Investigation the Dynamic Interaction between Two Collinear Cracks in the Functionally Graded Piezoelectric Materials Subjected to the Harmonic Anti-Plane Shear Stress Waves by Using the Non-Local Theory*

Jun LIANG**

In this paper, the non-local theory of elasticity is applied to obtain the dynamic interaction between two collinear cracks in functionally graded piezoelectric materials under the harmonic anti-plane shear stress waves for the permeable electric boundary conditions. To make the analysis tractable, it is assumed that the material properties vary exponentially with coordinate vertical to the crack. By means of the Fourier transform, the problem can be solved with the help of a pair of triple integral equations that the unknown variable is the jump of the displacement across the crack surfaces. These equations are solved by use of the Schmidt method. Unlike the classical elasticity solutions, it is found that no stress and electric displacement singularities are present at the crack tips. The non-local elastic solutions yield a finite stress at the crack tips, thus allows us to use the maximum stress as a fracture criterion. The finite stresses at the crack tips depend on the crack length, the distance between two cracks, the functionally graded parameter, the circular frequency of the incident waves and the lattice parameter of the materials, respectively.

Key Words: Collinear Crack, Non-Local Theory, Functionally Graded Piezoelectric Materials, Waves

1. Introduction

The coupling nature of piezoelectric materials has attracted wide applications in electric-mechanical and electric devices, such as electric-mechanical actuators, sensors and structures. Therefore, it is of great importance to study the electro-elastic interaction and fracture behavior of piezoelectric materials (1)–(6). On the other hand, the development of functionally graded materials (FGMs) has demonstrated that they have the potential to reduce the stress concentration and increase of fracture toughness. Consequently, the concept of FGMs can be extended to the piezoelectric materials to improve the reliability of piezoelectric materials and structures. Some applications of functionally graded piezoelectric materials (FGPMs) have been made (7). Recently, the fracture problems of FGPMs have been considered in Refs. (8)–(13). To our knowledge, Li and Weng (13) first applied the concept of fracture mechanics on a finite crack in a strip of functionally graded piezoelectric material. They found that the singular stress and the singular electric displacement at the crack tips in FGPMs carry the same forms as those in the homogeneous piezoelectric materials but the magnitudes of the intensity factors are dependent on the gradient of the FGPM properties. However, these solutions (8)–(13) contain stress and the electric displacement singularities at the crack tips. This is not reasonable according to the physical nature. In fact, the stress and the electric displacement fields near the crack tips should be finite. As a result of this, beginning with Griffith, all fracture criteria in practice today based on other considerations, e.g. energy, the J-integral (14) and the strain gradient theory (15). Now the main difficulty is remained ambiguous how to provide effective fracture criteria for functionally graded piezoelectric materials.

To overcome the stress singularities at the crack tips in the classical elastic fracture theory, Eringen (16)–(18) used the non-local theory to discuss the stresses near the tips.
of a sharp line crack in an isotropic elastic plate subject to uniform tension, shear and anti-plane shear, and the resulting solutions did not contain any stress singularities at the crack tips. The stress fields near the crack tips were finite. This allows us to use the maximum stress as a fracture criterion. In contrast to these local approaches, of zero-thickness interaction, the modern non-local continuum mechanics originated and developed in the last five decades. Eringen \cite{21} contributed not just the complete physics and mathematics of the non-local theory but also, in addition, shaped the theory into a concrete form making it viable for practical applications to boundary value problems. According to the non-local theory, the stress at a point \(X\) in a body depends not only on the strain at point \(X\) but also on that at all other points of the body. This is contrary to the classical theory that the stress at a point \(X\) in a body depends only on the strain at point \(X\). Eringen \cite{22} stated the basic theory of non-local elasticity. In Ref. \cite{22}, the basic theory of non-local elasticity was stated with emphasis on the difference between the non-local theory and the classical theory. The basic idea of non-local elasticity is to build a relationship between macroscopic mechanical quantities and microscopic physical quantities within the framework of continuum mechanics. The constitutive theory of non-local elasticity has been developed in Ref. \cite{22}, in which the elastic modulus is influenced by the microstructure of the material. Other results have been given by the application of non-local elasticity to the fields such as a dislocation near a crack \cite{23, 24} and fracture mechanics problems \cite{25, 26}. The results of those concrete problems that were solved display a rather remarkable agreement with experimental evidence. This can be used to predict the cohesive stress for various materials and the results close to those obtained in atomic lattice dynamics \cite{27, 28}. Recently, some static and dynamic fracture problems \cite{29}–\cite{35} in an isotropic elastic material, the functionally graded materials and the piezoelectric material have been studied by use of the non-local theory. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the functionally graded piezoelectric effects. To overcome the mathematical difficulties, a one-dimensional non-local kernel is used to instead of a two-dimensional one for the dynamic fracture problem to obtain the stress and the electric displacement fields near the crack tips. To make the analysis tractable, it is assumed that the material properties vary exponentially with coordinate vertical to the crack. Fourier transform is applied and a mixed boundary value problem is reduced to a pair of triple integral equations that the unknown variable is the jump of the displacement across the crack surfaces. To solve the triple integral equations, the jump of the displacement across the crack surface is expanded in a series of Jacobi polynomials and the Schmidt method \cite{36} is used. As expected, the solution in this paper does not contain the stress and the electric displacement singularities at the crack tips. The stress and the electric fields for the non-local theory are similar to that of the classical elasticity solution away from the crack tips. Near the crack tips, a lattice parameter tends to control the amplitude of the stress and the electric displacement fields.

