An Axisymmetric Contact Problem of an Elastic Half-Space Pressed onto a Rigid Foundation with a Parabolic-Ended Pit and Protrusion*

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A frictionless, axisymmetric contact problem is considered, where an elastic half-space is pressed onto a rigid foundation with a parabolic-ended pit or protrusion. It is necessary to find the extent of the contact region and contact stress between an elastic half-space and the rigid foundation. Also required is the normal displacement on the free surface. The problem is solved by using Papkovitch-Neuber equations in oblate spheroidal coordinates. The effects of various parameters on the contact stress are studied. Numerical results are compared with those for a flat-ended pit or protrusion, and they are found to be in agreement.

Key Words: Elasticity, Contact Stress, Elastic Half-Space, Mixed Boundary-Value Problem, Protrusion, Pit, Rigid Foundation,

1. Introduction

Accurate estimation of a contact stress is an important part of the design of structural and mechanical elements, and is central to assuring safety and reliability in the design of devices, machines and structures. Indeed, a number of experimental and theoretical methods for estimating the contact stress have been developed. It is well known that these methods include theoretical techniques based on solutions of punch problems which a large number of investigators have pursued vigorously. Most of the applied research in this field is concerned with contact problems for certain rigid punches and bases in contact with the plane face of an isotropic elastic medium over a circle and a circular annulus. In the problems for a circular annulus, these solutions have been obtained by the use of a certain series expansions^{(1)~(6)} with parameters for the annulus and appropriate polynomials for the displacement, and by the assumption

tion of the stress singularities^{(7)~(9)} at the contact region. On the other hand, there are particular cases in which the magnitude of the contact radius varies with the indentational force. It is pointed out that different conditions at the contact boundaries are presented for the so-called receding contact problem. Shibuya et al.^{(10)~(15)} solved the problems for a rigid base with a cylindrical and parabolic-ended protrusion or pit. They also paid attention to the concave punch problem. Gladwell and Hara⁽¹⁶⁾ formulated the contact problem for a rigid obstacle pressed between two dissimilar elastic half-spaces by reducing it to an infinite system of simultaneous equations, and determined the extent of the contact regions between the obstacle and each half-space. Barber⁽¹⁷⁾⁽¹⁸⁾ attempted to solve the concave punch problem. Selvadurai⁽¹⁹⁾ solved the circular inclusion problem for a transversely isotropic elastic medium with potential functions described by Elliott⁽²⁰⁾, and examined the extent of the separation zone.

In this paper an axisymmetric contact problem is considered, where an elastic half-space is pressed onto a rigid foundation with a parabolic-ended pit or protrusion. The method due to Gladwell⁽²¹⁾ and England⁽²²⁾

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is extended to derive the solution to the problem which is reduced to an infinite system of simultaneous equations. A corresponding result for a cylindrical pit or protrusion given by Shibuya et al.⁽¹⁰⁾ is also found, and it is shown to be in agreement with the results obtained as a special case of the present analysis.

2. Formulation

2.1 Stress functions

Following Ref. [16] we construct the harmonic stress functions $\phi(r, z)$ and $\psi(r, z)$ as Papkovitch-Neuber solutions in cylindrical coordinates (r, θ, z) . The normal displacement w_z , the normal stress σ_z and the shear stress τ_{rz} , supposed as being free of torsion, may be written in terms of ϕ and ψ as

$$\left. \begin{aligned} 2Gw_z &= \frac{\partial \phi}{\partial z} - z \frac{\partial \psi}{\partial z} + (3-4\nu)\psi \\ \sigma_z &= \frac{\partial^2 \phi}{\partial z^2} - z \frac{\partial^2 \psi}{\partial z^2} + 2(1-\nu) \frac{\partial \psi}{\partial z} \\ \tau_{rz} &= \frac{\partial^2 \phi}{\partial z \partial r} - z \frac{\partial^2 \psi}{\partial z \partial r} + (1-2\nu) \frac{\partial \psi}{\partial r} \end{aligned} \right\} \quad (1)$$

where G, ν are the shear modulus and Poisson's ratio respectively.

