Interaction Between Cracks Perpendicular to and on the Interface in Composite Materials*

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A plane elasto-static problem is solved for two bonded elastic and isotropic strips with a T-shaped crack. The T-shaped crack consists of a crack perpendicular to the interface and an interface crack. By means of the Airy stress function and Fourier transforms the problem is formulated as a system of integral equations. A new numerical method using only the Gauss-Chebyshev integration formula is proposed to evaluate the integral equations, although the complicated Jacobi polynomial expansion and integration formula were used in the previous literature. Stress intensity factors are obtained numerically for various materials and geometrical parameters, where the interaction between the cracks perpendicular to and on the interface is clearly shown.

Key Words: Composite Materials, Stress Intensity Factor, T-Shaped Crack, Interface, Delamination, Debonding, Numerical Method

1. Introduction

Composite materials consist of two or more constituents with different material constants and have interfaces between them. The local crack in composite materials therefore does not propagate similarly and the fracture behavior of composite materials is quite complicated^{(1)~(4)} compared with those of conventional materials. Thus, in considering the fracture behavior of composite materials, it is necessary to investigate the possible local crack growth patterns, the interaction between them and their relation to the strength of composite materials from a microscopic viewpoint. A set of basic local crack geometries was presented in the previous literature^{(5),(6)}, including a crack on or near the interface of a bimaterial. Most of the existing solutions to these

crack problems refer to a crack perpendicular to or along the interface^{(7)~(10)}. The kind of crack perpendicular to and along the interface of a bimaterial such as a T-shaped crack does not seem to have been well investigated except for Refs. 6, 11 and 12 within the knowledge of the authors. Goree and Venezia⁽¹¹⁾ solved this crack problem for two bonded half planes by lending it to a system of singular integral equations. Lu and Erdogan^{(6),(12)} solved the case of two bonded infinite strips or layers. In these studies, the Jacobi polynomial expansion, which is very effective for two bonded half planes with a single crack along the interface, and Gauss-Jacobi integration formula were used in the numerical process of the integral equations. However, this method was not effective due to the interaction between cracks perpendicular to and along the interface. The numerical evaluation process of these integral equations was cumbersome and it was pointed out⁽¹²⁾ that numerical convergence was not easily attainable. Further study for the numerical treatment including the accuracy of results is necessary together with the formulation of an auxiliary boundary condition related to the single-valuedness of displacement.

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In the present paper, a T-shaped crack between two bonded elastic and isotropic infinite strips is highlighted as a fundamental microscopic failure pattern in composite materials, and a simple numerical method is developed for the evaluation of the integral equations. The T-shaped crack consists of a crack perpendicular to the interface and an interface crack. Firstly, by means of the Airy stress function and Fourier transforms, the problem is reduced to a system of first and second kinds of singular integral equations^{(6),(11)} with Cauchy and generalized Cauchy kernels. Secondly, a new numerical method using only the Gauss-Chebyshev integration formula is proposed to treat the integral equations without any loss of the complex singularities at the interface crack tip. This method easily reduces the integral equations to a set of linear algebraic equations. Finally, numerical results are presented for the stress intensity factors at both crack tips and the opening displacement along the crack surface perpendicular to the interface.

2. Theoretical Analysis

A calculation model is shown in Fig. 1, where two elastic and isotropic infinite strips of width h_i with different material constants are bonded perfectly except for a T-shaped crack consisting of an interface crack of length $2b$ and a perpendicular crack of a . The Young's modulus and Poisson's ratio are denoted by E_i and ν_i , respectively. The subscripts $i=1, 2$ indicate the strips 1 and 2, respectively. The thickness of the adhesive layer is assumed to be zero. The interface crack is symmetric with respect to the x axis, and the perpendicular crack surface is subjected to a self-equilibrium pressure $P_0(x)$. It is well known that this loading condition corresponds to tensile load-

Fig. 1 Calculation model

ing at infinity parallel to the y axis from the fracture viewpoint, that is, for the intensity of singularities at crack tips⁽¹²⁾.