2. The Crack Model

It is assumed that there are two collinear cracks length 1 − \(l\) along the \(x\)-axis in the functionally graded materials as shown in Fig. 1. \(2l\) is the distance between the two cracks (The solution of two collinear cracks of length \(r = l\) in the piezoelectric materials can easily be obtained by a simple change in the numerical values of the present paper for crack length 1 − \(l = r = 1 > 0\)). A Cartesian coordinate system \((x, y)\) is positioned as shown in Fig. 1. In this paper, the harmonic elastic stress wave is vertically incident to the crack. Let \(\omega\) be the circular frequency of the incident wave. \(\omega^{(1)}(x, y, t)\) and \(\omega^{(2)}(x, y, t)\) \((j = 1, 2)\) are the mechanical displacement and the electric potential, respectively. \(\tau^{(1)}(x, y, t)\) and \(\tau^{(2)}(x, y, t)\) \((k = x, y, j = 1, 2)\) are the anti-plane shear stress field and in-plane electric displacement field, respectively. Also note that all quantities with superscript \(j\) \((j = 1, 2)\) refer to the upper half plane 1 and the lower half plane 2 as shown in Fig. 1, respect-

\[\begin{align*}
\text{Fig. 1 Two collinear cracks in the FGPMs under anti-plane shear}
\end{align*}\]
tively. Because the incident waves are the harmonic anti-plane shear stress waves, all field quantities of \( \omega = 0 \) specifically. Because the incident waves are the harmonic anti-plane shear stress waves, all field quantities of \( \omega = 0 \) and the normal electric displacement are assumed to be continuous across the crack surfaces. Here, the standard superposition technique was used in the present paper. So the boundary conditions of the present problem are (in this paper, we just consider the perturbation stress and the perturbation electric displacement fields):

\[
\begin{align*}
\tau_{gc}(x, 0^+) & = \tau_{gc}(x, 0^-) = -\tau_0, \quad |x| \leq 1 \\
D_{kc}^{(1)}(x, 0^+) & = D_{kc}^{(1)}(x, 0^-), \quad |x| < \ell, |x| > 1 \\
u^{(1)}(x, 0^+) & = u^{(1)}(x, 0^-), \quad |x| \leq 1 \\
u^{(j)}(x, y) & = \phi^{(j)}(x, y) = 0, \quad (j = 1, 2)
\end{align*}
\]

where \( \tau_0 \) is a magnitude of the incident wave.

3. Basic Equations of Non-Local Functionally Graded Piezoelectric Materials

In the absence of body forces and free charges, the basic equations of two-dimensional anti-plane of functionally graded piezoelectric materials, non-local elastic solid, with the variable shear modulus and the variable material density are

\[
\frac{\partial \tau_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial \tau_{y}^{(j)}(x, y)}{\partial y} = -\rho(y)\omega^2 u^{(j)}(x, y) \quad (j = 1, 2)
\]

\[
\frac{\partial D_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial D_{y}^{(j)}(x, y)}{\partial y} = 0
\]

\[
\tau_{x}^{(1)}(x, 0) = \tau_{x}^{(2)}(x, 0) = -\tau_0, \quad |x| \leq 1
\]

\[
D_{x}^{(1)}(x, 0) = D_{x}^{(2)}(x, 0), \quad |x| < \ell, |x| > 1
\]

\[
u^{(1)}(x, 0) = u^{(2)}(x, 0), \quad |x| \leq 1
\]

\[
u^{(j)}(x, y) = \phi^{(j)}(x, y) = 0, \quad (j = 1, 2)
\]

where \( \tau_0 \) is a magnitude of the incident wave.

3. Basic Equations of Non-Local Functionally Graded Piezoelectric Materials

In the absence of body forces and free charges, the basic equations of two-dimensional anti-plane of functionally graded piezoelectric materials, non-local elastic solid, with the variable shear modulus and the variable material density are

\[
\frac{\partial \tau_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial \tau_{y}^{(j)}(x, y)}{\partial y} = -\rho(y)\omega^2 u^{(j)}(x, y) \quad (j = 1, 2)
\]

\[
\frac{\partial D_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial D_{y}^{(j)}(x, y)}{\partial y} = 0
\]

\[
\tau_{x}^{(1)}(x, 0) = \tau_{x}^{(2)}(x, 0) = -\tau_0, \quad |x| \leq 1
\]

\[
D_{x}^{(1)}(x, 0) = D_{x}^{(2)}(x, 0), \quad |x| < \ell, |x| > 1
\]

\[
u^{(1)}(x, 0) = u^{(2)}(x, 0), \quad |x| \leq 1
\]

\[
u^{(j)}(x, y) = \phi^{(j)}(x, y) = 0, \quad (j = 1, 2)
\]

where \( \tau_0 \) is a magnitude of the incident wave.

