Reliability-Based Consideration of ASME's Scatter Factor Used in Low-Cycle Fatigue Design Based on Geometrically Similar Model Tests*

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In the low-cycle fatigue design based on testing of geometrically similar models, the scatter of fatigue life is taken into account using the factor for the statistical variation in test results, $K_{ss}=1.470-0.044r$, the formula given by the ASME Boiler and Pressure Vessel Code, where r is the number of replicate tests. In the present paper, this formula has been examined by calculating the upper confidence limit of the probability of failure P_{fu} for the design service cycles as determined using the above formula. It was found that P_{fu} varied with η and r , where η was the coefficient of variation of the fatigue life. An alternative formula was proposed which made P_{fu} independent of η and r .

Key Words: Fatigue, Low-Cycle Fatigue Design, Geometrically Similar Model Test, ASME Code, Reliability Engineering

1. Introduction

Consider the low-cycle fatigue design method based on testing of full-size or geometrically similar test models⁽¹⁾. According to this design method, the adequacy of a part to withstand cyclic loading is justified if the fatigue lives of the test models exceed a specified life. This specified life is a function of the test stress. When the test stress is equal to the design service stress, the specified life is given by $K_n N_D$, where N_D is the design service cycles and the factor K_n is given by ASME Boiler and Pressure Vessel Code⁽²⁾ as

$$K_n = K_s^{4.3} = (K_{st} \cdot K_{sf} \cdot K_{sc} \cdot K_{st} \cdot K_{ss})^{4.3} \quad (1)$$

where K_{st} is the factor for the difference in size between the model and the prototype, K_{sf} is the factor for the difference in surface finish between the model and the prototype, K_{sc} is the factor for the difference in design fatigue curves at various temperatures, K_{st}

is the factor for the difference between the test temperature and the service temperature. K_{ss} is the factor for the statistical variation in test results given by

$$K_{ss} = 1.470 - 0.044r \quad (2)$$

where r is the number of replicate tests, i.e., the number of models tested. Thus, the scatter of the fatigue life is taken into account in this design method through Eq.(2). The purpose of this paper is to examine Eq.(2) and propose an alternative equation.

2. Method of Examination

Two problems are involved in Eq.(2). First, is it reasonable to give K_{ss} independent of the magnitude of the scatter of the fatigue life? Second, is the effect of r taken account of adequately in Eq.(2)? To solve these problems we consider the case of $K_{st} = K_{sf} = K_{sc} = K_{st} = 1$. In this case Eq.(1) reduces to

$$K_n = (K_{ss})^{4.3} = (1.470 - 0.044r)^{4.3} \quad (3)$$

Suppose fatigue tests are conducted on r models at the design service stress, and the fatigue lives $N_1, N_2, \dots, N_r (N_1 \leq N_2 \leq \dots \leq N_r)$ are obtained. The part under design is accepted if $N_1 > K_n N_D$ where N_D is the design

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service cycles.* This is equivalent to determining N_D based on the minimum life N_1 as

$$N_D = \frac{N_1}{K_n} = \frac{N_1}{(1.470 - 0.044r)^{4.3}} \quad (4)$$

Suppose the probability of failure P_f for $N = N_D$ is calculated. If P_f is shown to be almost independent of r and the coefficient of variation of the fatigue life, η , Eq.(2) may be considered adequate. Assuming a three-parameter Weibull distribution, Uemura et al.⁽³⁾ calculated P_f . It was seen from their result that P_f varied with r . Then, we⁽⁴⁾ calculated P_f assuming a two-parameter Weibull and a log-normal distribution. The result showed that P_f varied not only with r but also with η .

In Refs.(3) and (4), P_f was evaluated as

$$P_f = \int_0^\infty F(N_D)g(N_D)dN_D \quad (5)$$

where $F(\cdot)$ denotes the distribution function of the fatigue life of the part under design, and $g(\cdot)$ denotes the probability density function of the design service life as determined by Eq.(4). Equation (5) gives the mean value of the probability of failure when design is conducted by many teams independently. In practice, however, each design team will be much more concerned with the reliability attained by their own design. For this purpose, use of the upper confidence limit of the probability of failure, P_{fu} , is appropriate. Thus, in this paper, we examine Eq.(2) using P_{fu} instead of using Eq.(5).