Based on the two-dimensional linear elasticity⁽¹³⁾, let $\varphi_i(x, y)$ represent the Airy stress function of the strip i , satisfying the equilibrium equation without body forces. Applying the Fourier transforms to the compatibility equation, that is, the biharmonic equation in terms of $\varphi_i(x, y)$, we obtain⁽¹⁴⁾

$$\varphi_i(x, y) = \frac{2}{\pi} \int_0^\infty \{[(A_i + B_i x)e^{-xs} + (C_i + D_i x)e^{xs}] \cos(sy) + G_i(1 + sy)e^{-ys} \times \cos(xs)\} ds, \quad (i=1, 2), \quad (1)$$

where $A_i(s)$, $B_i(s)$, $C_i(s)$, $D_i(s)$ ($i=1, 2$) and $G_i(s)$ ($G_1(s)=0$) are the nine unknown functions which are determined from the boundary and continuity conditions.

Using Eq. (1) and following relations between the Airy stress function and stress $\sigma(\tau)$, stress and strain $\epsilon(\gamma)$, and strain and displacement u ,

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \quad (2)$$

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y), \quad \epsilon_y = \frac{1}{E}(\sigma_y - \nu \sigma_x),$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}, \quad (3)$$

$$\epsilon_x = \frac{\partial u_x}{\partial x}, \quad \epsilon_y = \frac{\partial u_y}{\partial y}, \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad (4)$$

we can show the stress and displacement in the strip i as

$$\sigma_{ix} = -\frac{2}{\pi} \int_0^\infty \{[(A_i + B_i x)e^{-xs} + (C_i + D_i x)e^{xs}] s^2 \cos(sy) - G_i s^2 (ys - 1) e^{-ys} \cos(xs)\} ds, \quad (5.a)$$

$$\sigma_{iy} = -\frac{2}{\pi} \int_0^\infty \{[(A_i s + B_i x s - 2B_i) e^{-xs} + (C_i s + D_i x s + 2D_i) e^{xs}] s \cos(sy) - G_i s^2 (1 + ys) e^{-ys} \cos(xs)\} ds, \quad (5.b)$$

$$\tau_{ixy} = -\frac{2}{\pi} \int_0^\infty \{[(A_i s + B_i x s - B_i) e^{-xs} - (C_i s + D_i x s + D_i) e^{xs}] s \sin(sy) + G_i y s^3 e^{-ys} \sin(xs)\} ds, \quad (5.c)$$

$$u_{ix} = \frac{2}{E_i \pi} \int_0^\infty \{[(A_i + B_i x)(1 + \nu_i) s + (1 - \nu_i) B_i] e^{-xs} - [(C_i + D_i x)(1 + \nu_i) s - (1 - \nu_i) D_i] e^{xs}\} \cos(sy) ds + \frac{2}{E_i \pi} \int_0^\infty G_i s \times [ys(1 + \nu_i) - (1 - \nu_i)] e^{-ys} \sin(xs) ds, \quad (5.d)$$

$$u_{iy} = \frac{2}{E_i \pi} \int_0^\infty \{[(A_i + B_i x)(1 + \nu_i) s - 2B_i] e^{-xs} + [(C_i + D_i x)(1 + \nu_i) s + 2D_i] e^{xs}\} \sin(sy) ds + \frac{2}{E_i \pi} \int_0^\infty G_i s [2 + ys(1 + \nu_i)] e^{-ys} \cos(xs) ds, \quad (5.e)$$

$$(i=1, 2), \quad (5.f)$$

for the generalized plane stress condition. For the plane strain condition the E_i and ν_i should be replaced by

$$\frac{E_i}{(1-\nu_i^2)}, \quad \frac{\nu_i}{(1-\nu_i^2)}, \quad (6)$$

respectively, in Eq. (5). These replacements are true also of the following equations.

On account of the symmetry it is sufficient to consider the upper half of Fig. 1. Thus, the boundary and continuity conditions can be written as follows:

$$u_{1x}(0, y) = u_{2x}(0, y), \quad (b < y < \infty), \quad (7)$$

$$u_{1y}(0, y) = u_{2y}(0, y), \quad (b < y < \infty), \quad (8)$$

$$\sigma_{1x}(0, y) = \sigma_{2x}(0, y), \quad (0 \leq y < \infty), \quad (9)$$

$$\tau_{1xy}(0, y) = \tau_{2xy}(0, y), \quad (0 \leq y < \infty), \quad (10)$$

$$u_{1y}(x, 0) = 0, \quad (-h_1 \leq x \leq 0), \quad (11)$$

$$u_{2y}(x, 0) = 0, \quad (a < x \leq h_2), \quad (12)$$

$$\tau_{1xy}(x, 0) = 0, \quad (-h_1 \leq x \leq 0), \quad (13)$$

$$\tau_{2xy}(x, 0) = 0, \quad (0 \leq x \leq h_2), \quad (14)$$

$$\sigma_{1y}(-h_1, y) = 0, \quad (0 \leq y < \infty), \quad (15)$$

$$\tau_{1xy}(-h_1, y) = 0, \quad (0 \leq y < \infty), \quad (16)$$

$$\sigma_{2y}(h_2, y) = 0, \quad (0 \leq y < \infty), \quad (17)$$

$$\tau_{2xy}(h_2, y) = 0, \quad (0 \leq y < \infty), \quad (18)$$

$$\sigma_{2y}(x, 0) = -P_0(x), \quad (0 < x < a), \quad (19)$$

$$\sigma_{2x}(0, y) = 0, \quad (0 \leq y < b), \quad (20)$$

$$\tau_{2xy}(0, y) = 0, \quad (0 \leq y < b), \quad (21)$$

where, Eqs. (11), (13) and (14) are automatically satisfied by Eq. (5) due to the symmetry.

Generally, six of the unknown functions, $A_i(s)$, ..., may be eliminated by using six conditions (9), (10) and (15)-(18), and the remaining three unknown functions are determined from the mixed boundary conditions (7), (8) and (19)-(21), which give a system of singular integral equations. In practice, we define the three new functions $f_i^{(6)}$ in terms of the displacement derivatives as

$$f_1(x) = \frac{\partial u_{2y}(x, 0)}{\partial x}, \quad (22)$$

$$f_2(y) = \frac{\partial}{\partial y} [u_{2x}(0, y) - u_{1x}(0, y)], \quad (23)$$

$$f_3(y) = \frac{\partial}{\partial y} [u_{2y}(0, y) - u_{1y}(0, y)]. \quad (24)$$

Substituting from Eq. (5) into Eqs. (7)-(21) and using the above Eqs. (22)-(24), we can arrive at a system of singular integral equations for the new three functions f_i after lengthy manipulations as follows:

$$\begin{aligned} & \frac{1}{\pi} \int_0^a \frac{f_1(t)}{t-x} dt + \frac{1}{\pi} \int_0^a f_1(t) K_{11}(t, x) dt \\ & + \frac{1}{\pi} \int_0^b [f_2(t) K_{12}(t, x) + f_3(t) K_{13}(t, x)] dt \\ & = -\frac{2}{E_2} P_0(x), \quad (0 < x < a), \quad (25) \\ & \frac{1}{\pi} \int_0^b \frac{f_2(t)}{t-y} dt + \lambda f_3(y) + \frac{1}{\pi} \int_0^a f_1(t) K_{21}(t, y) dt \\ & + \frac{1}{\pi} \int_0^b [f_2(t) K_{22}(t, y) + f_3(t) K_{23}(t, y)] dt = 0, \end{aligned}$$

$$(0 \leq y < b), \quad (26)$$

$$\begin{aligned} & \frac{1}{\pi} \int_0^b \frac{f_3(t)}{t-y} dt - \lambda f_2(y) + \frac{1}{\pi} \int_0^a f_1(t) K_{31}(t, y) dt \\ & + \frac{1}{\pi} \int_0^b [f_2(t) K_{32}(t, y) \\ & + f_3(t) K_{33}(t, y)] dt = 0, \quad (0 \leq y < b), \quad (27) \end{aligned}$$

and

$$\int_0^a f_1(t) dt - \int_0^b f_3(t) dt = 0, \quad (28)$$

where λ , E_{11} and E_{12} are given by

$$\lambda = \frac{4 + E_{11} - E_{12}}{4 + E_{11} + E_{12}},$$

$$E_{11} = \frac{E_1}{E_2} (1 + \nu_2) - (1 + \nu_1),$$

$$E_{12} = \frac{E_1}{E_2} (3 - \nu_2) + (1 + \nu_1). \quad (29)$$

Equation (28) corresponds to the single-valuedness condition of displacement. It is not derived from the separate physical condition but directly from Eqs. (7)-(21). Equations (25)-(27) are related to the boundary conditions (19)-(21) along the crack surface. Although many unknown functions make the processes of deriving the integral equations (25)-(28) lengthy and complicated, they are straightforward and analytic processes such as solving linear algebraic equations and calculating integrals of elementary functions. The kernels K_{ij} are known functions containing Cauchy and generalized Cauchy singular functions. The unknown nine functions, $A_i(s)$, ..., the stress and displacement given in Eq. (5) may easily be expressed by the new functions f_i , which are the solution of the integral equations (25)-(28). The details of K_{ij} and the way in which to obtain them can be found in Ref. 15.

