometric for the permeable electric boundary conditions. The relations among the electric filed, the magnetic flux field and the dynamic stress field near the crack tips can be obtained. Numerical examples are provided to show the effect of the functionally graded parameter and the thickness of the strip upon the stress, the electric displacement and the magnetic flux intensity factors of the crack.

Key Words: Functionally Graded Piezoelectric/Piezomagnetic Materials, Two Collinear Interface Cracks

1. Introduction

The BaTiO3-CoFe2O4 composite possesses piezoelectric, piezomagnetic and magneto-electric effects, thereby making the composite sensitive to elastic, electric and magnetic fields. Fibrous and laminated composites made of piezoelectric-piezomagnetic materials have found increasingly wide engineering applications, particularly in the aerospace and automotive industries. The study of magneto-electroelastic mechanics has increased dramatically in recent years. However, the effort devoted to the fracture mechanics of magneto-electroelastic materials is far behind that devoted to other reliability studies. With increasingly wide application of piezoelectric and piezomagnetic composites in smart material systems, cavity or crack problems in magneto-electroelastic media have received considerable interest(1)–(12). Based on the extended Stroh formalism, the genera two-dimensional solutions to the magnetoelectroelastic fracture problem are obtained by Wang and Mai(13), involving five analytic functions of different variables. In the Ref. (5), Gao et al., first derived the general solution under arbitrary loads by reducing the generalized 2D fracture problem in the magnetoelectroelastic solids to an equivalent interface crack problem in two-elastic anisotropic media, which can be solved with well-established method. Liu et al.(10) derived a 2D Green’s functions of an elliptical cavity in an infinite magnetoelectroelastic plane subjected to a mechanical-electric-magnetic line load, respectively. On the other hand, the development of functionally graded materials has demonstrated that they have the potential to reduce the stress concentration and increase of fracture toughness. Consequently, some application of functionally graded piezoelectric materials have been made(13), (14). Recently, the fracture problems of functionally graded piezoelectric materials have been considered in Refs. (15)– (20). Li and Weng(19) first considered the static anti-plane problem of a finite crack in functionally graded piezoelectric material strip. Their results showed that the singular stress and electric displacements in functionally graded piezoelectric materials carry the same forms as those in the homogeneous piezoelectric materials but the magnitudes of the intensity factors depend significantly upon the gradient of the functionally graded piezoelectric ma-
terials properties. More recently, the concept of functionally graded materials was firstly extended to the piezoelectric/piezomagnetic materials to improve the reliability of piezoelectric/piezomagnetic materials and structures in Ref. (21). The results also showed that the singular stress, the singular electric displacements and the singular magnetic flux in functionally graded piezoelectric/piezomagnetic materials carry the same forms as those in the homogeneous piezoelectric/piezomagnetic materials but the magnitudes of the intensity factors depend significantly upon the gradient of the functionally graded piezoelectric/piezomagnetic materials properties. To our knowledge, the magneto-electro-elastic behavior of functionally graded piezoelectric/piezomagnetic material strip with two collinear interface cracks subjected to an anti-plane shear stress loading has not been studied by using Schmidt method (22). Thus, the present work is an attempt to fill this information needed. Here, we just attempt to give a theoretical solution for this problem.

In this paper, the magneto-electro-elastic behavior of two collinear interface cracks between two dissimilar functionally graded piezoelectric/piezomagnetic material layers subjected to an anti-plane shear stress loading is investigated using the Schmidt method (20) – (22). To make the analysis tractable, it is assumed that the material properties vary exponentially with coordinate vertical to the crack. Fourier transform is applied and a mixed boundary value problem is reduced to a pair of triple integral equations. To solve the triple integral equations, the jump of the displacements across the crack surfaces is expanded in a series of Jacobi polynomials. This process is quite different from that adopted in previous works (1) – (10), (15) – (19). Numerical solutions are obtained for the stress, the electric displacement and the magnetic flux intensity factors for the permeable interface crack surface conditions.

2. Formulation of the Problem

It is assumed that there are two collinear interface cracks of length 1 – b between two dissimilar functionally graded piezoelectric/piezomagnetic material layers of finite thickness as shown in Fig. 1. 2b is the distance between the two cracks (The solution of two collinear cracks of length d – b in anisotropic materials can easily be obtained by a simple change in the numerical values of the present paper for crack length 1 – b/d, d > b > 0). h₁ and h₂ are the thickness of the upper layer and the lower layer, respectively. The functionally graded piezoelectric/piezomagnetic materials boundary-value problem for anti-plane shear is considerably simplified if we consider only the out-of-plane displacement, the in-plane electric fields and the in-plane magnetic fields. As discussed in Wang and Mai’s paper (23), for the Mode-I crack in the piezoelectric materials, since a flaw in engineering materials is always a notch of finite thickness rather than a slit crack, the electrically impermeable boundary is a reasonable one for engineering problems. However, two collinear Mode-III interface cracks in functionally graded piezoelectric/piezomagnetic materials are considered in the present paper. As discussed in Ref. (24), since no opening displacement exists for the present anti-plane problem, the crack surfaces can be assumed to be in perfect contact. Accordingly, permeable condition will be enforced in the present study, i.e., both the electric potential and the normal electric displacement are assumed to be continuous across the crack surfaces. So the boundary conditions of the present problem are: (In this paper, we just consider the perturbation field)

