Stress Concentrations of a Sandwich Plate with a Hole or Notch*

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Nowadays, due to the development of the sandwich core series and the higher degree of accuracy required for each thickness of sandwich plate, optimum design of the structural sandwich plate has been needed. We have found several papers relating to the optimum design of the laminated plates, but not for the optimum design of a sandwich plate with a hole or notch. Hereby, this paper presents the characteristics and some example data of the stress concentrations of a sandwich plate in a wide sphere of dimensions in order to determine the optimum design of a sandwich plate with a hole or notch. In analyzing the structural sandwich plate, we apply two bending theories, of which we name one the simple bending theory and the other the advanced bending theory. By using FEM analysis, which is based on these two bending theories, we determine stress concentrations of a sandwich plate, taking into account both the accuracy and cost efficiency of the calculations. We also examine the stress concentrations and reveal the characteristics of the stress concentrations under two types of bending loads: anticlastic bending load and cylindrical bending load. To show examples of stress concentrations of a sandwich plate, we use the simple bending theory under the anticlastic bending load.

Key Words: Sandwich Plate, Stress Concentration, FEM, Optimum Design, Simple Bending Theory, Advanced Bending Theory

1. Introduction

For the optimum design of a sandwich plate with a hole or notch, much data regarding the stress concentrations of a sandwich plate is needed. These days, with the development of computers, it is not so difficult to investigate the stress concentrations of a sandwich plate by FEM analysis. Thus, this paper offers the characteristics and data of the stress concentrations of a sandwich plate with a hole or notch.

In analyzing the structural sandwich plate, we apply two bending theories, of which we name one the simple bending theory and the other the advanced bending theory. By using FEM analysis, which is based on the two bending theories, we determine stress concentrations, taking into account both the

accuracy and cost efficiency of the calculations. With regard to the accuracy, we cannot find a significant difference between the results of the two bending theories. However, with regard to the cost efficiency, we find that the simple bending theory is more useful for investigating the stress concentrations. We also examine the stress concentrations under two types of load conditions of bending: anticlastic bending load and cylindrical bending load, and we reveal the specific characteristics of stress concentrations with various rigidity and thickness of core.

Lastly, we show some examples of the stress concentration factors of sandwich plates with holes or notches under the anticlastic bending load by using the simple bending theory.

2. Symbols and Definitions

This paper primarily uses the following symbols and definitions.

x, y, z^* : rectangular coordinates in face

x, y, z : rectangular coordinates in core

u, v, w : face displacements in the x -, y -, z^* -directions

u', v', w' : core displacements in the x -, y -, z -directions

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- t_f : face thickness
 $2t_c$: core thickness
 $2h$: distance between centerlines of opposite faces
 E_f : modulus of elasticity in face
 G_c : modulus of rigidity in isotropic core
 G_{zx} : modulus of rigidity in zx -plane of orthotropic core
 G_{yz} : modulus of rigidity in yz -plane of orthotropic core
 ν_f : Poisson ratio in face
 L : span of the sandwich plate
 $2B$: width of the sandwich plate
 R : radius of hole
 α : stress concentration factor
 e : distance between center of plate and center of eccentric hole
 $2e_v$: distance between centers of two holes which lie on the vertical axis of the sandwich plate
 $2e_h$: distance between centers of two holes which lie on the horizontal axis of the sandwich plate
 $2b_v$: length of slot
 d_v : depth of notch

Suffixes

- s : simple bending theory
 a : advanced bending theory
 f : face of sandwich plate
 c : core of sandwich plate
 F : finite element method
 O : theoretical analysis
 AN : anticlastic bending
 CY : cylindrical bending

3. Bending Analysis of the Structural Sandwich Plate by FEM

In analyzing the stress concentration of a sandwich plate, we apply two bending theories of the sandwich plate, one of which is the simple bending theory, where the face bending rigidity is not considered, and the other is the advanced bending theory,

Fig. 1 Displacements in simple bending theory

where it is considered.