3. Basic Equations of Non-Local Functionally Graded Piezoelectric Materials

In the absence of body forces and free charges, the basic equations of two-dimensional anti-plane of functionally graded piezoelectric materials, non-local elastic solid, with the variable shear modulus and the variable material density are

\[
\frac{\partial \tau_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial \tau_{y}^{(j)}(x, y)}{\partial y} = -\rho(y)\omega^2 u^{(j)}(x, y) \quad (j = 1, 2)
\]

\[
\frac{\partial D_{x}^{(j)}(x, y)}{\partial x} + \frac{\partial D_{y}^{(j)}(x, y)}{\partial y} = 0
\]

\[
\tau_{x}^{(1)}(x, 0) = \tau_{x}^{(2)}(x, 0) = -\tau_0, \quad |x| \leq 1
\]

\[
D_{x}^{(1)}(x, 0) = D_{x}^{(2)}(x, 0), \quad |x| < \ell, |x| > 1
\]

\[
u^{(1)}(x, 0) = u^{(2)}(x, 0), \quad |x| \leq 1
\]

\[
u^{(j)}(x, y) = \phi^{(j)}(x, y) = 0, \quad (j = 1, 2)
\]

where \( \tau_0 \) is a magnitude of the incident wave.

The only difference from the classical electric-elastic theory and the non-local theory is in the stress and the electric displacement constitutive equations (8) and (9) in which the stress \( \tau_{x}^{(j)}(X) \) and the electric displacement \( D_{k}^{(j)}(X) \) at a point \( X \) depends on \( u^{(j)}(X) \) and \( \phi^{(j)}(X) \), at all points \( X \) of the body. For the isotropic functionally graded piezoelectric materials there exist only three material parameters, \( c_{44}^{(j)}(X' - X) \), \( c_{15}^{(j)}(X' - X) \) and \( c_{11}^{(j)}(X' - X) \). The integrals in Eqs. (8) and (9) are over the volume \( V \) of the body enclosed within a surface \( \partial V \). As discussed in Refs. (38) and (39), it can be assumed in the form of \( c_{44}^{(j)}(X' - X) \), \( c_{15}^{(j)}(X' - X) \) and \( c_{11}^{(j)}(X' - X) \) for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful

\[
(c_{44}^{(j)}(X' - X), c_{15}^{(j)}(X' - X), c_{11}^{(j)}(X' - X)) = (c_{44}^{(j)}, c_{15}^{(j)}, c_{11}^{(j)}, \rho_0) e^{i\omega 
\]

where \( c_{44}^{(j)}, c_{15}^{(j)}, c_{11}^{(j)}, \rho_0 \) are the shear modulus, the piezoelectric coefficient, the dielectric parameter and the mass density along \( y = 0 \), respectively. They are constants. \( \lambda \) is the functionally graded parameter. \( \lambda \neq 0 \) is the case for the functionally graded materials. When \( \lambda = 0 \), it will return to the homogeneous piezoelectric material case in Ref. (33).

Substituting Eqs. (10) and (11) into Eqs. (8) and (9) yield

\[
\tau_{x}^{(j)}(X) = \int_{V} \alpha(X' - X)\omega^{(j)}(X')dV(X'), \quad (k = x, y)
\]

\[
D_{k}^{(j)}(X) = \int_{V} \alpha(X' - X)\omega^{(j)}(X') \quad (k = x, y)
\]

where

\[
\alpha^{(j)}(X') = c_{44}^{(j)}u^{(j)} + c_{15}^{(j)}\phi^{(j)}, \quad (k = x, y)
\]

\[
D_{k}^{(j)}(X') = c_{11}^{(j)}u^{(j)} - c_{11}^{(j)}\phi^{(j)}, \quad (k = x, y)
\]

The expressions (14) and (15) are the classical constitutive equations for the functionally graded piezoelectric materials.
4. The Triple Integral Equation

Substituting Eqs. (12) and (13) into Eqs. (6) and (7) and using the Green-Gauss theorem leads to (16)–(18):

\[
\int_V \sigma(x',y') \, dV = \int_V \rho(x',y') \, dV + \int_{\partial V} \sigma(x',y') \, dS.
\]

This integral vanishes as \( x \to \infty \) and \( y \to \infty \), which implies that the electric field \( \mathbf{E} \) and the displacement field \( \mathbf{D} \) vanish at infinity. Hence, the line integrals in Eqs. (16) and (17) vanish. So it can be shown that the general solutions of Eqs. (16) and (17) are identical to that of

\[
\int_V \sigma(x',y') \, dV = \int_V \rho(x',y') \, dV + \int_{\partial V} \sigma(x',y') \, dS = 0
\]

almost everywhere.