To understand the singularities of the functions f_i , let f_i be described⁽¹⁶⁾ by

$$f_1(t) = \frac{g_1(t)}{t^{\alpha_1} (a-t)^{\beta_1}},$$

$$f_2(t) + if_3(t) = \frac{g_2(t)}{t^{\alpha_2} (b-t)^{\beta_2}},$$

$$[(0 \leq \operatorname{Re}(\alpha_i, \beta_i) < 1); i = 1, 2], \quad (30)$$

where, both g_1 and g_2 satisfy the Hölder condition, and are not zero at the crack tips. Both α_i and β_i describe the singularities at the crack tips. Substituting from Eq. (30) into Eqs. (25)-(27) and following Muskhelishvili⁽¹⁶⁾, we have

$$\alpha_1 = \alpha_2 = 0, \quad (31)$$

$$\beta_1 = \frac{1}{2}, \quad (32)$$

$$\beta_2 = \frac{1}{2} + i\omega, \quad \omega = \frac{1}{2\pi} \ln \left(\frac{1+\lambda}{1-\lambda} \right). \quad (33)$$

Here, Eq. (31) indicates that the intersecting point of the cracks, i. e., the origin, ($x=0, y=0$), is not a singular point. That is, the functions f_i must be bounded at

($x=0, y=0$). Hence, only two points, ($x=a, y=0$) and ($x=0, y=b$) need to be considered for the stress intensity factors.

The stress intensity factors k_i are defined by

$$k_1(a) = \lim_{x \rightarrow a^+} \sqrt{2(x-a)} \sigma_{2y}(x, 0), \quad (34)$$

$$k_1(b) + ik_2(b) = \lim_{y \rightarrow b^+} \sqrt{2(y-b)^{1/2}} [\sigma_{2x}(0, y) + i\tau_{2xy}(0, y)]. \quad (35)$$

Substituting from Eq. (5) with f_i and (30)-(33) into the above definitions we obtain

$$k_1(a) = -\frac{E_2}{\sqrt{2}} g_1(a), \quad (36)$$

$$k_1(b) + ik_2(b) = -\frac{a_{11}E_1}{\sqrt{2}} \sqrt{1-\lambda^2} g_2(b),$$

$$a_{11} = \frac{1}{E_{12}} + \frac{1}{4 + E_{11}}. \quad (37)$$

The displacement and relative displacements along the cracks become

$$u_{2y}(x, 0) = -\int_x^a f_1(x) dx, \quad (38)$$

$$u_{2x}(0, y) - u_{1x}(0, y) = -\int_y^b f_2(y) dy, \quad (39)$$

$$u_{2y}(0, y) - u_{1y}(0, y) = -\int_y^b f_3(y) dy, \quad (40)$$

from Eqs. (22)-(24). Thus, once the integral equations (25)-(28) are solved, the stress intensity factors, displacement and relative displacements can be easily calculated from Eqs. (36)-(40).

Here, it should be noted that the above equations such as integral equations (25)-(28), stress intensity factors (36) and (37), and displacements (38)-(40) are basically the same to those in Ref. 6 but that Eq. (28) is derived in a different manner.

3. Numerical Method

The integral equations (25)-(28) are complicated, and it seems to be intractable to solve them analytically. In the remainder of this paper a numerical method will be considered. It is known that β_2 , the singularity of both f_2 and f_3 , is complex as given in Eqs. (30) and (33). A combination of Jacobi polynomial expansion and Gauss-Jacobi integration formula was therefore used to evaluate the integral equations numerically in the previous papers^{(11),(12)}. However, the interaction between cracks perpendicular to and on the interface made their calculations quite cumbersome, although the Jacobi polynomial expansion is very effective⁽⁹⁾ for two bonded half planes with only one interface crack. The convergence difficulties were indicated in Ref. 12, and further investigation may be necessary to assess the accuracy of their numerical results.

Thus, here we propose a new numerical method using only the Gauss-Chebyshev integration formula. The new method consists of three steps: (a) divide

the unknown complex function $g_2(t)$ given in Eq. (30) into its real and imaginary parts; (b) separate the oscillatory part of the singularity of f_2 and f_3 from the weight function of numerical integrals; and (c) reduce the integral equations to a set of linear algebraic equations by use of the Gauss-Chebyshev integration formula. According to this new method, the numerical processes become considerably simplified and both a good convergence and accuracy of numerical results may be expected.