3.1 Analysis by the Simple Bending Theory

3.1.1 Assumption of Displacements

Using the simple bending theory of a sandwich plate, we assume the following displacements in the x -, y -, z -directions of the face and core, as shown in Fig. 1.

$$u = u_0, v = v_0, w \quad (1)$$

$$u' = \theta_x z, v' = \theta_y z, w' = w \quad (2)$$

Here, u_0 and v_0 are represented by the rotational displacements θ_x and θ_y , as follows.

$$u_0 = \theta_x h, v_0 = \theta_y h \quad (3)$$

3.1.2 Finite Element Method

a) Displacements We express the displacements of an arbitrary point (x, y) in a triangular element, as shown in Fig. 2, by Eq. (4).

$$\begin{bmatrix} w \\ \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \quad (4)$$

Let the area of a triangular element be A_{ijk} and the areas of three triangles, which are made of three corner points i, j and k , and an arbitrary point (x, y) as in Fig. 2, be A_i, A_j and A_k . Then, we can rewrite Eq. (4) as

$$d_s = Nu_s \quad (5)$$

where

$$d_s = [w \ \theta_x \ \theta_y]^T, N = [N_i \ N_j \ N_k],$$

$$N_i = N_i \times I_{(3 \times 3)}, N_i = A_i / A_{ijk},$$

$$u_s = [u_{si}^T \ u_{sj}^T \ u_{sk}^T]^T, u_{si} = [w_i \ \theta_{xi} \ \theta_{yi}]^T \quad (6)$$

b) Stiffness Matrix The strain and stress of the face and core can be obtained from the displacements expressed as Eq. (5). By using the principle of the virtual work, we obtain the following relation.

$$P_s = K_s u_s \quad (7)$$

where

$$P_s = [P_{si}^T \ P_{sj}^T \ P_{sk}^T]^T, P_{si} = [P_i \ M_{xi} \ M_{yi}]^T,$$

$$K_s = K_{sf} + K_{sc} \quad (8)$$

The stiffness matrices K_{sf} and K_{sc} are obtained by the following formulae.

$$K_{sf} = 2 \int_{V_f} B_{sf}^T D_{f2} B_{sf} dV_f, \quad K_{sc} = \int_{V_c} B_{sc}^T D_{c2} B_{sc} dV_c \quad (9)$$

Fig. 2 Triangular element

where

$$\begin{aligned} B_{sf} &= D_{sf1} N, \quad B_{sc} = D_{sc1} N, \\ D_{sf1} &= h \begin{bmatrix} 0 & \partial/\partial x & 0 \\ 0 & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}, \quad D_{f2} = \bar{E}_f \begin{bmatrix} 1 & \nu_f & 0 \\ \nu_f & 1 & 0 \\ 0 & 0 & \bar{\nu} \end{bmatrix}, \\ D_{sc1} &= \begin{bmatrix} \partial/\partial x & 1 & 0 \\ \partial/\partial y & 0 & 1 \end{bmatrix}, \quad D_{c2} = \begin{bmatrix} G_{zx} & 0 \\ 0 & G_{yz} \end{bmatrix}, \\ \bar{E}_f &= E_f / (1 - \nu_f^2), \quad \bar{\nu} = (1 - \nu_f) / 2 \end{aligned} \quad (10)$$

3.2 Analysis by Advanced Bending Theory

3.2.1 Assumption of Displacements Using the advanced bending theory of a sandwich plate, we assume the following displacements in the x -, y - and z -directions of the face and core, as shown in Fig. 3.

$$u = u_0 - z^* \frac{\partial w}{\partial x}, \quad v = v_0 - z^* \frac{\partial w}{\partial y}, \quad w \quad (11)$$

$$u' = \frac{z}{t_c} u_c, \quad v' = \frac{z}{t_c} v_c, \quad w' = w \quad (12)$$

Then, from Eq.(11) and the boundary conditions of z^* which is from $-t_f/2$ to $t_f/2$, u_f and u_c in Fig. 3 can be represented by Eq.(13).

$$u_f = u_0 - \frac{t_f}{2} \frac{\partial w}{\partial x}, \quad u_c = u_0 + \frac{t_f}{2} \frac{\partial w}{\partial x} \quad (13)$$

v_f and v_c can also be expressed similarly to Eq.(13).

3.2.2 Finite Element Method

a) Displacements We include the partial derivatives of the nodal displacement w_x and w_y , into the displacements of an arbitrary point (x, y) in Fig. 2. Then the displacements of (x, y) are given as equation (14).

$$\begin{bmatrix} w \\ w_x \\ w_y \\ \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \quad (14)$$

where

$$w_x = \partial w / \partial x, \quad w_y = \partial w / \partial y \quad (15)$$

Equation (14) can be rewritten into Eq.(16) as a vector formula,

$$d_a = Xa \quad (16)$$

where d_a and a denote the vectors of the left- and right-hand side, respectively, and X the matrix of the right-hand side in Eq.(14).