\[
\begin{align*}
\tau \phi^{(1)}(x,0^{-}) = & \tau \phi^{(2)}(x,0^{+}) = -\tau_0, b \leq |x| \\
\tau \psi^{(1)}(x,0^{-}) = & \tau \psi^{(2)}(x,0^{+}) = 0, \ |x| > b \\
\phi^{(1)}(x,0) = & \phi^{(2)}(x,0), D \phi^{(2)}(x,0) = D \phi^{(2)}(x,0), \ |x| < \infty \\
\psi^{(1)}(x,0) = & \psi^{(2)}(x,0), B \psi^{(2)}(x,0) = B \psi^{(2)}(x,0), \ |x| < \infty
\end{align*}
\]

\[
\begin{align*}
\tau \phi^{(1)}(x,h_1) = & \tau \phi^{(2)}(x,-h_2) = 0 \\
D \phi^{(2)}(x,h_1) = & D \phi^{(2)}(x,-h_2) = 0, \ |x| < \infty \\
B \psi^{(2)}(x,h_1) = & B \psi^{(2)}(x,-h_2) = 0 \\
\phi^{(1)}(x,y) = & \phi^{(2)}(x,y) = 0 \text{ for } (x^2 + y^2)^{1/2} \rightarrow \infty \\
\psi^{(1)}(x,y) = & \psi^{(2)}(x,y) = 0 \\
\end{align*}
\]

In this paper, τ₀ is the magnitude of the anti-plane shear loading. Also note that all quantities with superscript i (i = 1, 2) refer to the upper layer 1 and the lower layer 2 as shown in Fig. 1, respectively.

Crack problems in the non-homogeneous piezoelectric/piezomagnetic materials do not appear to be analytically tractable for arbitrary variations of material properties. Usually, one tries to generate the forms of non-homogeneities for which the problem becomes tractable. Similar to the treatment of the crack problem for isotropic non-homogeneous materials in Refs. (25) – (27), we assume the material properties are described by:

![Fig. 1 Geometry and coordinate system for two collinear interface cracks](image)
\[
\begin{align*}
\begin{cases}
\varepsilon^{(1)}_{11} = e^{(1)}_{11} \varepsilon^{(1)}_{11} + e^{(1)}_{11} \varepsilon^{(1)}_{11} + e^{(1)}_{11} \varepsilon^{(1)}_{11}, \\
\varepsilon^{(2)}_{11} = e^{(2)}_{11} \varepsilon^{(2)}_{11} + e^{(2)}_{11} \varepsilon^{(2)}_{11}, \\
\varepsilon^{(1)}_{11} = e^{(1)}_{11} \varepsilon^{(1)}_{11} + e^{(1)}_{11} \varepsilon^{(1)}_{11}, \\
\varepsilon^{(2)}_{11} = e^{(2)}_{11} \varepsilon^{(2)}_{11} + e^{(2)}_{11} \varepsilon^{(2)}_{11}, \\
\end{cases}
\end{align*}
\]

where \( e^{(1)}_{11}, e^{(2)}_{11}, e^{(1)}_{11} \), and \( e^{(2)}_{11} \) are the shear modulus, the piezomagnetic coefficient, the piezoelectric coefficient, and the functionally graded parameter of two dissimilar functionally graded piezoelectric/piezomagnetic material layers, respectively.

The constitutive equations for the mode III crack can be expressed as

\[
\begin{align*}
\tau^{(i)}_{yk} &= e^{(i)}_{11} \varepsilon^{(i)}_{11} + e^{(i)}_{11} \varepsilon^{(i)}_{11} + e^{(i)}_{11} \varepsilon^{(i)}_{11}, \\
D^{(i)}_{k} &= e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11}, \\
B^{(i)}_{k} &= e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11}, \\
\end{align*}
\]

The anti-plane governing equations are

\[
\begin{align*}
\frac{\partial^{2}}{\partial x^{2}} u^{(i)} + \beta^{(i)} \frac{\partial^{2}}{\partial y^{2}} u^{(i)} + e^{(i)}_{11} \frac{\partial^{2}}{\partial y^{2}} \phi^{(i)} + e^{(i)}_{11} \frac{\partial^{2}}{\partial y^{2}} \phi^{(i)} &= 0, \\
D^{(i)}_{k} &= e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11}, \\
B^{(i)}_{k} &= e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11} - e^{(i)}_{11} \varepsilon^{(i)}_{11}, \\
\end{align*}
\]