What now remains is to solve the integrodifferential equations (18) and (19) for the anti-plane mechanical displacement \( u^{(j)}(x,y) \) and the electric potential \( \phi^{(j)}(x,y) \). It is impossible to obtain a rigorous solution at the present stage. It seems obvious that in the solution of such a problem we encounter serious if not unsurmountable mathematical difficulties and will have to resort to an approximate procedure. In the given problem, as discussed in Refs. (40) and (41), it is assumed that the non-local interaction in the \( y \)-direction is ignored. This is a purely assumption for mathematical tractability. In view of our assumptions, it can be given as

\[
\alpha(x',y') = \alpha_0(0,0) \delta(x'-y')
\]

where \( \alpha_0(0,0) = \frac{1}{\sqrt{\pi}}(\beta/a) \exp[-(\beta/a)^2(x'+y')^2] \). \( \beta \) is a constant and can be determined by experiment. \( a \) is the characteristic length. The characteristic length may be selected according to the range and sensitivity of the physical phenomena. For instance, for the perfect crystals, \( a \) may be taken as the lattice parameter. For a granular material, \( a \) may be considered to be the average granular distance and for a fiber composite, the fiber distance, etc. In the present paper, \( a \) is taken as the lattice parameter.

Substituting Eq. (20) into Eqs. (18) and (19), and using the Fourier transform with \( x \) can be given as follows:

\[
\tilde{\alpha}_0(j,k) = 2 \pi \int_{0}^{\infty} \int_{0}^{\infty} \tilde{u}^{(j)}(s,t) e^{-s^2} \, ds \, dt = 2 \pi \int_{0}^{\infty} \int_{0}^{\infty} \tilde{w}^{(j)}(s,t) e^{-s^2} \, ds \, dt
\]

where \( A_1(s) e^{-s^2} \), \( B_1(s) e^{-s^2} \), \( A_2(s) e^{-s^2} \), and \( B_2(s) e^{-s^2} \) are to be determined from the boundary conditions. \( \gamma_1 = \lambda + \frac{\sqrt{2} + 4\sqrt{2}c_1}{c_1}\), \( \gamma_2 = \frac{\lambda + \sqrt{2} + 4\sqrt{2}c_1}{2} \)

\( c_1 = \sqrt{\mu_1/\rho_1} \).

**JSME International Journal** Series A, Vol. 49, No. 4, 2006
Substituting Eqs. (23) and (24) into Eqs. (14) and (15), it can be obtained

\[
\alpha'_{y(x,y)} = \frac{2e^{iy}}{\pi} \int_0^\infty \left[ \mu_0 \gamma_1 A_1(s) e^{-\gamma_1 y} + e_{150} \gamma_2 B_2(s) e^{-\gamma_2 y} \right] \cos(sx) ds
\]

(25)

\[
D_y^{(y)}(x,y) = \frac{2e^{iy}}{\pi} \int_0^\infty \gamma_2 B_1(s) e^{-\gamma_2 y} \cos(sx) ds
\]

(26)

where \( \mu_0 = \frac{\varepsilon_{150}}{\varepsilon_{110}} \). According to the boundary conditions (2) – (4), it can be obtained that \( \alpha'_{y(x,y)} = \alpha'_{y(x,0')} \) and \( D_y^{(y)}(x,0') = D_y^{(y)}(x,0) \). So we have

\[
-\mu_0 \gamma_1 A_1(s) + e_{150} \gamma_2 B_2(s) = 0
\]

(29)

\[
B_1(s) + B_2(s) = 0
\]

(30)

To solve the problem, the jump of the displacement across the crack surfaces is defined as follows:

\[
f(x) = w^{(1)}(x,0') - w^{(2)}(x,0)
\]

(31)

Substituting Eqs. (23) and (24) into Eq. (31), applying the boundary conditions (3) and (4) and the Fourier transform, it can be obtained

\[
f(s) = A_1(s) - A_2(s)
\]

(32)

\[
\frac{\varepsilon_{150}}{\varepsilon_{110}} A_1(s) + B_1(s) - B_2(s) = 0
\]

(33)

By solving four equations (29), (30) and (32), (33) with four unknown functions, it can be obtained that

\[
B_1(s) = -B_2(s), \quad A_1(s) = -A_2(s)
\]

(34)

\[
B_1(s) = \frac{\varepsilon_{150}}{\varepsilon_{110}} A_1(s), \quad A_1(s) = \tilde{f}(s)/2
\]

(35)