The nodal displacements u_{ai} , u_{aj} and u_{ak} at nodal points i , j and k , respectively, can be expressed as Eq. (17) as a vector formula.

$$u_a = ca \quad (17)$$

where

$$\begin{aligned} u_a &= [u_{ai}^T \quad u_{aj}^T \quad u_{ak}^T]^T, \\ u_{ai} &= [w_i \quad w_{xi} \quad w_{yi} \quad \theta_{xi} \quad \theta_{yi}]^T, \\ c &= [X_i^T \quad X_j^T \quad X_k^T]^T \end{aligned} \quad (18)$$

By calculating a from Eq.(17) and substituting it into Eq.(16), we get the next equation.

$$d_a = Xc^{-1} u_a \quad (19)$$

Equation (19) is to be used for calculating the strain and stress of a sandwich plate by FEM.

b) Stiffness Matrix By calculating the strain and stress of the face and core from displacements in Eq.(19), and using the principle of the virtual work, we can obtain the following relations.

$$P_a = K_a u_a \quad (20)$$

where

$$\begin{aligned} P_a &= [P_{ai}^T \quad P_{aj}^T \quad P_{ak}^T]^T, \\ P_{ai} &= [P_i \quad M_{fxi} \quad M_{fyi} \quad M_{xi} \quad M_{yi}]^T, \\ K_a &= K_{af} + K_{ac} \end{aligned} \quad (21)$$

The stiffness matrices K_{af} and K_{ac} are obtained by the following formulae.

$$\begin{aligned} K_{af} &= 2 \int_{V_f} B_{af}^T D_{f2} B_{af} dV_f, \\ K_{ac} &= \int_{V_c} B_{ac}^T D_{c2} B_{ac} dV_c \end{aligned} \quad (22)$$

where

$$B_{af} = D_{af1} Xc^{-1}, \quad B_{ac} = D_{ac1} Xc^{-1},$$

Fig. 3 Displacements in advanced bending theory

$$D_{af1} = \begin{bmatrix} 0 & -z^* \partial/\partial x & 0 & h \partial/\partial x & 0 \\ 0 & 0 & -z^* \partial/\partial y & 0 & h \partial/\partial y \\ 0 & -z^* \partial/\partial y & -z^* \partial/\partial x & h \partial/\partial y & h \partial/\partial x \end{bmatrix} \quad (23)$$

$$D_{ac1} = \frac{h}{t_c} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

4. Comparison of Two Bending Theories for Stress Concentration

4.1 In the Case of a Sandwich Plate without a Hole or Notch

4.1.1 Comparison of the z -direction Displacements by Simple Bending Theory and Advanced Bending Theory On the simply supported square sandwich plate under a uniform transverse load q_0 , the maximum displacements in the z -direction by the simple and advanced bending theories are obtained as Eq. (24) and Eq. (25), respectively.

$$w_{os(\max)} = \frac{16L^4 q_0}{\pi^6 D'} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{(i+j)/2-1} \times \frac{\{1 + D' \pi^2 (i^2 + j^2) / FL^2\}}{ij(i^2 + j^2)^2} \quad (24)$$

$$w_{oa(\max)} = \frac{16L^4 q_0}{\pi^6} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (-1)^{(i+j)/2-1}$$

$$\times \frac{\{1 + A(\pi^2/L^2)(i^2 + j^2)\}}{ij(i^2 + j^2)^2 \{2D_f + D' + 2D_f A(\pi^2/L^2)(i^2 + j^2)\}} \quad (25)$$

where

$$D' = 2h^2 t_f \bar{E}_f, D_f = t_f^3 \bar{E}_f / 12,$$

$$A = t_f t_c \bar{E}_f / G_c, F = 2t_c G_c,$$

$i, j = \text{odd number}$

One of the calculated results on the ratios of the two displacements, or Eq. (24) and Eq. (25), for various lengths and thicknesses of the core and face is shown in Fig. 4. We understand that as L/t_f becomes high and E_f/G_c becomes low, the ratio of $w_{oa(\max)}/w_{os(\max)}$ approaches 1; at $L/t_f = 1250$, it is over 99% in Fig. 4, where the vertical axis shows the ratio of the two maximum displacements in the z -direction, i.e., $w_{oa(\max)}/w_{os(\max)}$, and the horizontal axis shows L/t_f .