The general expressions for the displacements (9)–(11) is solved using the Fourier integral transform technique. The general expressions for the displacement components, the electric potentials and the magnetic potentials can be obtained as follows:

\[
\begin{align*}
\left\{\begin{array}{l}
\psi^{(1)}(x,y) = e^{(1)}_{11} \psi^{(1)}(x,y), \\
\psi^{(2)}(x,y) = e^{(2)}_{11} \psi^{(2)}(x,y), \\
\end{array}\right.
\end{align*}
\]

3. Solution

Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for \( 0 \leq x < \infty, -\infty \leq y < \infty \) only. The system of above governing equations (9)–(11) is solved using the Fourier integral transform technique. The general expressions for the displacement components, the electric potentials and the magnetic potentials can be obtained as follows:
\[ 
\tau^{(2)}_{yx}(x,y) = \frac{2\beta^{(2)}y}{\pi} \int_0^{\infty} \gamma_1 \left[ \mu^{(2)}_1 [A_2(s)e^{-y_2s} - B_2(s)e^{-y_2s}] \right. \\
+ \left. \gamma_2 \left[ \mu^{(2)}_1 [C_2(s)e^{-y_2s} - B_2(s)e^{-y_2s}] \right. \\
+ \left. \gamma_2 \left[ \mu^{(2)}_1 [E_2(s)e^{-y_2s} - F_2(s)e^{-y_2s}] \right. \right] \cos(sx)ds 
\]

(17)

\[ 
D^{(2)}_y(x,y) = -\frac{2e^{\beta^{(2)}y}}{\pi} \int_0^{\infty} \gamma_2 \left[ \mu^{(2)}_1 \{C_2(s)e^{y_2s}] \\
- D_2(s)e^{y_2s}] + \mu^{(2)}_1 \{E_2(s)e^{y_2s}] \\
- F_2(s)e^{y_2s}\right] \cos(sx)ds 
\]

(18)

\[ 
B^{(2)}_y(x,y) = -\frac{2e^{\beta^{(2)}y}}{\pi} \int_0^{\infty} \gamma_2 \left[ \mu^{(2)}_1 \{C_2(s)e^{y_2s}] \\
- D_2(s)e^{y_2s}] + \mu^{(2)}_1 \{E_2(s)e^{y_2s}] \\
- F_2(s)e^{y_2s}\right] \cos(sx)ds 
\]

(19)

To solve the problem, the jump of the displacements across the crack surfaces is defined as follows:

\[ f(x) = u^{(1)}(x,0) - u^{(2)}(x,0) \]

(20)

Substituting Eqs. (12) and (13) into Eq. (20), and applying the Fourier transform and the boundary conditions (1–3), it can be obtained:

\[ A_1(s) + B_1(s) - A_2(s) - B_2(s) = \tilde{f}(s) \]

(21)

\[ a^{(0)}_0 \{A_1(s) + B_1(s)] - a^{(2)}_0 \{A_2(s) + B_2(s)] + C_1(s) \\
+ D_1(s) - C_2(s) - D_2(s) = 0 \]

(22)

\[ a^{(1)}_0 \{A_1(s) + B_1(s)] - a^{(2)}_0 \{A_2(s) + B_2(s)] + E_1(s) \\
+ F_1(s) - E_2(s) - F_2(s) = 0 \]

(23)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s)e^{-y_1h}] \\
+ D_1(s)e^{-y_1h}] + \left. \gamma_2 \left[ \mu^{(1)}_1 [E_1(s)e^{-y_1h}] \\
- F_1(s)e^{-y_1h}\right] \right] = 0 \]

(24)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s)e^{-y_1h}] \\
- D_1(s)e^{-y_1h}] + \left. \gamma_2 \left[ \mu^{(1)}_1 [E_1(s)e^{-y_1h}] \\
- F_1(s)e^{-y_1h}\right] \right] = 0 \]

(25)

\[ a^{(1)}_1 \{C_1(s)e^{-y_1h}] - a^{(2)}_1 \{D_1(s)e^{-y_1h}] \\
+ \gamma_1 \left[ \mu^{(1)}_1 [E_1(s)e^{-y_1h}] \\
- F_1(s)e^{-y_1h}\right] = 0 \]

(26)

\[ \mu^{(2)}_0 \{A_2(s)e^{-y_2h}] - B_2(s)e^{y_2h}] + a^{(2)}_1 \{E_2(s)e^{-y_2h}] \\
- F_2(s)e^{-y_2h}] = 0 \]

(27)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s)e^{-y_1h}] \\
+ D_1(s)e^{-y_1h}] + \left. \gamma_2 \left[ \mu^{(1)}_1 [E_1(s)e^{-y_1h}] \\
- F_1(s)e^{-y_1h}\right] \right] = 0 \]