Substituting Eqs. (25) – (28) into Eqs. (12) and (13) and applying \( \alpha \) from Eq. (20), it can be obtained

\[
\tau_{y(x,y)} = -\frac{2e^{iy}}{\pi} \frac{1}{\sqrt{\beta(a)}} \int_0^\infty \left[ \mu_0 \gamma_1 e^{-\gamma_1 y} - \frac{\varepsilon_{150}}{\varepsilon_{110}} \gamma_2 e^{-\gamma_2 y} \right] \tilde{f}(s) ds
\]

(36)

\[
D_y^{(y)}(x,y) = -\frac{2e^{iy}}{\pi} \frac{1}{\sqrt{\beta(a)}} \int_0^\infty \gamma_2 \tilde{f}(s) ds
\]

(37)

We carry out integrations on \( x' \). To this end we note the following integral(42),

\[
I = \int_0^\infty e^{\alpha x'} \cos(x' + x) dx' = \left( \frac{\pi}{\alpha} \right)^{1/2} \exp(-\alpha^2/4\pi) \cos(x)
\]

(38)

Hence

\[
\tau_{y(x,0)} = -\frac{2}{\pi} \int_0^\infty \left[ \mu_0 \gamma_1 - \frac{\varepsilon_{150}}{\varepsilon_{110}} \gamma_2 \right] e^{-\alpha^2/2\pi} \tilde{f}(s) \cos(sx) ds
\]

(39)

\[
D_y^{(y)}(x,0) = -\frac{2e_{150}}{\pi} \int_0^\infty \gamma_2 e^{-\alpha^2/2\pi} \tilde{f}(s) \cos(sx) ds
\]

(40)

Applying the boundary conditions (2) – (4), it can be obtained:

\[
\int_0^\infty \tilde{f}(s) \cos(sx) ds = 0, \quad |x| < l \quad |x| > 1
\]

(41)

Since the only difference between the classical and the non-local equations is in the introduction of the function \( g(s) = e^{-\alpha^2/2\pi} \), it is logical to utilize the classical solution to convert the system equations (41) and (42) to an integral equation of the second kind that is generally better behaved. For \( a = 0 \), then \( g(s) = 1 \) and Eqs. (41) and (42) reduce to the triple integral equations for the same problem as in classical piezoelectric materials. To determine the unknown function \( \tilde{f}(s) \), the triple integral equations (41) and (42) must be solved.

5. Solution of the Triple Integral Equations

The triple integral equations (41) and (42) can not be transformed into the second Fredholm integral equation, because the kernel of the second kind Fredholm integral equation in Ref. (18) is divergent. The kernel of the second kind Fredholm integral equation in Ref. (18) can be written as follows:

\[
L(x,u) = (xu)^{1/2} \int_0^\infty k(x') J_0(x't_0) J_0(ut_0) dt_0, \quad 0 \leq u, t_0 \leq 1
\]

where \( J_n(x) \) is the Bessel function of order \( n \).

\[
k(x') = -\Phi(x'), \quad \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-t^2) dt, \quad J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \frac{1}{2} \sqrt{\pi}) \quad \text{for} \quad x \gg 0
\]

\[
\lim_{t_0 \to \infty} k(x') = 0 \quad \text{for} \quad x' = \frac{a}{2l} \neq 0,
\]

(\( l \) is the length of the crack.)

The limit of \( \frac{tk(x') J_0(x't_0) J_0(ut_0)}{J_0(x't_0) J_0(ut_0)} \) is unequal to zero for \( t \to \infty \). So the kernel \( L(x,u) \) in Ref. (18) is divergent. Of course, the triple integral equations (41) and (42) can be considered to be a single integral equation of the first kind

Series A, Vol. 49, No. 4, 2006
with a discontinuous kernel as discussed in Ref. (16). It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. For overcoming the difficulty, the Schmidt method (35) is used to solve the triple integral equations (41) and (42). The jump of the displacement across the crack surfaces is represented by the following series:

\[
f(x) = \sum_{n=0}^{\infty} a_n F_n(x) \frac{1}{s} J_{n+1} \left( s \frac{1-b}{2} \right),
\]

where \(a_n\) is unknown coefficients to be determined and \(F_n^{(1)}(x)\) is a Jacobi polynomial (44). The Fourier transform (43) of Eqs. (43) and (44) is

\[
\tilde{f}(s) = \sum_{n=0}^{\infty} a_n F_n G_n(s) \frac{1}{s} J_{n+1} \left( s \frac{1-b}{2} \right)
\]

where \(F_n = 2 \sqrt{\pi} \Gamma(n+1) \Gamma(n+1+\frac{1}{2}) / n!\), and

\[
G_n(s) = \left\{ \begin{array}{ll}
(-1)^n \cos \left( \frac{s+1+b}{2} \right), & n = 0, 2, 4, 6, \ldots \\
(-1)^{n+1} \sin \left( \frac{s+1+b}{2} \right), & n = 1, 3, 5, 7, \ldots 
\end{array} \right.
\]

\(\Gamma(x)\) and \(J_n(x)\) are the Gamma and Bessel functions, respectively.