3. Examination of ASME's Formula for K_{ss}

3.1 Case of two-parameter Weibull distribution

Suppose that the fatigue life of the part under design follows a two-parameter Weibull distribution function as

$$F(N) = 1 - \exp\left[-\left(\frac{N}{\beta}\right)^\alpha\right] \quad (6)$$

It is also assumed that the shape parameter α is known from existing data but that the scale parameter β is unknown. This type of assumption concerning the distribution parameters is typical in practice as may be seen in Ref.(5). When β is unknown, the true value of the probability of failure P_f for a particular design life N_D given by Eq.(4) cannot be calculated, but the upper confidence limit of P_f can be calculated as follows.

As stated previously, we are considering the case of $K_{st} = K_{sf} = K_{sc} = K_{st} = 1$. Hence, Eq.(6) gives the fatigue life distribution of not only the actual component but also the test models. So, the distribution

function of the minimum life, $F_1(N_1)$, is related to $F(N)$ as

$$1 - F_1(N_1) = [1 - F(N_1)]^r \quad (7)$$

From Eqs.(6) and (7), it follows that

$$F_1(N_1) = 1 - \exp\left[-r\left(\frac{N_1}{\beta}\right)^\alpha\right] \quad (8)$$

The upper 100 γ percentile value of $F_1(N_1)$ is found from Eq.(8) to be $\beta[-(\ln \gamma)/r]^{1/\alpha}$. That is,

$$P\left[N_1 \leq \beta\left(-\frac{\ln \gamma}{r}\right)^{1/\alpha}\right] = 1 - \gamma \quad (9)$$

where $P[\]$ denotes the probability of the event in the bracket. Multiplying both sides of the inequality in the bracket of Eq.(9) by N_D and arranging, it follows

$$P\left[\frac{N_D}{\beta} \leq \frac{N_D}{N_1}\left(-\frac{\ln \gamma}{r}\right)^{1/\alpha}\right] = 1 - \gamma$$

which may be rewritten as

$$P\left[1 - \exp\left\{-\left(\frac{N_D}{\beta}\right)^\alpha\right\} \leq 1 - \exp\left\{\left(\frac{N_D}{N_1}\right)^\alpha \frac{\ln \gamma}{r}\right\}\right] = 1 - \gamma \quad (10)$$

The left-hand side of the inequality in the bracket of Eq.(10) is just the true value of the probability of failure for a particular design service life N_D . The right-hand side of the inequality is a statistic, which can be calculated from N_1 . According to Eq.(10), the probability that this statistic takes a value larger than the true value of P_f is $1 - \gamma$. Hence, this statistic stands for the upper confidence limit of the probability of failure, P_{fu} , at the confidence level $1 - \gamma$. That is,

$$P_{fu} = 1 - \exp\left[\frac{\ln \gamma}{r} \left(\frac{N_D}{N_1}\right)^\alpha\right] \quad (11)$$

In terms of $K_n (= N_1/N_D)$, Eq.(11) is written as

$$P_{fu} = 1 - \exp\left[\frac{\ln \gamma}{r} \cdot \frac{1}{K_n^\alpha}\right] \quad (12)$$

Equation(11) denotes the upper confidence limit of P_f based on the minimum life N_1 , and should be distinguished from the upper confidence limit of P_f based on all the data $N_1, N_2, \dots, N_r$. The latter is given by⁽⁶⁾

$$P_{fu} = 1 - \exp\left[-\frac{\chi^2_{2r}(2r)}{2r} \cdot \frac{N_D^\alpha}{(\sum N_i^\alpha)/r}\right] \quad (13)$$

where $\chi^2_{2r}(2r)$ is the upper 100 γ percentile value of the chi-square distribution with the degrees of freedom $2r$.

Let us examine whether P_{fu} is almost independent of the coefficient of variation of the fatigue life η , and the number of test models r when Eq.(2) is used. Substituting Eq.(3) into Eq.(12), P_{fu} is expressed as a function of r , α and $1 - \gamma$. The shape parameter α was determined from η as follows. η is related to α as

$$\eta = \left[\frac{\Gamma(1+2/\alpha)}{\{\Gamma(1+1/\alpha)\}^2} - 1\right]^{1/2} \quad (14)$$

Equation(14) can be approximated with good accuracy in the range of $1 \leq \alpha \leq 50$ by⁽⁷⁾

$$\eta = \alpha^{-0.93} \quad (15)$$

or

$$\alpha = \eta^{-1/0.93} \quad (16)$$

* K_n is a kind of so-called scatter factor. It should be noted, however, that K_n is defined on the basis of the minimum life instead of the mean life.