First, using the following linear transforms,

$$\begin{aligned} t &= \frac{a}{2}(t_1 + 1), & x &= \frac{a}{2}(s_1 + 1), \\ (0 < t, x < a; & -1 < t_1, s_1 < 1), \\ t &= \frac{b}{2}(t_2 + 1), & y &= \frac{b}{2}(s_2 + 1), \\ (0 \leq t, y < b; & -1 \leq t_2, s_2 < 1), \end{aligned} \quad (41)$$

we normalize the integral equations (25)-(28) as follows:

$$\begin{aligned} &\frac{1}{\pi} \int_{-1}^1 \frac{f^*_1(t_1)}{t_1 - s_1} dt_1 + \frac{a}{2\pi} \int_{-1}^1 f^*_{11}(t_1) K_{11}^*(t_1, s_1) dt_1 \\ &+ \frac{b}{2\pi} \int_{-1}^1 [f^*_{22}(t_2) K_{12}^*(t_2, s_1) + f^*_{33}(t_2) K_{13}^*(t_2, s_1)] dt_2 \\ &= -\frac{2}{E_2} P_0^*(s_1), \end{aligned} \quad (42)$$

$$\begin{aligned} &\frac{1}{\pi} \int_{-1}^1 \frac{f^*_{22}(t_2)}{t_2 - s_2} dt_2 + \lambda f^*_{33}(s_2) + \frac{a}{2\pi} \int_{-1}^1 f^*_{11}(t_1) \\ &\times K_{21}^*(t_1, s_2) dt_1 + \frac{b}{2\pi} \int_{-1}^1 [f^*_{22}(t_2) K_{22}^*(t_2, s_2) \\ &+ f^*_{33}(t_2) K_{23}^*(t_2, s_2)] dt_2 = 0, \end{aligned} \quad (43)$$

$$\begin{aligned} &\frac{1}{\pi} \int_{-1}^1 \frac{f^*_{33}(t_2)}{t_2 - s_2} dt_2 - \lambda f^*_{22}(s_2) + \frac{a}{2\pi} \int_{-1}^1 f^*_{11}(t_1) \\ &\times K_{31}^*(t_1, s_2) dt_1 + \frac{b}{2\pi} \int_{-1}^1 [f^*_{22}(t_2) K_{32}^*(t_2, s_2) \\ &+ f^*_{33}(t_2) K_{33}^*(t_2, s_2)] dt_2 = 0, \end{aligned} \quad (44)$$

$$a \int_{-1}^1 f^*_{11}(t_1) dt_1 - b \int_{-1}^1 f^*_{33}(t_2) dt_2 = 0, \quad (45)$$

where the superscript * denotes the quantity after transformation, although f^*_{ij} , K^*_{ij} and P^*_0 are the same as f_{ij} , K_{ij} and P_0 , except that the arguments of f^*_{ij} , K^*_{ij} and P^*_0 are t_1, s_1, t_2 and s_2 .

Next, following the previous procedure⁽¹²⁾ we set f^*_{11} as

$$f^*_{11}(t_1) = \frac{g^*_{11}(t_1)}{(1-t_1^2)^{1/2}}, \quad (46)$$

$$g^*_{11}(-1) = 0. \quad (47)$$

A new transformation is applied to f^*_{22} and f^*_{33} , which is the core part of this paper. $g^*_{22}(t_{2i})$ is defined from Eq. (30) with a transformation (41) and a little modification as follows:

$$g^*_{22}(t_2) = \left(\frac{2}{b}\right)^{1/2} g_2\left[\frac{b}{2}(t_2 + 1)\right]. \quad (48)$$

The complex function $g^*_{22}(t_{2i})$ is divided into its real part g^*_{2R} and an imaginary part g^*_{2I} ,

$$g^*_{22}(t_2) = g^*_{2R}(t_2) + ig^*_{2I}(t_2). \quad (49)$$

Substituting from Eq. (49) through Eq. (48) into Eq.