2.2 A rigid base with a parabolic-ended pit

The problem configuration is shown in Fig. 1(a). The profile of the pit for $z=0$ may then be taken as $w_z = f(r)$ where, for infinitesimal elasticity theory, there is no loss in generality if one assumes that $f(r) = -D + r^2/2R$. If the frictionless rigid base is out of contact with the half-space over the annulus $z=0, s \leq r < l$, then the boundary conditions on $z=0$ may be written

$$\left. \begin{aligned} \text{(i)} \quad (w_z)_{z=0} &= -D + r^2/2R \quad (0 \leq r \leq s) \\ \text{(ii)} \quad (w_z)_{z=0} &= 0 \quad (l \leq r \leq \infty) \\ \text{(iii)} \quad (\sigma_z)_{z=0} &= 0 \quad (s < r < l) \\ \text{(iv)} \quad (\tau_{rz})_{z=0} &= 0 \quad (0 \leq r < \infty) \\ \text{(v)} \quad \sigma_z &= -p_0, \tau_{rz} = 0 \quad \text{at } z \rightarrow \infty \end{aligned} \right\} \quad (2)$$

Equation (2) states that the inner radius s is dependent upon the magnitude of p_0, D, R and l . The solution for the half-space may be obtained as the superposition of two fields. The first corresponds to a uniform stress field in the z -direction, namely $\sigma_z = -p_0$, for which

$$\left. \begin{aligned} 2(1+\nu)Gu_r &= \nu r p_0 \\ 2(1+\nu)Gw_z &= -z p_0 \\ \sigma_z &= -p_0, \sigma_r = \sigma_\theta = \tau_{rz} = 0 \end{aligned} \right\} \quad (3)$$

The second may be obtained from a Papkovitch-Neuber solution, which for axisymmetric problems may satisfy the boundary conditions as follows:

$$\left. \begin{aligned} \text{(i)} \quad (w_z)_{z=0} &= -D + r^2/2R \quad (0 \leq r \leq s) \\ \text{(ii)} \quad (w_z)_{z=0} &= 0 \quad (l \leq r < \infty) \\ \text{(iii)} \quad (\sigma_z)_{z=0} &= p_0 \quad (s < r < l) \\ \text{(iv)} \quad (\tau_{rz})_{z=0} &= 0 \quad (0 \leq r < \infty) \\ \text{(v)} \quad \text{All stress components} &\text{vanish at infinity} \end{aligned} \right\} \quad (4)$$

Equation (1) and condition (iv) yield

$$\partial \phi / \partial z = -(1-2\nu)\psi \quad (5)$$

so that the boundary values are

$$(w_z)_{z=0} = \zeta \psi, (\sigma_z)_{z=0} = \partial \psi / \partial z \quad (6)$$

where $\zeta = (1-\nu)/G$. Following Ref. [16] the stress function ψ is now built up as the sum of three parts; thus

$$\psi = \psi_1 + \psi_2 + \psi_3 \quad (7)$$

The oblate spheroidal coordinates (ξ, η) are related to the cylindrical coordinates (r, z) by the equations

$$r = l\{(1+\xi^2)(1-\eta^2)\}^{1/2}, z = l\xi\eta \quad (8)$$

for $z=0$, Eq. (8) leads to

$$\left. \begin{aligned} \xi &= 0, \eta = (1-r^2/l^2)^{1/2}, (r \leq l) \\ \xi &= (r^2/l^2 - 1)^{1/2}, \eta = 0, (r > l) \end{aligned} \right\} \quad (9)$$