Substituting Eq. (45) into Eqs. (41) and (42), respectively, Eq. (42) can be automatically satisfied. Then the remaining equation (41) reduces to the form

\[
2 \sum_{n=0}^{\infty} a_n F_n \int_0^\infty \frac{1}{s} \left( \mu_0 \gamma_1 - \frac{e_{50}^2}{e_{110}} \gamma_2 \right) e^{-s \frac{s^2 \pi^2}{\omega^2}} \times G_n(s) J_{n+1} \left( s \frac{1-b}{2} \right) \cos(sx) ds = r_{00}, \quad |x| \leq 1
\]

For a large \(s\), the integrands of Eq. (46) are almost decreases exponentially. So they can be evaluated numerically. Equation (46) can now be solved for the coefficients \(a_n\) by the Schmidt method (36). It can be seen in Refs. (30) – (36). Here, it was omitted.

### 6. Numerical Calculations and Discussion

To check the numerical accuracy of the Schmidt method, the values of \(2 \sum_{n=0}^{9} a_n E_n(x)/(\pi \tau_0)\) with \(a/\beta = 0.002\), \(\omega / \gamma_1 = 0.2\), \(l = 0.1\), \(\lambda = 0.4\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(2 \sum_{n=0}^{9} a_n E_n(x)/(\pi \tau_0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.12</td>
<td>0.101175E+01</td>
</tr>
<tr>
<td>0.2</td>
<td>0.101274E+01</td>
</tr>
<tr>
<td>0.3</td>
<td>0.994597E+00</td>
</tr>
<tr>
<td>0.4</td>
<td>0.98441E-03</td>
</tr>
<tr>
<td>0.5</td>
<td>0.984467E-03</td>
</tr>
<tr>
<td>0.6</td>
<td>0.984566E-03</td>
</tr>
<tr>
<td>0.7</td>
<td>0.987161E-00</td>
</tr>
<tr>
<td>0.8</td>
<td>0.106602E+01</td>
</tr>
<tr>
<td>0.9</td>
<td>0.102331E+01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n)</th>
<th>(2a_n/(\pi \gamma_1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.122944E+00</td>
</tr>
<tr>
<td>1</td>
<td>-0.648176E-02</td>
</tr>
<tr>
<td>2</td>
<td>-0.385196E-02</td>
</tr>
<tr>
<td>3</td>
<td>-0.194075E-02</td>
</tr>
<tr>
<td>4</td>
<td>-0.964677E-03</td>
</tr>
<tr>
<td>5</td>
<td>-0.437951E-03</td>
</tr>
<tr>
<td>6</td>
<td>-0.216221E-03</td>
</tr>
<tr>
<td>7</td>
<td>-0.125342E-03</td>
</tr>
<tr>
<td>8</td>
<td>-0.623451E-04</td>
</tr>
<tr>
<td>9</td>
<td>-0.354138E-04</td>
</tr>
</tbody>
</table>

\(E_n(x) = F_n \int_0^\infty \frac{1}{s} \left( \mu_0 \gamma_1 - \frac{e_{50}^2}{e_{110}} \gamma_2 \right) e^{-s \frac{s^2 \pi^2}{\omega^2}} G_n(s) J_{n+1} \left( s \frac{1-b}{2} \right) \times \cos(sx) ds\) are given in Table 1 for \(a/\beta = 0.002\), \(\omega / \gamma_1 = 0.2\), \(l = 0.1\), \(\lambda = 0.4\). In Table 2, the values of the coefficients \(a_n\) for \(a/\beta = 0.002\), \(\omega / \gamma_1 = 0.3\), \(l = 0.1\), \(\lambda = 0.4\).
significant: in Figs. 2 – 13. The following observations are very
of the stress and the electric displacement fields are plot-
materials(33) but the magnitudes of the stress and electric dis-
same forms as those in a homogeneous piezoelectric ma-
the functionally graded piezoelectric materials. Here, we just attempt
firstly extended to solve the fracture problem of function-
displacement fields in
results converge to the classical ones when far away from
larities are present at the crack tips, and also the present
is found that no stress and electric displacement singularities at the crack tips. At
x < 1, \( \tau^{(1)}_{yx} / \tau_0 \) is very close to negative unity, and for
x > 1, \( \tau^{(1)}_{yx} / \tau_0 \) possesses finite values diminishing from a fi-
te value at x = 1 to zero at x = \( \infty \). In all computations, the
the piezoelectric material is assumed to be the commercially
available piezoelectric PZT-4. The material constants of
PZT-4 are \( c_{440} = 2.56 \times 10^{10} \) N/m\(^2\), \( e_{150} = 12.7 \) (c/m\(^2\)) and
\( e_{110} = 64.6 \times 10^{-10} \) c/(V/m), respectively. The results
of the stress and the electric displacement fields are plotted
in Figs. 2 – 13. The following observations are very
significant:

(i) The traditional concepts of the non-local theory are
firstly extended to solve the fracture problem of function-
ally graded piezoelectric materials. Here, we just attempt
to give a theoretical solution for this problem. It can be
found that the stress and electric displacement fields in
the functionally graded piezoelectric materials carry the
same forms as those in a homogenous piezoelectric mate-
rials(33) but the magnitudes of the stress and electric dis-
placement fields are dependent on the gradient of the func-
tionally graded piezoelectric material properties.