(30) we obtain (see Eqs. (31), (33) and (41))

$$f_2^*(t_2) + if_3^*(t_2) (\equiv f_2(t) + if_3(t)) \\ = \frac{1}{(1-t_2)^{1/2}} \{ [g_{2R}^*(t_2) \cos \theta - g_{2I}^*(t_2) \sin \theta] \\ + ig_{2R}^*(t_2) \sin \theta + g_{2I}^*(t_2) \cos \theta \}, \quad (50)$$

$$\theta = \omega \ln \left(\frac{2}{b(1-t_2)} \right). \quad (51)$$

Thus,

$$f_2^*(t_2) = \frac{1}{(1-t_2)^{1/2}} [g_{2R}^*(t_2) \cos \theta - g_{2I}^*(t_2) \sin \theta], \quad (52)$$

$$f_3^*(t_2) = \frac{1}{(1-t_2)^{1/2}} [g_{2R}^*(t_2) \sin \theta + g_{2I}^*(t_2) \cos \theta]. \quad (53)$$

The oscillatory singularity is contained in θ of $\sin \theta$ and $\cos \theta$. A solution without considering the oscillatory singularity corresponds to the case with $\theta=0$ in Eqs. (52) and (53). Multiplying the right sides of Eqs. (52) and (53) by $(1+t_2)^{1/2}(1-t_2)^{-1/2}$, we can use the following Gauss-Chebyshev integration formula⁽¹⁷⁾,

$$\frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{(t-s)} \frac{1}{(1-t^2)^{1/2}} dt = \frac{1}{n} \sum_{i=1}^n \frac{f(t_i)}{t_i - s_j}, \quad (54)$$

$$\frac{1}{\pi} \int_{-1}^1 \frac{f(t)}{(1-t^2)^{1/2}} K(t, s) dt = \frac{1}{n} \sum_{i=1}^n f(t_i) K(t_i, s_j), \quad (55)$$

$$t_i = \cos \left(\frac{2i-1}{2n} \pi \right), \quad (i=1, 2, \dots, n),$$

$$s_j = \cos \left(\frac{j\pi}{n} \right), \quad (j=1, 2, \dots, n-1). \quad (56)$$

Finally, substituting from Eqs. (46), (52) and (53) into the integral equations (42)-(45) and using the integration formulas (54)-(56), we obtain a set of linear algebraic equations with the unknowns $g_i^*(t_i)$, $g_{2R}^*[t_{2i}, (s_{2j})]$ and $g_{2I}^*[t_{2i}, (s_{2j})]$ as follows:

$$\frac{1}{N_1} \sum_{i=1}^{N_1} g_{1i}^*(t_{1i}) \left[\frac{1}{t_{1i} - s_{1j}} + \frac{a}{2} K_{11}^* \right] \\ + \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} [g_{2R}^*(t_{2i}) (K_{12}^* \cos \theta_i \\ + K_{13}^* \sin \theta_i) + g_{2I}^*(t_{2i}) (K_{13}^* \cos \theta_i \\ - K_{12}^* \sin \theta_i)] = -\frac{2}{E_3} P_0^*(s_{1j}), \quad (57)$$

$$\frac{1}{N_2} \sum_{i=1}^{N_2} \frac{(1+t_{2i})^{1/2}}{t_{2i} - s_{2j}} [g_{2R}^*(t_{2i}) \cos \theta_i \\ - g_{2I}^*(t_{2i}) \sin \theta_i] + \frac{\lambda}{(1-s_{2j})^{1/2}} [g_{2R}^*(s_{2j}) \sin \theta_j \\ + g_{2I}^*(s_{2j}) \cos \theta_j] + \frac{a}{2N_1} \sum_{i=1}^{N_1} g_{1i}^*(t_{1i}) K_{21}^* \\ + \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} [g_{2R}^*(t_{2i}) (K_{22}^* \cos \theta_i \\ + K_{23}^* \sin \theta_i) + g_{2I}^*(t_{2i}) (K_{23}^* \cos \theta_i \\ - K_{22}^* \sin \theta_i)] = 0, \quad (58)$$

$$\frac{1}{N_2} \sum_{i=1}^{N_2} \frac{(1+t_{2i})^{1/2}}{t_{2i} - s_{2j}} [g_{2R}^*(t_{2i}) \sin \theta_i + g_{2I}^*(t_{2i}) \cos \theta_i] \\ - \frac{\lambda}{(1-s_{2j})^{1/2}} [g_{2R}^*(s_{2j}) \cos \theta_j - g_{2I}^*(s_{2j}) \sin \theta_j]$$

$$+ \frac{a}{2N_1} \sum_{i=1}^{N_1} g_{1i}^*(t_{1i}) K_{31}^* + \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} \\ \times [g_{2R}^*(t_{2i}) (K_{32}^* \cos \theta_i + K_{33}^* \sin \theta_i) \\ + g_{2I}^*(t_{2i}) (K_{33}^* \cos \theta_i - K_{32}^* \sin \theta_i)] = 0, \quad (59)$$