Similarly the coordinates (ξ_1, η_1) are related to the circle $r=s$ by the equations

$$\left. \begin{aligned} r &= s\{(1+\xi_1^2)(1-\eta_1^2)\}^{1/2} \\ z &= s\xi_1\eta_1 \end{aligned} \right\} (\xi_1 \geq 0) \quad (10)$$

and for $z=0$

$$\left. \begin{aligned} \xi_1 &= 0, \eta_1 = (1-r^2/s^2)^{1/2}, (r \leq s) \\ \xi_1 &= (r^2/s^2 - 1)^{1/2}, \eta_1 = 0, (r > s) \end{aligned} \right\} \quad (11)$$

We now note that σ, τ are related to ξ, η by the equations

$$\left. \begin{aligned} \sigma &= \xi(1+\xi^2-\eta^2)^{-1/2} \\ \tau &= \eta(1+\xi^2-\eta^2)^{-1/2} \end{aligned} \right\} (\sigma \geq 0) \quad (12)$$

and for $z=0$

$$\left. \begin{aligned} \sigma &= 0, \tau = (l^2/r^2 - 1)^{1/2}, (r \leq l) \\ \sigma &= (1-l^2/r^2)^{1/2}, \tau = 0, (r > l) \end{aligned} \right\} \quad (13)$$

We now represent $\psi_i (i=1, 2, 3)$ in the basic form for a solution of Laplace's equation and unknown

Fig. 1 Geometry of the rigid base

coefficients c_k, d_k by applying Kelvin's theorem⁽¹⁶⁾ and Eq. (8) to (13), namely:

$$\psi_1 = lp_0 \sum_{k=0}^{\infty} c_k P_{2k}(\eta_1) Q_{2k}(i\xi_1) / Q'_{2k}(i0) \quad (14)$$

$$\psi_2 = lp_0 (1 - \sigma^2 + \tau^2)^{1/2} \sum_{k=0}^{\infty} d_k P_{2k}(\sigma) Q_{2k}(i\tau) / Q'_{2k}(i0) \quad (15)$$

$$\psi_3 = 2lp_0 P_1(\eta) Q_1(i\xi) / \pi \quad (16)$$

where $P_{2k}(x)$ is the Legendre polynomial, $Q_{2k}(iy)$ is the associated Legendre function of the second kind, $i = \sqrt{-1}$, and $Q'_{2k}(i0) = dQ_{2k}(i\xi) / d\xi|_{\xi=0}$.

The z -derivative expressed in terms of $\xi_1, \eta_1, \xi, \eta, \sigma$ and τ has the form

$$\frac{\partial}{\partial z} = \begin{cases} \partial / \partial \eta_1 \partial \xi_1, (\xi_1 = 0, r < s) \\ \partial / \partial \xi_1 \partial \eta_1, (\eta_1 = 0, r > s) \\ \partial / \partial \eta \partial \xi, (\xi = 0, r < l) \\ \partial / \partial \xi \partial \eta, (\eta = 0, r > l) \\ (1 + \tau^2) \partial / \partial \tau \partial \sigma, (\sigma = 0, r < l) \\ (1 - \sigma^2) \partial / \partial \sigma \partial \tau, (\tau = 0, r > l) \end{cases} \quad (17)$$