(28)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s)e^{-y_1h}] \\
- D_1(s)e^{-y_1h}] + \left. \gamma_2 \left[ \mu^{(1)}_1 [E_1(s)e^{-y_1h}] \\
- F_1(s)e^{-y_1h}\right] \right] = 0 \]

(29)

A superposed bar indicates the Fourier transform throughout the paper. Substituting Eqs. (14)–(19) into Eqs. (1) and (2), it can be obtained:

\[ \gamma_1 \left[ \mu^{(1)}_1 [A_1(s) - B_1(s)] + \gamma_1 \left[ \mu^{(1)}_1 [C_1(s) - D_1(s)] \\
+ \gamma_2 \left[ \mu^{(2)}_1 [A_2(s) - B_2(s)] \right] \right] \]

(30)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s) - D_1(s)] + \gamma_1 \left[ \mu^{(1)}_1 [E_1(s) - F_1(s)] \\
+ \gamma_2 \left[ \mu^{(2)}_1 [C_2(s) - D_2(s)] + \gamma_2 \left[ \mu^{(2)}_1 [E_2(s) - F_2(s)] \right] \right] \right] = 0 \]

(31)

\[ \gamma_1 \left[ \mu^{(1)}_1 [C_1(s) - D_1(s)] + \gamma_1 \left[ \mu^{(1)}_1 [E_1(s) - F_1(s)] \\
+ \gamma_2 \left[ \mu^{(2)}_1 [C_2(s) - D_2(s)] + \gamma_2 \left[ \mu^{(2)}_1 [E_2(s) - F_2(s)] \right] \right] \right] = 0 \]

(32)

By solving twelve Eqs. (21)–(31) with twelve unknown functions \( A_1(s), B_1(s), C_1(s), D_1(s), E_1(s), F_1(s), A_2(s), B_2(s), C_2(s), D_2(s), E_2(s) \) and \( F_2(s) \) and applying the boundary conditions (1) and (2), it can be obtained:

\[ \frac{2}{\pi} \int_0^\infty \tilde{f}(s) \cos(sx)ds = 0, \quad x > 0, 0 < x < b \]

(33)

\[ \frac{2}{\pi} \int_0^\infty g_1(s) \tilde{f}(s) \cos(sx)ds = -\tau_0, \quad b \leq x \leq 1 \]

(34)

where \( g_1(s) \) is a known function (see Appendix). When the properties of the upper layer and the lower layer are continuous along the crack line, \( \beta_1 = -e^{(1)}_{140}/2 \). To determine the unknown function \( \tilde{f}(s) \), the above a pair of triple integral equations (33) and (34) must be solved.

4. Solution of the Triple Integral Equations

The Schmidt method\(^{(20)}\)–(22) is used to solve the triple integral equations (33) and (34). The jump of the displacements across the crack surfaces is represented by the following series:

\[ f(x) = \sum_{n=0}^{\infty} b_n F_n \left( \frac{1}{2} \right) \left( \frac{x - 1 + b}{2} \right) \left( \frac{1 - b}{2} \right) \]

(35)

for \( b \leq x \leq 1 \)

\[ f(x) = u^{(1)}(x,0) - u^{(2)}(x,0) = 0, \]

(36)

for \( x > 1, 0 < x < b \)

where \( b_n \) are unknown coefficients to be determined and \( F_n^{\left( \frac{1}{2} \right)}(x) \) is a Jacobi polynomial\(^{(28)}\). The Fourier transform of Eqs. (35) and (36) is\(^{(29)}\)

\[ \tilde{f}(s) = \sum_{n=0}^{\infty} b_n F_n G_n(s) \left( \frac{1}{2} \right) \left( \frac{x - 1 + b}{2} \right) \left( \frac{1 - b}{2} \right) \]

(37)

where

\[ F_n = 2 \sqrt{\pi} \frac{n + 1}{n!} \]
\[ G_n(s) = \begin{cases} (-1)^n \cos \left( \frac{1+b}{2} \right), & n=0,2,4,6,... \\ (-1)^{n+1} \sin \left( \frac{1+b}{2} \right), & n=1,3,5,7,... \end{cases} \]

\( \Gamma(x) \) and \( J_n(x) \) are the Gamma and Bessel functions, respectively.

Substituting Eq. (37) into Eqs. (33) and (34), respectively. It can be shown that Eq. (33) is automatically satisfied. After integration with respect to \( x \) in \([b,x] \), Eq. (34) reduces to

\[
\int_0^\infty \frac{1}{s} J_n(sa) \sin(x) ds = \frac{\beta_1}{2(n+1)} \sin \left( \frac{1+b}{2} \right) \sin(sx) ds
\]

\[
\times \cos \left( \frac{(n+1)\pi}{2} \right) \cos \left( \frac{n\pi}{2} \right)
\]