$$(j=1, 2, \dots, N_2-1),$$

$$\frac{a}{N_1} \sum_{i=1}^{N_1} g_{1i}^*(t_{1i}) - \frac{b}{N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} [g_{2R}^*(t_{2i}) \sin \theta_i \\ + g_{2I}^*(t_{2i}) \cos \theta_i] = 0, \quad (60)$$

where, t_{1i} , t_{2i} , s_{1j} and s_{2j} are given by

$$t_{ki} = \cos \left(\frac{2i-1}{2N_k} \pi \right), \quad (i=1, 2, \dots, N_k; k=1, 2),$$

$$s_{kj} = \cos \left(\frac{j\pi}{N_k} \right), \quad (j=1, 2, \dots, N_k-1; k=1, 2). \quad (61)$$

The subscripts such as $1i$, $2i$, and $2j$ denote two indices; they do not mean $1 \times i$, $2 \times i$, nor $2 \times j$. Moreover, $g_{2R}^*(s_{2j})$ and $g_{2I}^*(s_{2j})$ in Eqs. (58) and (59) are approximated by

$$g_{2R}^*(s_{2j}) = \frac{1}{2} [g_{2R}^*(t_{2j}) + g_{2R}^*(t_{2j+1})], \\ g_{2I}^*(s_{2j}) = \frac{1}{2} [g_{2I}^*(t_{2j}) + g_{2I}^*(t_{2j+1})]. \quad (62)$$

It can be seen that $t_{2j} < s_{2j} < t_{2j+1}$ as shown in Eq. (61), and $g_{2R}^*(s_{2j})$ and $g_{2I}^*(s_{2j})$ satisfy the Hölder condition in the closed region ($-1 \leq s_2 \leq 1$). Thus, Eq. (62) becomes accurate for a large N_2 . The numerical results given in Table 1 verify that satisfactory results are obtained for $N_2=50$ and that this approximation is appropriate, though indirectly.

Equations (47) and (57)-(60) give us N_1+2N_2-1 algebraic equations with N_1+2N_2 unknowns of $g_{1i}^*(t_{1i})$, $g_{2R}^*(t_{2i})$ and $g_{2I}^*(t_{2i})$ after substitution of Eq. (62) into Eqs. (58) and (59). One more condition is necessary to solve the algebraic equations. The following condition

$$\sigma_{2x}(0, 0) = 0, \quad (63)$$

at the free origin is selected as an auxiliary condition in the present paper. Equation (63) corresponds to Eq. (58) with $s_{2j} = -1$. The shear stress at the origin is automatically satisfied to be zero due to the symmetry of this problem. Thus, the final step is to solve the algebraic equations, (47), (57)-(60) and (63).

The functions $g_1(a)$ and $g_2(b)$ in Eqs. (36) and (37) for the stress intensity factors yield to

$$g_1(a) = \frac{\sqrt{a}}{2} g_{1i}^*(1), \quad (64)$$

$$g_2(b) = \frac{\sqrt{b}}{\sqrt{2}} [g_{2R}^*(1) + ig_{2I}^*(1)], \quad (65)$$

using Eqs. (30)-(35), (46), (48) and (49).

4. Numerical Results and Discussion

Inserting the material constants, loading condition and geometrical parameters into the Eqs. (41), (57)-(60) and (63), one can obtain a set of discrete values of $g_{1i}^*(t_{1i})$, $g_{2R}^*(t_{2i})$ and $g_{2I}^*(t_{2i})$. The stress

intensity factors and the displacements along the cracks can then be obtained from Eqs. (36)–(40), (64) and (65). In this section the plane strain condition and a constant pressure P_0 are assumed. The material constants of $\nu_1 = \nu_2 = 0.3$, $E_1/E_2 = 1/3, 3, 20$ and 100 , are used, which simulates actual composite materials such as tungsten-copper, glass-epoxy and carbon-epoxy. The strips 1 and 2 are assumed to have the same width h , i. e., $h_1 = h_2 = h$. The numerical results are presented in Table 1 and Figs. 2–6.

A good numerical convergence is obtained as exhibited in Table 1. Both N_1 and N_2 are set to be N . Here, $k'_1(a)$ is the nondimensional Mode I stress intensity factor at the perpendicular crack tip, $k'_1(b)$ and $k'_2(b)$ are the nondimensional Mode I and II stress intensity factors at the interface crack tip, respectively, and a and b are the perpendicular crack length and the half of the interface crack length, respectively. It is seen that both $k'_1(a)$ and $k'_1(b)$ converge with N increasing and that good accuracy is achieved for $N = 50$. All of the following numerical results are obtained with $N = 50$.