The resulting expressions for the displacement and contact stress on $z=0$ are

$$\begin{aligned} \frac{(w_z)_{z=0}}{lp_0 \zeta} &= \begin{cases} \sum_{k=0}^{\infty} \{ \bar{c}_k P_{2k}(\eta_1) + (2i/\pi)(1 + \tau^2)^{1/2} \\ \times \bar{d}_k Q_{2k}(i\tau) \} - 2\eta/\pi, (r \leq s) \\ (2/\pi) [i \sum_{k=0}^{\infty} \{ \bar{c}_k Q_{2k}(i\xi_1) + (1 + \tau^2)^{1/2} \\ \times \bar{d}_k Q_{2k}(i\tau) \} - \eta], (s < r < l) \\ \sum_{k=0}^{\infty} \{ (2i/\pi) \bar{c}_k Q_{2k}(i\xi_1) + (1 - \sigma^2)^{1/2} \\ \times \bar{d}_k P_{2k}(\sigma) \}, (l \leq r) \end{cases} \quad (18) \\ \frac{(\sigma_z)_{z=0}}{p_0} &= \begin{cases} (l/s\eta_1) \sum_{k=0}^{\infty} c_k P_{2k}(\eta_1), (r \leq s) \\ 0 (s < r < l) \\ \{ (1 - \sigma^2)^{3/2} / \sigma \} \sum_{k=0}^{\infty} d_k P_{2k}(\sigma) \\ - (2/\pi) \{ 1/\xi - \tan^{-1}(1/\xi) \} - 1, (l \leq r) \end{cases} \quad (19) \end{aligned}$$

where

$$\frac{\bar{c}_k}{c_k} = \frac{\bar{d}_k}{d_k} = \frac{Q_{2k}(i0)}{Q'_{2k}(i0)} = -\frac{\pi}{2} \{ P_{2k}(0) \}^2 \quad (20)$$

and

$$\begin{cases} 2iQ_{2k}(+i0) = \pi P_{2k}(0) \\ Q'_{2k}(+i0) = i/P_{2k}(0) \end{cases} \quad (21)$$

The condition that $(\sigma_z)_{z=0}|_{r=s-} = 0$ at the inner boundary now yields

$$\sum_{k=0}^{\infty} c_k P_{2k}(0) = 0 \quad (22)$$

We then use the following approximations:

$$\begin{aligned} 2iQ_{2k}(i\xi_1)/\pi &\approx (1 - \sigma^2)^{1/2} \sum_{m=0}^{\infty} a_{km} P_{2m}(\sigma) \\ 2i(1 + \tau^2)^{1/2} Q_{2k}(i\tau)/\pi &\approx \sum_{m=0}^{\infty} b_{km} P_{2m}(\eta_1) \\ -2\eta/\pi &\approx \sum_{m=0}^{\infty} e_m P_{2m}(\eta_1) \end{aligned} \quad (23)$$

The use of the relations between $\xi_1, \eta_1, \sigma, \tau$ shows that

$$a_{km} = (s/l) b_{km} \quad (24)$$

Thus, using Eqs.(23) and (24) in Eq.(18) we can see

immediately that the expression for $(w_z)_{z=0}$ is

$$\frac{(w_z)_{z=0}}{lp_0 \zeta} = \begin{cases} \sum_{k=0}^{\infty} (\bar{c}_k + \sum_{m=0}^{\infty} \bar{d}_m b_{km} + e_k) P_{2k}(\eta_1), & (r \leq s) \\ (1 - \sigma^2)^{1/2} \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} \bar{c}_m a_{km} + \bar{d}_k \right) P_{2k}(\sigma), & (l \leq r) \end{cases} \quad (25)$$

We now write $f(r)$ as

$$-D + \frac{r^2}{2R} = \left(-D + \frac{s^2}{3R} \right) P_0(\eta_1) - \frac{s^2}{3R} P_2(\eta_1) \quad (26)$$

Making use of Eqs.(25) and (26), and equating the coefficients of $P_{2k}(\eta_1), P_{2k}(\sigma)$ in the boundary conditions (i) and (ii) respectively, we find that

$$\begin{aligned} \sum_{k=0}^{\infty} \bar{d}_k b_{km} + \bar{c}_m + e_m &= \left(-D + \frac{s^2}{3R} \right) \frac{\delta_{0m}}{lp_0 \zeta} - \frac{s^2 \delta_{1m}}{3R lp_0 \zeta} \\ \sum_{k=0}^{\infty} \bar{c}_k a_{km} + \bar{d}_m &= 0 \end{aligned} \quad (m=0, 1, 2, \dots) \quad (27)$$