Thus the semi-infinite integral in Eqs. (41) and (42) can be evaluated directly. Equation (38) can now be solved for the coefficients \( b_n \) by the Schmidt method\(^{21,22} \). For brevity, Eq. (38) can be rewritten as

\[
\int_0^\infty b_n F_n \int_0^\infty \frac{1}{s} g_1(x) G_n(s) J_{n+1} \left( \frac{s-1+b}{2} \right) \sin(sx) ds = \gamma_0(x-b),
\]

for \( b \leq x \leq 1 \)

From the relationships\(^{28} \)

\[
\int_0^\infty \frac{1}{s} J_n(sa) \sin(x) ds = \left\{ \begin{array}{ll} \frac{\sin \left[ \sin^{-1}(b/a) \right]}{a^n}, & a > b \\ \frac{\cos \left[ \sin^{-1}(b/a) \right]}{a^n}, & a < b \end{array} \right\}
\]

\[
\int_0^\infty \frac{1}{s} J_n(sa) \cos(x) ds = \left\{ \begin{array}{ll} \frac{\cos \left[ \sin^{-1}(b/a) \right]}{a^n}, & a > b \\ \frac{\sin \left[ \sin^{-1}(b/a) \right]}{a^n}, & a < b \end{array} \right\}
\]

and

\[
\int_0^\infty \frac{1}{s} \left[ J_n(sa) \sin(x) \right] ds = \left\{ \begin{array}{ll} \frac{\beta_1}{2(n+1)} \cos \left( \frac{1+b}{2} \right), & a > b \\ \frac{\beta_1}{2(n+1)} \sin \left( \frac{1+b}{2} \right), & a < b \end{array} \right\}
\]

\[
\times \cos \left( \frac{(n+1)\pi}{2} \right) \cos \left( \frac{n\pi}{2} \right)
\]

\[
\int_0^\infty \frac{1}{s} \left[ J_n(sa) \cos(x) \right] ds = \left\{ \begin{array}{ll} \frac{\beta_1}{2(n+1)} \sin \left( \frac{1+b}{2} \right), & a > b \\ \frac{\beta_1}{2(n+1)} \cos \left( \frac{1+b}{2} \right), & a < b \end{array} \right\}
\]

\[
\times \sin \left( \frac{(n+1)\pi}{2} \right) \sin \left( \frac{n\pi}{2} \right)
\]

The coefficients \( b_n \) are known, so that the entire perturbation stress field, the perturbation electric displace-
ment field and the magnetic flux can be obtained. However, in fracture mechanics, it is of importance to determine the perturbation stress, the perturbation electric displacement and the perturbation magnetic flux in the vicinity of the crack tips. In the case of the present study, \( \tau_{gc}^{(1)}, \tau_{gc}^{(2)} \), \( D_y^{(1)}, D_y^{(2)} \), \( B_y^{(1)} \) and \( B_y^{(2)} \) along the crack line can be expressed respectively as

\[
\tau_{gc}^{(1)}(x,0) = \tau_{gc}^{(2)}(x,0) = \frac{2}{\pi} \sum_{n=0}^{\infty} b_n F_n \\
\times \int_0^\infty \frac{1}{s} g_1(s) G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
= \frac{2b_1}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^\infty G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
+ \frac{2}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^\infty \left[ \frac{1}{s} g_1(s) - \beta_1 \right] G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
\times \cos(xs) ds
\]  

(48)

\[
B_y^{(1)}(x,0) = B_y^{(2)}(x,0) = \frac{2}{\pi} \sum_{n=0}^{\infty} b_n F_n \\
\times \int_0^\infty \frac{1}{s} g_2(s) G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
= \frac{2b_2}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^\infty G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
+ \frac{2}{\pi} \sum_{n=0}^{\infty} b_n F_n \int_0^\infty \left[ \frac{1}{s} g_2(s) - \beta_2 \right] G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds \\
\times \cos(xs) ds
\]  

(49)

where \( g_2(s) \) and \( g_3(s) \) are known functions (see Appendix). \( \lim g_2(s)/s = \beta_2 \), \( \lim g_3(s)/s = \beta_3 \). Where \( \beta_2 \) and \( \beta_3 \) are two constants which depend on the properties of the materials (see Appendix). When the properties of the upper layer and the lower layer are continuous along the crack line, \( \beta_2 = -q_{150}^{(1)}/2 \) and \( \beta_3 = -q_{150}^{(1)}/2 \).