4.1.2 Accuracies of FEM calculation We calculated the maximum displacement in the z -direction by FEM at the value of $L/t_f = 1250$ in Fig. 4. We show the ratio of the maximum displacements in the z -direction by FEM and the theoretical analysis using the simple bending theory in Fig. 5, and that of advanced bending theory in Fig. 6. Here, N is the modulus

Fig. 4 Ratio of the maximum displacements in the z -direction (advanced and simple bending theories)

Fig. 5 Ratio of the maximum displacements in the z -direction (FEM and theoretical analyses in simple bending theory)

Fig. 6 Ratio of the maximum displacements in the z -direction (FEM and theoretical analyses in advanced bending theory)

which gives the number of elements in FEM calculation in terms of $2N^2$. Hence, the value of N by which FEM analysis has at least 95% accuracy must be higher than 7 for simple bending theory, and 3 for advanced bending theory.

4.1.3 Core Rigidity We also present a ratio of the maximum displacements in the z -direction with various values of core rigidity and Young's moduli at the face in Fig. 7, where the vertical axis is shown by the ratio of w (maximum displacement in the z -direction by FEM and theoretical analysis) and $w_{G_c \rightarrow \infty}$ (displacement in the z -direction when core rigidity becomes infinite in theoretical analysis), and the horizontal axis is shown by $\log(D_0^*)$, that is, the ratio of core face bending rigidities. If D_0^* becomes higher than 200, the accuracy of FEM analysis is reduced. However, the displacement of the theoretical analysis approaches the solution of a homogeneous plate in which the core rigidity is not considered. Therefore, we will use the solution of FEM analysis for values of D_0^* below 200, and may use that of theoretical analysis in a homogeneous plate for the values of D above 200.

4.2 In the Case of a Sandwich Plate with a Hole

On the Simply supported square sandwich plate with one hole at the center under a bending moment, we calculated the stress concentrations, by FEM, that were executed with 92 elements by the simple bending theory and 30 elements by the advanced bending theory. The calculation conditions of the sandwich plate

and load are as follows.

$$2R/L=0.1, L/t_f=1250, t_c/t_f=6.25,$$

$$t_f=0.8 \text{ mm}, E_f/G_c=720, M_x=800 \text{ kgf}\cdot\text{mm} \quad (27)$$

The values determined by FEM analysis are presented in Table 1. From this table, we can see that the difference between the two bending theories is only 0.6%; hence, we can say that the simple bending theory is sufficient for investigating the stress concentrations of a sandwich.

5. Comparison of the Stress Concentrations Due to Two Types of Bending Loads

By using the two types of bending loads which are defined as the anticlastic bending load and the cylindrical bending load of a homogeneous plate, we investigated the stress concentrations of a sandwich plate with a hole. In Fig. 8, the stress concentration factors of anticlastic bending and cylindrical bending at point a are shown. In Fig. 9, the difference between the two maximum stress concentrations of Fig. 8. is shown. In Fig. 10, we show how the stress concentration $\sigma_{\theta\theta}$ changes at the point b , that lies on the edge of a hole along the horizontal centerline, for the case of the anticlastic bending.

From Figs. 8 and 9, we understand that the stress concentrations of anticlastic bending $\sigma_{AN(\max)}$ is higher than that of cylindrical bending $\sigma_{CY(\max)}$ for all values of R . Extending this into very small R , or an infinite sandwich plate, we can judge that $\sigma_{AN(\max)}$ is also higher than $\sigma_{CY(\max)}$. From Fig. 10, it can be seen that $\sigma_{\theta\theta}$ approaches a negative fixed value in an infinite sandwich plate. From this relations, we know that the stress concentration of anticlastic bending is higher than that of cylindrical bending in a general sandwich plate, which is the opposite result to that of a homogeneous plate. We also know that the stress concentrations of a sandwich plate is more similar to that in tension than that in the bending of a homogeneous

Table 1 Stress values of two bending theories by FEM

[STRESS UNIT; kgf/mm^2]

BENDING THEORY	NUMBER OF ELEMENTS	MAXIMUM STRESS; $\sigma_{x(\max)}$	AVERAGE STRESS; σ_0	$\frac{\sigma_{x(\max)}}{\sigma_0}$
SIMPLE THEORY	92	0.173	0.103	1.68
ADVANCED THEORY	30	0.172	0.103	1.67

Fig. 7 Ratio of displacements vs. the rigidity ratio D_0^*

Fig. 8 Stress concentration factor under anticlastic and cylindrical bending loads

plate.