and note that the problem in this section is reduced to an infinite system of simultaneous equations(27). Eq. (27) is solved as follows. If s/l is assumed to be known, then Eq.(27) provides $2N+2$ simultaneous nonhomogeneous linear equations for the $2N+2$ unknown $\bar{c}_m$ and $\bar{d}_m$. These unknowns may be written in terms of α_{mj} and β_{mj} ($j=0, 1$) as

$$\begin{aligned} \bar{c}_m &= \alpha_{m0} + \alpha_{m1}/\zeta p_0 \\ \bar{d}_m &= \beta_{m0} + \beta_{m1}/\zeta p_0 \end{aligned} \quad (28)$$

so that Eqs.(20) and (22) yield

$$\sum_{k=0}^{\infty} \frac{\alpha_{k0} + \alpha_{k1}/\zeta p_0}{P_{2k}(0)} = 0 \quad (29)$$

and the value $p_0 \zeta$ is then determined. Similarly, all the coefficients $c_m, \bar{c}_m, d_m, \bar{d}_m$ are obtained by Eqs.(20) and (28). The displacement and the contact stress are then evaluated.

2.3 A rigid base with a parabolic-ended protrusion

We now consider the problem for a rigid base with a parabolic-ended protrusion, as shown in Fig. 1(b), of which the profile for $z=0$ may be taken as $w_z = f(r)$ for infinitesimal elasticity theory. The analysis of the preceding section may now be applied, and provides that $f(r) = D - r^2/2R$. If the half-space is in contact with the rigid base over $z=0, r \leq s, r \geq l$, then the boundary conditions on $z=0$ may be written as

$$\begin{aligned} \text{(i)} \quad (w_z)_{z=0} &= D - r^2/2R \quad (0 \leq r \leq s) \\ \text{(ii)} \quad (w_z)_{z=0} &= 0 \quad (l \leq r < \infty) \\ \text{(iii)} \quad (\sigma_z)_{z=0} &= 0 \quad (s < r < l) \\ \text{(iv)} \quad (\tau_{rz})_{z=0} &= 0 \quad (0 \leq r < \infty) \\ \text{(v)} \quad \sigma_z &= -p_0, \tau_{rz} = 0 \text{ at } z \rightarrow \infty \end{aligned} \quad (30)$$

Equation(30) states that the contact is frictionless and the boundary radius $r=l$ depends upon the values of p_0, s, D, R . Taking it into consideration that the condition of compression at the end of the contact region, namely $(\sigma_z)_{z=0}|_{r=l-} = 0$, we find that

Table 1 Coefficients $\bar{c}_k$ and $\bar{d}_k$

k	s/l=0.4		s/l=0.6		s/l=0.8	
	$\bar{c}_k$	$\bar{d}_k$	$\bar{c}_k$	$\bar{d}_k$	$\bar{c}_k$	$\bar{d}_k$
0	4.141540D-02	-1.066662D-02	1.190086D-01	-4.673501D-02	2.986985D-01	-1.609855D-01
1	1.997557D-02	1.225638D-04	5.483392D-02	1.337641D-03	1.275577D-01	9.759629D-03
2	-5.316071D-04	-2.339596D-06	-3.216777D-03	-6.428636D-05	-1.339905D-02	-1.019323D-03
3	1.405170D-05	5.697640D-08	2.165452D-04	3.960131D-06	1.976964D-03	1.373236D-04
4	-4.397759D-07	-1.583382D-09	-1.725370D-05	-2.789177D-07	-3.378681D-04	-2.088120D-05
5	1.495207D-08	4.780497D-11	1.485930D-06	2.129825D-08	5.906314D-05	3.298297D-06
6	-5.337541D-10	-1.527188D-12	-1.317312D-07	-1.696545D-09	-9.692122D-06	-5.007038D-07
7	1.958185D-11	5.064258D-14	1.135434D-08	1.339095D-10	1.372876D-06	6.706123D-08
8	-7.127384D-13	-1.688130D-15	-8.668923D-10	-9.568913D-12	-1.522689D-07	-7.153147D-09
9	2.317381D-14	5.122250D-17	5.003191D-11	5.274997D-13	1.147590D-08	5.247256D-10
10	-4.889127D-16	-1.031230D-18	-1.568878D-12	-1.604922D-14	-4.335153D-10	-1.945094D-11