From the relationship

\[
\cos \left( \frac{s+1+b}{2} \right) \cos(xs) \\
= \frac{1}{2} \left[ \cos \left( \frac{s+1+b}{2} - x \right) + \cos \left( \frac{s+1+b}{2} + x \right) \right]
\]

\[
\sin \left( \frac{s+1+b}{2} \right) \cos(xs) \\
= \frac{1}{2} \left[ \sin \left( \frac{s+1+b}{2} - x \right) + \sin \left( \frac{s+1+b}{2} + x \right) \right]
\]

the singular parts of the stress field, the electric displacement and the magnetic flux near the crack tips in Eqs. (48)–(50) can be expressed respectively as follows (x > 1 or x < b):

\[
\tau = \frac{b_1}{\pi} \sum_{n=0}^{\infty} b_n F_n H_n(b,x)
\]

(51)

\[
D = \frac{b_2}{\pi} \sum_{n=0}^{\infty} b_n F_n H_n(b,x)
\]

(52)

\[
B = \frac{b_3}{\pi} \sum_{n=0}^{\infty} b_n F_n H_n(b,x)
\]

(53)

where \( H_n(b,x) = \left\{ \begin{array}{ll} (-1)^{n+1} R(b,x,n), & 0 < x < b \\ -R(b,x,n), & x > 1 \end{array} \right. \)

\[
R(b,x,n) = \frac{2}{[1+b-2x]^n} \\
\sqrt{[1+b-2x]^2 - (1-b)^2} \\
\sqrt{[1+b-2x]^2 + \sqrt{[1+b-2x]^2 - (1-b)^2}^n}
\]

At the left tip of the right crack, the stress intensity factor \( K_L \), the electric displacement intensity factor \( K_{LD} \) and the magnetic intensity factor \( K_{LB} \) can be written as follows, respectively.

\[
K_L = \lim_{x \to b^-} \sqrt{2(b-x) \cdot \tau}
\]

(54)

\[
K_{LD} = \lim_{x \to b^-} \sqrt{2(b-x) \cdot D}
\]

(55)

\[
K_{LB} = \lim_{x \to b^-} \sqrt{2(b-x) \cdot B}
\]

(56)

At the right tip of the right crack, the stress intensity factor \( K_R \), the electric displacement intensity factor \( K_{RD} \) and the magnetic flux intensity factor \( K_{RB} \) can be written as follows, respectively.

\[
K_L = \lim_{x \to b^-} \sqrt{2(b+x) \cdot \tau}
\]

(54)

\[
K_{RD} = \lim_{x \to b^-} \sqrt{2(b+x) \cdot D}
\]

(55)

\[
K_{RB} = \lim_{x \to b^-} \sqrt{2(b+x) \cdot B}
\]

(56)
\[
K_R = \lim_{x \to 1} \sqrt{2(x-1)} \cdot \tau
\]
\[=-\frac{\beta_1}{\pi} \sqrt{\frac{2}{1-b}} \sum_{n=0}^{\infty} \frac{b_n F_n}{h_n F_n} (57)\]

\[
K_R^D = \lim_{x \to 1} \sqrt{2(x-1)} \cdot D
\]
\[=-\frac{\beta_2}{\pi} \sqrt{\frac{2}{1-b}} \sum_{n=0}^{\infty} \frac{b_n F_n}{h_n F_n} = \frac{\beta_2}{\beta_1} K_R (58)\]

\[
K_R^B = \lim_{x \to 1} \sqrt{2(x-1)} \cdot D
\]
\[=-\frac{\beta_3}{\pi} \sqrt{\frac{2}{1-b}} \sum_{n=0}^{\infty} \frac{b_n F_n}{h_n F_n} = \frac{\beta_3}{\beta_1} K_R (59)\]

6. Numerical Calculations and Discussion

To check the numerical accuracy of the Schmidt method, the values of \(\sum_{n=0}^{9} b_n E_n(x) \left( \frac{2}{\pi \tau_0} \right)\) and \(U(x)/\tau_0\) in Eq. (43) are given in Table 1 for \(b = 0.1\), \(h_1 = 0.1\), \(h_2 = 6.0\), \(\beta^{(1)} = 0.2\) and \(\beta^{(2)} = 0.4\) (Material-I/Material-II). In Table 2, the values of the coefficients \(b_n \left( \frac{2}{\pi \tau_0} \right)\) are given for \(b = 0.1\), \(h_1 = 0.1\), \(h_2 = 6.0\), \(\beta^{(1)} = 0.2\), \(\beta^{(2)} = 0.4\) (Material-I/Material-II). As discussed in the works\((21),(22),(30)\) and the above discussion, it can be seen that the Schmidt method performs satisfactorily if the first ten terms in the infinite series in Eq. (43) are retained. At \(b \leq x \leq 1, y = 0\), it can be obtained that \(\sum_{n=0}^{9} b_n E_n(x) \left( \frac{2}{\pi \tau_0} \right)\) is close to \(-(x-b)\). Hence, the solution of present paper can also be proved to satisfactory the boundary conditions (1). In all computations, according to Refs. (1), (2) and (9), the constants of materials-I are assumed to be that \(s_{14}^{(1)} = 44.0 \text{ (GPa)}\), \(e_{150}^{(1)} = 5.8 \text{ (C/m²)}\), \(\epsilon_{110}^{(1)} = 5.64 \times 10^{-9} \text{ (C²/Nm²)}\), \(q_{150}^{(1)} = 275.0 \text{ (N/Am)}\), \(d_{11}^{(1)} = 0.005 \times 10^{-9} \text{ (Ns/VC)}\), \(\mu_{110}^{(1)} = -297.0 \times 10^{-6} \text{ (Ns²/C²)}\) and the constants of materials-II are assumed to be that \(e_{440}^{(2)} = 34.0 \text{ (GPa)}\), \(e_{150}^{(2)} = 4.8 \text{ (C/m²)}\), \(e_{110}^{(2)} = 4.64 \times 10^{-9} \text{ (C²/Nm²)}\), \(q_{150}^{(2)} = 195.0 \text{ (N/Am)}\), \(d_{11}^{(2)} = 0.004 \times 10^{-9} \text{ (Ns/VC)}\), \(\mu_{110}^{(2)} = -201.0 \times 10^{-6} \text{ (Ns²/C²)}\). The normalized non-homogeneity constants \(\beta^{(i)} (i = 1, 2)\) are varied between \(-2\) and \(2\), which covers the most of the practical cases. The results of the present paper are shown in Figs. 2–10. From the results, the following observations are very significant (In the present paper, the units of the variables \(b, h_1\) and \(\beta^{(i)}\) are all dimensionless \((i = 1, 2)\):