(ii) For \( a/\beta \neq 0 \), it can be proved that the semi-infinite
integrations and the series in Eqs. (47) and (48) are con-
vergent for any variable \( x \). So the stress and the electric
displacement fields are finite all along the crack line, thus
allows us to use the maximum stress as a fracture criterion.
Contrary to the classical piezoelectric theory solution, it is
found that no stress and electric displacement singularities are present at the crack tips, and also the present
results converge to the classical ones when far away from
the crack tips as shown in Figs. 2 – 7. The maximum stress
and the electric displacement do not occur at the crack tips, but slightly away from it as shown in Figs. 3, 4 and Figs. 6,
7. This phenomenon has been thoroughly substantiated in Ref. (44). The distance between the crack tip and the
maximum stress point is very small, and it depends on the
crack length, the frequency of the incident stress waves,
the functionally graded parameter and the lattice parameter.

(iii) The stresses and the electric displacements at the
crack tips become infinite as the lattice parameter \( a \to 0 \).
This is the classical continuum limit of square root sin-
ularity. This can be shown from Eq. (46). For \( a \to 0 \),
\( g(x) = e^{-\frac{x^2}{\psi^2}} = 1 \), Eq. (46) will reduce to the triple integral
equations for the same problem in classical functionally

\[
\times \int_0^\infty \frac{1}{s} \left( \mu_0 \gamma_1 - \frac{\epsilon_{150}^2}{\epsilon_{110}} \gamma_2 \right) e^{-\frac{s^2}{\psi^2}} \times G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds
\]

\[
D_y = D_y^{(1)}(x,0) = -2e_{150} \sum_{n=1}^{\infty} a_n F_n
\]

\[
\times \int_0^\infty \frac{\gamma_2}{s} e^{-\frac{s^2}{\psi^2}} G_n(s) J_{n+1} \left( \frac{1-b}{2} \right) \cos(xs) ds
\]

So long as \( a/\beta \neq 0 \), the semi-infinite integration and
the series in Eqs. (47) and (48) are convergent for any
variable \( x \). Equations (47) and (48) give finite stress and
electric displacement all along \( y = 0 \), so there is no stress
and electric displacement singularities at the crack tips. However, for \( a/\beta = 0 \), we have the classical stress and
electric displacement singularities at the crack tips. At
\( l < x < 1 \), \( \tau^{(1)}_{yx} / \tau_0 \) is very close to negative unity, and for
\( x > 1 \), \( \tau^{(1)}_{yx} / \tau_0 \) possesses finite values diminishing from a fi-
te value at \( x = 1 \) to zero at \( x = \infty \). In all computations, the
the piezoelectric material is assumed to be the commercially
available piezoelectric PZT-4. The material constants of
PZT-4 are \( c_{440} = 2.56 \times 10^{10} \) N/m\(^2\), \( e_{150} = 12.7 \) (c/m\(^2\)) and
\( e_{110} = 64.6 \times 10^{-10} \) c/(V/m), respectively. The results
of the stress and the electric displacement fields are plotted
in Figs. 2 – 13. The following observations are very
significant:

(i) The traditional concepts of the non-local theory are
firstly extended to solve the fracture problem of function-
ally graded piezoelectric materials. Here, we just attempt
to give a theoretical solution for this problem. It can be
found that the stress and electric displacement fields in
the functionally graded piezoelectric materials carry the
same forms as those in a homogenous piezoelectric mate-
rials(33) but the magnitudes of the stress and electric dis-
placement fields are dependent on the gradient of the func-
tionally graded piezoelectric material properties.

(ii) For \( a/\beta \neq 0 \), it can be proved that the semi-infinite
integrations and the series in Eqs. (47) and (48) are con-
vergent for any variable \( x \). So the stress and the electric
displacement fields are finite all along the crack line, thus
allows us to use the maximum stress as a fracture criterion.
Contrary to the classical piezoelectric theory solution, it is
found that no stress and electric displacement singularities are present at the crack tips, and also the present
results converge to the classical ones when far away from
the crack tips as shown in Figs. 2 – 7. The maximum stress
and the electric displacement do not occur at the crack tips, but slightly away from it as shown in Figs. 3, 4 and Figs. 6,
7. This phenomenon has been thoroughly substantiated in Ref. (44). The distance between the crack tip and the
maximum stress point is very small, and it depends on the
crack length, the frequency of the incident stress waves,
graded piezoelectric materials. These triple integral equations can be solved by using the singular integral equation for the same problem as in the local functionally graded piezoelectric materials problem. However, the stress and the electric displacement singularities are present at the crack tips in the local functionally graded piezoelectric materials problem as well known.