$$\sum_{k=0}^{\infty} d_k P_{2k}(0) = 2/\pi \quad (31)$$

The profile of the protrusion can be written as

$$D - \frac{r^2}{2R} = \left(D - \frac{s^2}{3R}\right) p_0(\eta_1) + \frac{s^2}{3R} P_2(\eta_1) \quad (32)$$

so that the problem in this section is reduced to an infinite system of simultaneous equations for the unknowns $\bar{c}_m$ and $\bar{d}_m$ as follows:

$$\left. \begin{aligned} \sum_{k=0}^{\infty} \bar{d}_k b_{km} + \bar{c}_m + \bar{e}_m \\ = \left(D - \frac{s^2}{3R}\right) \frac{\delta_{0m}}{l p_0 \zeta} + \frac{s^2 \delta_{1m}}{3R l p_0 \zeta} \\ \sum_{k=0}^{\infty} \bar{c}_k a_{km} + \bar{d}_m = 0 \end{aligned} \right\} \quad (m=0, 1, 2, \dots) \quad (33)$$

From Eqs.(20), (28) and (31) it is seen that the value $p_0 \zeta$ is evaluated by the equation

$$\sum_{k=0}^{\infty} \frac{\beta_{k0} + \beta_{k1}/\zeta p_0}{P_{2k}(0)} = -1 \quad (34)$$

and the coefficients $\bar{c}_m$ and $\bar{d}_m$ are then determined.

3. Numerical results

It is now required to find $\bar{c}_k$ and $\bar{d}_k$ so that Eqs. (27) and (33) are satisfied. For given values of R , D , l , p_0 in Eq.(27) and of R , D , s , p_0 in Eq.(33), the contact radii s and l are the unknowns which need to be determined, respectively. As in all contact problems of this type, the present problems are solved in an inverse manner, i. e. by assuming that s in Eq.(27) and l in Eq.(33) rather than the value $p_0 \zeta$ for the uniformly compressive stress are prescribed.

The results for $\bar{c}_k$ and $\bar{d}_k$ obtained from Eq.(27) with $R/l=20$, $D/l=0.1$ are shown in Table 1, in which $a \times 10^{-b}$ is written as $aD-b$. It shows that convergence becomes slower with an increasing s/l .

The contact stresses for $s/l=0.8$ were obtained by using 7 terms in Eqs.(27) and (33) so as to investigate the effects of the terms N on their convergence in actual computation. These were found to provide

Fig. 2 Distribution of $(w_z)_{z=0}$ and $(\sigma_z)_{z=0}$

apparent numerical convergence. For N we take $N=11$ since this appears to give a sufficient numerical result for all cases considered below.

3.1 Results for a rigid base with the pit

Some results for $(w_z)_{z=0}$ and $(\sigma_z)_{z=0}$ are shown in Fig. 2 for given values of $s/l=0.2, 0.4, 0.6$ and 0.8 . The figure shows that the value of the r -derivative of $(w_z)_{z=0}$ for all s/l becomes infinite as $r \rightarrow l^-$, and tends to s/R as $r \rightarrow s^+$. It is therefore found that smooth contact starts at $r=s$. On the other hand, we find that the contact stress becomes zero at the inner boundary and singular at the edge of the pit, namely $r=l$. It is also seen from the figure that the stress for $r/l \geq 2$ is in approximate agreement with the result for a uniformly stressed field given by Eq.(3), namely $(\sigma_z)_{z=0} = -p_0$. Thus, it can be considered that there is no disturbance of the stress field due to the presence of the pit for given values of D/l and R/l where $r/l \geq 2$.