(i) From the results, it can be shown that the singular stress, the singular electric displacement and the singular magnetic flux in functionally graded piezoelectric/piezomagnetic materials carry the same forms as those in the homogeneous piezoelectric/piezomagnetic materials or in the homogeneous piezoelectric materials but the magnitudes of the intensity factors depend significantly upon the gradient of the functionally graded piezoelectric/piezomagnetic materials properties as discussed in Ref. (19).

(ii) The electro-magneto-elastic coupling effects can be obtained as shown in Eqs. (54)–(59). For the electric displacement and the magnetic flux intensity factors, they have the same changing tendency as the stress intensity factor as shown in Figs. 2–4. However, the amplitude values of the electric displacement filed, the magnetic flux field and the stress field are different. The amplitude values of the electric displacement and the magnetic flux fields are very small as shown in Figs. 3 and 4. The re-

| Table 1 | Values of \(\sum_{n=0}^{9} b_n E_n(x) \left( \frac{2}{\pi \tau_0} \right)\) and \(U(x)/\tau_0 = -(x-b)\) for \(b = 0.1, h_1 = 0.1, h_2 = 6.0, \beta^{(1)} = 0.2\) and \(\beta^{(2)} = 0.4\) (Material-I/Material-II) |
|---|---|---|
| \(x\) | \(\sum_{n=0}^{9} b_n E_n(x) \left( \frac{2}{\pi \tau_0} \right)\) | \(U(x)/\tau_0 = -(x-b)\) |
| 0.51 | -0.409041E+00 | -0.41 |
| 0.56 | -0.460147E+00 | -0.46 |
| 0.61 | -0.510341E+00 | -0.51 |
| 0.66 | -0.560338E+00 | -0.56 |
| 0.71 | -0.610125E+00 | -0.61 |
| 0.76 | -0.659608E+00 | -0.66 |
| 0.81 | -0.709579E+00 | -0.71 |
| 0.86 | -0.759731E+00 | -0.76 |
| 0.90 | -0.800180E+00 | -0.80 |
| 0.99 | -0.890921E+00 | -0.89 |

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Values of (b_n \left( \frac{2}{\pi \tau_0} \right)) for (b = 0.1, h_1 = 0.1, h_2 = 6.0, \beta^{(1)} = 0.2) and (\beta^{(2)} = 0.4) (Material-I/Material-II)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(n)</td>
<td>(b_n \left( \frac{2}{\pi \tau_0} \right))</td>
</tr>
<tr>
<td>0</td>
<td>0.243007E-01</td>
</tr>
<tr>
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<td>-0.191008E-02</td>
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<tr>
<td>2</td>
<td>-0.559481E-03</td>
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<tr>
<td>3</td>
<td>-0.800573E-04</td>
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<tr>
<td>4</td>
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</tr>
<tr>
<td>5</td>
<td>-0.203147E-04</td>
</tr>
<tr>
<td>6</td>
<td>0.113426E-04</td>
</tr>
<tr>
<td>7</td>
<td>0.973421E-05</td>
</tr>
<tr>
<td>8</td>
<td>-0.437842E-05</td>
</tr>
<tr>
<td>9</td>
<td>0.134276E-05</td>
</tr>
</tbody>
</table>

Fig. 2 The stress intensity factor versus \(h_2\) for \(b = 0.1, h_1 = 0.1, h_2 = 6.0, \beta^{(1)} = 0.2\) and \(\beta^{(2)} = 0.4\) (Material-I/Material-II)
results of the electric displacement and the magnetic flux intensity factors can be directly obtained from the results of the stress intensity factors through Eqs. (54)–(59). This means that an applied mechanical load alone can produce the electric displacement and magnetic flux singularities. The results of the electric displacement and the magnetic flux intensity factors of the other cases have been omitted in the present paper.