The relative error of Eq.(15) is at most 3% in the range of $1 \leq \alpha \leq 50$. For a given value of η , α was calculated by Eq.(16). The number of test models r was varied in the range of $1 \leq r \leq 5$ so that the ASME Code requirement of $K_{st} \cdot K_{sf} \cdot K_{sc} \cdot K_{ss} \geq 1.25$ was satisfied. Since $K_{st} = K_{sf} = K_{sc} = K_{ss} = 1$ is assumed in the present study, this requirement reduces to $K_{ss} = 1.470 - 0.044r \geq 1.25$ or $1 \leq r \leq 5$.

P_{fu} was calculated for the confidence level of $1 - \gamma = 0.95$ and $1 - \gamma = 0.90$, and is shown in Fig. 1. It is seen from Fig. 1 that P_{fu} varies considerably with η , and also varies with r when η is small. This result indicates that Eq.(2) leaves room for improvement.

3.2 Case of log-normal distribution

Suppose that the fatigue life of the part under design follows a log-normal distribution function as

$$F(N) = \Phi\left(\frac{\ln N - \mu_L}{\sigma_L}\right) \quad (17)$$

where μ_L and σ_L are the mean value and the standard deviation of $\log N$, respectively. $\Phi(\cdot)$ is the standard normal distribution function defined by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp\left(-\frac{X^2}{2}\right) dX \quad (18)$$

It is also assumed that σ_L is known from existing data but that μ_L is unknown. This assumption is essentially equivalent to one in which α is known but β is unknown as in the case of a two-parameter Weibull distribution.

From Eqs.(7) and (17), the distribution function of the minimum life N_1 is given by

$$F_1(N_1) = 1 - \left[1 - \Phi\left(\frac{\ln N_1 - \mu_L}{\sigma_L}\right)\right]^r \quad (19)$$

The upper 100 γ percentile value of $F_1(N_1)$ is found from Eq.(19) to be $\mu_L + \sigma_L \Phi^{-1}(1 - \gamma^{1/r})$. That is,

$$P[\ln N_1 \leq \mu_L + \sigma_L \Phi^{-1}(1 - \gamma^{1/r})] = 1 - \gamma \quad (20)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse function of $\Phi(\cdot)$. Adding $\ln N_D$ to both sides of the inequality in the bracket of Eq.(20) and arranging, it follows that

$$P\left[\frac{\ln N_D - \mu_L}{\sigma_L} \leq \frac{\ln N_D - \ln N_1}{\sigma_L} + \Phi^{-1}(1 - \gamma^{1/r})\right] = 1 - \gamma \quad (21)$$

Noting that $\Phi(t)$ is a monotonously increasing function of t , Eq.(21) is equivalent to

$$P\left[\Phi\left(\frac{\ln N_D - \mu_L}{\sigma_L}\right) \leq \Phi\left(\frac{\ln N_D - \ln N_1}{\sigma_L} + \Phi^{-1}(1 - \gamma^{1/r})\right)\right] = 1 - \gamma \quad (22)$$

The left-hand side of the inequality in the bracket of Eq.(22) is just the true value of the probability of failure P_f for a particular design service life N_D . Due to similar reasoning as in the case of Eq.(10), the right hand-side of the inequality stands for the upper confidence limit of the probability of failure, P_{fu} , at the confidence level $1 - \gamma$. That is,

$$P_{fu} = \Phi\left[\frac{\ln N_D - \ln N_1}{\sigma_L} + \Phi^{-1}(1 - \gamma^{1/r})\right] \quad (23)$$

In terms of $K_n (= N_1/N_D)$, Eq.(23) is written as

$$P_{fu} = \Phi\left[-\frac{\ln K_n}{\sigma_L} + \Phi^{-1}(1 - \gamma^{1/r})\right] \quad (24)$$

Equation(23) gives the upper confidence limit of P_f based on the minimum life N_1 , and should be distinguished from the upper confidence limit of P_f based on all the data $N_1, N_2, \dots, N_r$. The latter is given by⁽⁶⁾

$$P_{fu} = \Phi\left[\frac{\ln N_D - (1/r) \sum \ln N_i}{\sigma_L} + \frac{\Phi^{-1}(1 - \gamma)}{\sqrt{r}}\right] \quad (25)$$

Let us examine whether P_{fu} is almost independent of η and r when Eq.(2) is used. Substituting Eq.(3) into Eq.(24), P_{fu} is expressed as a function of r , σ_L and $1 - \gamma$. σ_L was calculated for a given value of η as

$$\sigma_L = [\ln(1 + \eta^2)]^{1/2} \quad (26)$$

The result of the numerical calculation was shown in Fig. 2 for $1 - \gamma = 0.95$ and $1 - \gamma = 0.90$. As in Fig. 1, P_{fu} varies considerably with η and also varies with r when η is small. Thus, it is again indicated that Eq.(2) leaves room for improvement.