(iv) The results of the stress and the electric displacement fields at the crack tips tend to decrease with the increase of the lattice parameter as shown in Figs. 8, 9 and in Table 3. This is the same as shown in Ref. (18). For the electric displacement field, it has the same changing tendency as the stress field as shown in Figs. 2 – 11. However, the amplitude values of the electric displacement filed and the stress field are different. The amplitude values of the electric displacement filed is very small as shown in Figs. 5 – 7, 9 and 11.

(v) The stress field and the electric displacement field at the crack tips decrease with increase of the function-
Fig. 10 The stress at the crack tips versus \( x \) for \( a/\beta = 0.002, l = 0.1 \) and \( \omega/c_1 = 0.2 \).

Fig. 11 The electric displacement at the crack tips versus \( \lambda \) for \( a/\beta = 0.002, l = 0.1 \) and \( \omega/c_1 = 0.2 \).

Fig. 12 The stress at the crack tips versus \( \omega/c_1 \) for \( l = 0.1, a/\beta = 0.002 \) and \( \lambda = 0.4 \).

Fig. 13 The stress at the crack tips versus \( l \) for \( \omega/c_1 = 0.2, a/\beta = 0.002 \) and \( \lambda = 0.4 \).

Table 4 The stress at the crack tips versus \( x \) for \( l = 0.1, a/\beta = 0.002 \) and \( \omega/c_1 = 0.2 \) as shown in Fig. 10.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \tau_{xy}^{(1)}(1,0)/\tau_0 )</th>
<th>( \tau_{xy}^{(2)}(1,0)/\tau_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.0</td>
<td>0.173863E+02</td>
<td>0.139385E+02</td>
</tr>
<tr>
<td>-1.6</td>
<td>0.169030E+02</td>
<td>0.125991E+02</td>
</tr>
<tr>
<td>-1.2</td>
<td>0.163696E+02</td>
<td>0.125625E+02</td>
</tr>
<tr>
<td>-0.8</td>
<td>0.157779E+02</td>
<td>0.121076E+02</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.151818E+02</td>
<td>0.116697E+02</td>
</tr>
<tr>
<td>0.0</td>
<td>0.141719E+02</td>
<td>0.109285E+02</td>
</tr>
<tr>
<td>0.4</td>
<td>0.130035E+02</td>
<td>0.105768E+02</td>
</tr>
<tr>
<td>0.8</td>
<td>0.117119E+02</td>
<td>0.913132E+01</td>
</tr>
<tr>
<td>1.2</td>
<td>0.106389E+02</td>
<td>0.835456E+01</td>
</tr>
<tr>
<td>1.6</td>
<td>0.971957E+01</td>
<td>0.769204E+01</td>
</tr>
<tr>
<td>2.0</td>
<td>0.893412E+01</td>
<td>0.712491E+01</td>
</tr>
</tbody>
</table>

Materially graded parameter \( \lambda \) as shown in Figs. 10, 11 and in Table 4. This phenomenon is the same as the results in Ref. (45). Hence, the stress intensity factors can be reduced by adjusting the functionally graded parameter \( \lambda \). When \( \lambda = 0 \), the present problem will revert to the same problem as studied in Ref. (33). The results of the present paper are the same as ones in Ref. (33) as shown in Figs. 10 and 11. Certainly, the solving process of the present paper is similar with ones of Refs. (30) – (35). However, the material properties and the basic equations of the present paper are more complex than ones of Refs. (27) – (29).

(vii) From results as shown in Fig. 12, the stress fields near the crack tips increase with increase of the circular frequency of the incident waves until reaching the first peak value at \( \omega/c_1 \approx 0.90 \), then they decrease with increase in magnitude. This variation tendency similar to one as in the local functionally graded piezoelectric material fracture problem as well known.

(viii) The stress and the electric displacement fields near the inner crack tips are larger than ones near the outer crack tips as shown in Figs. 2 – 13 and in Tables 3 and 4. From results as shown in Fig. 13, the stress fields at the crack tip decrease with increase of the crack length, i.e., the interaction of two collinear crack decreases with increase of the distance between two collinear cracks.

7. Conclusion

In the present paper, the traditional concepts of the non-local theory are firstly extended to solve the fracture problem of functionally graded piezoelectric materials. As expected, the solution in this paper does not contain the stress and the electric displacement singularities at the crack tips. It can be obtained that the solution of the
The present paper yields a finite stress at the crack tips, thus allows us to use the maximum stress as a fracture criterion.

Acknowledgements

The author is grateful for the financial support by the National Natural Science Foundation of China (90405016).

References