(iii) The stress intensity factors tend to decrease with increase in the thickness of the functionally graded piezoelectric/piezomagnetic material layers, and then they tend to constants for the different cases, respectively. These constants are equal to the stress intensity factors of the interface cracks in infinite functionally graded piezoelectric/piezomagnetic material plane for the different composite cases. From the results, it can be also concluded that the thickness effects on the stress, the electric dis-
inner crack tips are bigger than those at the outer crack tips. However, the intensity factors at the inner and outer crack tips are almost overlapped for \( b \geq 0.5 \) as shown in Fig. 7.

(v) The stress intensity factors tend to decrease with increase in the functionally graded parameters \( \beta^{(i)} (i = 1, 2) \) as shown in Figs. 8–10. When the material properties of the upper layer-1 and the lower layer-2 along the crack line are continuous, it can be obtained the same conclusion as shown in Fig. 10. This means that, by adjusting the functionally graded parameters, the stress fields near the crack tips can be reduced in engineering practices.

(vi) The solution of the present paper can revert to the one of the problem, which the material properties of the upper layer-1 and the lower layer-2 along the crack line are continuous as shown in Figs. 6 and 10.

Acknowledgements

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Appendix

The functions of \( g_1(s) \), \( g_2(s) \) and \( g_3(s) \) can be obtained by the operation of the follow matrixes.

\[
[X_1] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
[X_2] = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{bmatrix}
\]

\[
[X_3] = \begin{bmatrix}
\mu_0^{(1)} & e_{150}^{(1)} & q_{150}^{(1)} \\
0 & e_{110}^{(1)} & d_{110}^{(1)} \\
0 & d_{110}^{(1)} & \mu_{110}^{(1)} \\
\mu_0^{(2)} & e_{150}^{(2)} & q_{150}^{(2)} \\
0 & e_{110}^{(2)} & d_{110}^{(2)} \\
0 & d_{110}^{(2)} & \mu_{110}^{(2)} \\
\end{bmatrix}
\]

\[
[X_4] = \begin{bmatrix}
-\mu_0^{(1)} - e_{150}^{(1)} - q_{150}^{(1)} \\
0 & e_{110}^{(1)} & d_{110}^{(1)} \\
0 & d_{110}^{(1)} & \mu_{110}^{(1)} \\
\end{bmatrix}
\]

\[
[X_5] = (1 + e^{-2y_2 h_1}) [X_1]
\]

\[
-\frac{y_1}{y_2} \left( 1 - e^{-2y_1 h_2} \right) \left( 1 + e^{-2y_2 h_1} \right) [X_2][X_4]^{-1} [X_3].
\]

\[
[X_7] = \gamma_1 \left( 1 - e^{-2y_1 h_1} \right) [X_5][X_6]^{-1}
\]

\[
g_1(s) = \alpha_1(s), \quad g_2(s) = \alpha_2(s), \quad g_3(s) = \alpha_3(s).
\]
The constants of $\beta_1, \beta_2$ and $\beta_3$ can be obtained by the operation of the follow matrixes.

$$[Y_1] = \begin{bmatrix} 1 & 0 & 0 \\ a_0^{(1)} & 1 & 0 \\ a_1^{(1)} & 0 & 1 \end{bmatrix},$$

$$[Y_2] = \begin{bmatrix} -1 & 0 & 0 \\ -a_0^{(2)} & -1 & 0 \\ -a_1^{(2)} & 0 & -1 \end{bmatrix},$$

$$[Y_3] = \begin{bmatrix} \mu_0^{(1)} & e_{150}^{(1)} & q_{150}^{(1)} \\ 0 & e_{110}^{(1)} & q_{110}^{(1)} \\ 0 & d_{110}^{(1)} & \mu_{110}^{(1)} \end{bmatrix},$$

$$[Y_4] = \begin{bmatrix} \mu_0^{(2)} & e_{150}^{(2)} & q_{150}^{(2)} \\ 0 & e_{110}^{(2)} & q_{110}^{(2)} \\ 0 & d_{110}^{(2)} & \mu_{110}^{(2)} \end{bmatrix},$$

$$[Y_5] = \begin{bmatrix} -\mu_0^{(1)} & -e_{150}^{(1)} & -q_{150}^{(1)} \\ 0 & e_{110}^{(1)} & d_{110}^{(1)} \\ 0 & d_{110}^{(1)} & \mu_{110}^{(1)} \end{bmatrix},$$

$$[Y_6] = [Y_1] - [Y_2][Y_4]^{-1}[Y_3],$$

$$[Y_3][Y_6]^{-1} = \begin{bmatrix} \beta_1 & * & * \\ \beta_2 & * & * \\ \beta_3 & * & * \end{bmatrix}.$$

References


(23) Wang, B.L. and Mai, Y.W., Impermeable Crack and...