6. Data of the Stress Concentration Factor

For investigating the stress concentration factor, we used the simple bending theory under anticlastic bending load. The dimensions of the sandwich plate were as follows.

Fig. 9 Maximum stress difference between anticlastic and cylindrical bending

Fig. 10 Stress σ_{yb}

Fig. 11 Stress concentration factors of sandwich plates with one hole at the center of the plate (1st)

$$L/2B=2.0, L/t_f=750, t_f=0.8 \text{ mm} \quad (28)$$

A HITAC-M-200 H computer was used for these calculations.

6.1 In the Case of One Hole Which Lies on the Center of Plate

We present an example, shown in Fig. 11, where the stress concentration factors of point a (see Fig. 8) are shown against G_{zx}/G_{yz} , G_{yz}/G_{zx} and G_c/E_f . From this figure, we know that the stress concentration factor is higher at lower R/B and G_c/E_f for isotropic core, or $G_{zx}=G_{yz}=G_c$, and it becomes a maximum at G_{zx}/G_{yz} equal to G_{yz}/G_{zx} , or $G_{zx}/G_{yz}=G_{yz}/G_{zx}=1$, for an orthotropic core. In Figs. 12 and 13, it is shown how the stress concentration factor changes with various core thicknesses. These show that the derivative values of α with respect to t_c/t_f are higher at lower values of E_f/G_c and t_c/t_f . That is very important factor for the optimum design of a sandwich plate when considering the stress concentration factor.

6.2 In the Case of One Eccentric Hole Which Lies on the Vertical Axis of Plate

In Figs. 14 and 15 the stress concentration factors

Fig. 12 Stress concentration factors of sandwich plates with one hole at the center of the plate (2nd)

Fig. 13 Stress concentration factors of sandwich plates with one hole at the center of the plate (3rd)

of sandwich plates with an eccentric hole are presented. When R/B and e/B are low, we cannot find an effect of strip width on the stress concentration factors at point a, but when they are high, it is obviously appears.

6.3 In the Case of Two Holes Which Lie on the Vertical or Horizontal Axis of Plate

6.3.1 We present in Fig. 16 the stress concentration factors of sandwich plates with two holes that lie on the vertical axis of plate. When e_v/B is low, the effect of interference between the two holes on the

Fig. 14 Stress concentration factors of sandwich plates with one eccentric hole on the vertical axis (1 st)

Fig. 15 Stress concentration factors of sandwich plates with one eccentric hole on the vertical axis (2 nd)

Fig. 16 Stress concentration factors of sandwich plates with two holes on the vertical axis

stress concentration factor is obvious. However, with e_v/B becoming high, this effect disappears. Thus, at the value of $e_v/B=0.5$, the stress concentration factor at point a is almost equal to that of point b.

6.3.2 Figure 17 shows the stress concentration factors of sandwich plates with two holes which lie on the horizontal axis of the plate. For low e_h/B , the stress concentration factors are low by the interference between the two holes. But as e_h/B becomes high, the stress concentration factor also becomes high, because the interference effect disappears.

6.4 In the Case of a Slot or Notch Which Lies on the Vertical Axis of Plate

We show the stress concentration factors of sandwich plate with a slot and notch in Figs. 18 and 19, respectively. The stress concentration factor of a slot becomes high as b_v/R becomes high, and that of notch becomes low as R/d_v becomes high.

7. Conclusions

From the resulting obtained in this study, it was

Fig. 17 Stress concentration factors of sandwich plates with two holes on the horizontal axis

Fig. 18 Stress concentration factors of sandwich plates with a slot

Fig. 19 Stress concentration factors of sandwich plates with a notch

found that the simple bending theory was sufficient for investigating the stress concentrations of a sandwich plate, and that the characteristics of stress concentration in the bending of a sandwich plate was more similar to that in tension than in the bending of a homogeneous plate. As the rigidity of an isotropic core became higher, the stress concentration factor became lower. Also, for a sandwich plate with an orthotropic core, it was more effective to design to make the rigidity of the core in the direction of the

anticlastic bending moment large than to make that in the orthogonal direction large, because the stress concentration factor of the former was slightly lower than that of the latter.

Under the anticlastic and cylindrical bending loads, we found that the stress concentration factor of anticlastic bending was higher than that of cylindrical bending, which was the opposite result to that of a homogeneous plate.

Lastly, we showed some example data of the stress concentration factors of a sandwich plate with holes or notches, that would become very important data for the optimum design of a sandwich plate.

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