4. Proposal of Alternative Formula

It is reasonable to determine K_{ss} so that P_{fu} becomes independent of r and η . Such K_{ss} can be obtained as follows. In the case of the two-parameter Weibull distribution, Eq.(12) yields

$$K_n = K_{ss}^{4.3} = \left[\frac{1}{r} \frac{\ln \gamma}{\ln(1 - P_{fu})}\right]^{1/\alpha} \quad (27)$$

Equation (27) can be used to determine K_{ss} by sub-

Fig. 1 Upper confidence limit of the probability of failure, P_{fu} , as evaluated using ASME's K_{ss} (case of two-parameter Weibull distribution).

Fig. 2 Upper confidence limit of the probability of failure, P_{fU} , as evaluated using ASME's K_{ss} (case of log-normal distribution).

stituting the allowable probability of failure P_{fa} for P_{fU} . Then, the resulting value of P_{fU} varies with neither r nor η . Equation (27) is rewritten as

$$\alpha \ln K_n = 4.3 \alpha \ln K_{ss} = -\ln r + \ln \left[\frac{\ln \gamma}{\ln(1 - P_{fU})} \right] \quad (28)$$

and is shown in Fig. 3. r was varied in the range of $1 \leq r \leq 10$ in Fig. 3. For comparison, the value of $\alpha \ln K_n$ based on ASME's K_{ss} was shown for $\eta=0.2$ (i. e. $\alpha=5.64$).

In the case of the log-normal distribution, Eq. (24) gives

$$K_n = K_{ss}^{4.3} = \exp[\sigma_L \{ \Phi^{-1}(1 - P_{fU}) + \Phi^{-1}(1 - \gamma^{1/r}) \}] \quad (29)$$

Equation (29) can be used to determine K_{ss} by substituting P_{fa} for P_{fU} . Then, the resulting value of P_{fU} varies with neither r nor η . Eq. (29) is rewritten as

$$\frac{\ln K_n}{\sigma_L} = \frac{4.3 \ln K_{ss}}{\sigma_L} = \Phi^{-1}(1 - P_{fU}) + \Phi^{-1}(1 - \gamma^{1/r}) \quad (30)$$

and is shown in Fig. 4. For comparison, the value of $(\ln K_n)/\sigma_L$ based on ASME's K_{ss} was shown for $\eta=0.2$ (i. e. $\sigma_L=0.198$).

It is noted that the ordinate of Fig. 3, $\alpha \ln K_n$ and that of Fig. 4, $(\ln K_n)/\sigma_L$ are approximately equal to each other, since $\alpha \approx 1/\eta$ and $\sigma_L \approx \eta$ due to Eqs. (16) and (26).

Consider how to use Eq. (27) and Eq. (29) in practice. If the allowable probability of failure P_{fa} is

Fig. 3 Alternative formula proposed in this paper (case of two-parameter Weibull distribution).

Fig. 4 Alternative formula proposed in this paper (case of log-normal distribution).

specified, the K_{ss} -value can be determined by substituting P_{fa} for P_{fU} in these equations. However, this type of application is associated with difficulties in specifying P_{fa} and choosing the distribution form of the fatigue life. A more practical application is to use these equations to modify the K_{ss} -value for changes in r and η . In this type of application, the value of P_{fa} specified and the distribution form chosen do not affect the modified K_{ss} -value seriously.

5. Conclusions

ASME's factor $K_{ss} = 1.470 - 0.044r$ used in low-cycle fatigue design was examined by calculating the

upper confidence limit of the probability of failure, P_{fu} . The following conclusions were obtained.

(1) P_{fu} as calculated using $K_{ss}=1.470-0.044r$ varied considerably with the coefficient of variation of the fatigue life, η , and also with the number of the test models r , both in the cases of the two-parameter Weibull distribution and the log-normal distribution (Fig. 1, Fig. 2). Hence, the safety margin varies with η and r when ASME's formula for K_{ss} is used.

(2) Equation (27) and Eq.(29) were proposed as alternative formulae for K_{ss} for the case of the two-parameter Weibull distribution and for the case of the log-normal distribution, respectively. When Eqs.(27) and (29) are used, P_{fu} becomes independent of η and r .

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