Figures 3 and 4 show the variations of $(\sigma_z)_{z=0}$ with R/l and D/l respectively. Fig. 3 shows that both $(\sigma_z)_{z=0}$ and p_0 increase with an increasing D/l . On the other hand, it is seen from Fig. 4 that $(\sigma_z)_{z=0}/p_0$ increases with an increasing R/l and comes close to the result for a flat-ended pit, namely for $R/l = \infty$.

3.2 Results for a rigid base with the protrusion

The distributions of $(\sigma_z)_{z=0}$ and $(w_z)_{z=0}$ for given values of $D/l=0.1$ and $R/l=20$ are shown in Fig. 5. The figure shows that the elastic half-space is in perfect contact with the protrusion and the value of the r -derivative of $(w_z)_{z=0}$ becomes infinite at the edge of the protrusion, namely for $r=s+$. Smooth contact starts at $r=l$, and then for all s/l , $(w_z)_{z=0}=0$

Fig. 3 Variation of $(\sigma_z)_{z=0}$ with D/l

Fig. 4 Variation of $(\sigma_z)_{z=0}$ with R/l

Fig. 5 Distribution of $(w_z)_{z=0}$ and $(\sigma_z)_{z=0}$

where $r \geq l$. On the other hand, it is seen from the figure that the contact stress has the singularity at $r=s-$, and $(\sigma_z)_{z=0}/p_0$ at the center portion of the protrusion decreases with an increasing s/l . The contact stress for $r/l \geq 1$ comes close to $-p_0$ as r/l increases; whereas, for all s/l , $(\sigma_z)_{z=0}=0$ where $s < r \leq l$. In this case it is interesting to note that $(\sigma_z)_{z=0} \approx -p_0$ where $r/l \geq 1.5$, and then the stress field is almost insensitive to the presence of the protrusion.

Figures 6 and 7 show the variations of $(\sigma_z)_{z=0}$ with D/l and R/l respectively. It is seen from Fig. 6 that the uniform stress p_0 decreases and $(\sigma_z)_{z=0}/p_0$ increases with a decreasing D/l . Fig. 7 shows that $(\sigma_z)_{z=0}/p_0$ for $R/l \geq 60$ is approximated by the result for $R/l = \infty$, namely, for a flat-ended protrusion problem.

The results for a particular case mentioned above, which is obtained from Eqs. (27) and (33) with $R/l = \infty$ correspond to those for flat-ended pit and protrusion problems. Comparisons for the relation between D , l , s , p_0 are made with the results obtained by Shibuya et al. and are given in Fig. 8. As shown from the figure, the present results are indistinguishable from those for the flat-ended pit and protrusion.

Fig. 6 Variation of $(\sigma_z)_{z=0}$ with D/l

Fig. 7 Variation of $(\sigma_z)_{z=0}$ with R/l

Fig. 8 Relation between D, l, s, p_0 for particular $R/l = \infty$

4. Conclusion

The solution is presented for the axisymmetric contact problem for an elastic half-space pressed onto a rigid base with a parabolic-ended pit or protrusion. The distributions of the surface displacement and the contact stress are shown. Especially, the variations of contact stress and the size of the ring-shaped separation area with a uniform stress parameter and some geometrical variables are studied.

The following conclusions were reached:

- (1) The contact stress becomes singular at the edge of the pit, and the presence of the pit does not have a significant effect on the stress where $r/l \geq 2$.
- (2) $(\sigma_z)_{z=0}/p_0$ for $r > l$ is insensitive to the magnitude of the contact boundary s/l .
- (3) The contact stress has a singularity at the edge of the protrusion. The stress field, where $r/l \geq 1.5$, is almost insensitive to the presence of the protrusion and becomes uniform.
- (4) For $r/l > 1$, $(\sigma_z)_{z=0}/p_0$ is insensitive to the curvature parameter of the protrusion, namely R/l .

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