A comparison between the present (solid line) and previous⁽¹²⁾ (dotted line) results for $k'_1(a)$ is given

Table 1 Effect of the number N ($\equiv N_1 = N_2$) on the convergence of the stress intensity factors: $k'_1(a) = k_1(a)/(P_0\sqrt{a/2})$, $k'_1(b) = k_i(b)/(P_0\sqrt{b/2})$; $\nu_1 = \nu_2 = 0.3$, $E_1 = 3E_2$, $b = 0.5h$, $h_1 = h_2 = h = 10a$

	$N=20$	30	40	50
$k'_1(a)$	1.5382	1.5416	1.5427	1.5438
$k'_1(b)$	-.0107	-.0109	-.0110	-.0110
$k'_2(b)$	.0525	.0526	.0526	.0526

Fig. 2 Stress intensity factors at the tip of a crack perpendicular to the interface: $k'_1(a) = k_1(a)/(P_0\sqrt{a/2})$; $h_1 = h_2 = h$

in Fig. 2 together with the ones using the finite element method (FEM)⁽¹⁵⁾. The previous results were obtained by means of the Jacobi polynomial expansion, and the ones with FEM were obtained from the crack opening displacements. It is shown that the present results are nearer to those with FEM.

The effects of the perpendicular crack length on the stress intensity factors at the interface crack tip are shown in Fig. 3 with the same conditions as Fig. 2. The $k'_2(b)$ increases at a nearly constant rate and the interface has an increasing tendency to the shear failure as the perpendicular crack propagates. On the other hand, $k'_1(b)$ is negative and its absolute value shows the maximum. It appears to become positive and increase just before the fracture of strip 2.

Figures 4 and 5 show the effects of the ratio b/a (half of the interface crack length to the perpendicular crack length) on the stress intensity factors at the both crack tips. Figure 4 gives $k'_1(a)$ at the perpendicular crack tip. The $k'_1(a)$ decreases with E_1/E_2 increasing. It increases with b/a increasing and converges to a constant with $b/a > 5$. This implies that strip 1 does not try to close the cracked strip 2 for $b/a > 5$. Figure 5 shows $k'_1(b)$ at the interface crack tip. When b/a is very small, both $k'_1(b)$ and $k'_2(b)$ are positive. Then $k'_2(b)$ decreases to zero monotonically with b/a increasing. However $k'_1(b)$ becomes negative first, presents the maximum absolute value and finally converges to zero with b/a increasing. Both $k'_1(b)$ and $k'_2(b)$ are practically zero for $b/a > 5$. This implies that the interface crack tends to propagate for small b/a , and tends to stop for a relatively large b/a . These

Fig. 3 Stress intensity factors at the tip of an interface crack: $k'_1(b) = k_i(b)/(P_0\sqrt{b/2})$; $\nu_1 = \nu_2 = 0.3$, $b = 0.5h$, $h_1 = h_2 = h$

Fig. 4 Effect of the half crack length ratio b/a on the stress intensity factor at the tip of a crack perpendicular to the interface: $k'_1(a) = k_1(a)/(P_0\sqrt{a/2})$; $\nu_1 = \nu_2 = 0.3$, $b = 0.5h$, $h_1 = h_2 = h = 10a$

Fig. 6 Displacement $u_y(x)$ along the surface of a crack perpendicular to the interface: $u_0 = P_0a/E_2$; $\nu_1 = \nu_2 = 0.3$, $h_1 = h_2 = h = 10a$

Fig. 5 Effect of the half crack length ratio b/a on the stress intensity factor at the tip of an interface crack: $k'_1(b) = k_1(b)/(P_0\sqrt{b/2})$; $\nu_1 = \nu_2 = 0.3$, $b = 0.5h$, $h_1 = h_2 = h = 10a$

numerical results are useful for the discussion of the interface failure in composite materials. It is also pointed out that the absolute values of both $k'_1(b)$ and $k'_2(b)$ are large for large E_1/E_2 .

Figure 6 gives the displacement $u_y(x)$ along the perpendicular crack nondimensionalized by $u_0 (= P_0a/E_2)$. It is seen that $u_y(x)/u_0$ is small for large E_1/E_2 and small b .

5. Conclusions

(1) In the present paper, a new numerical method using only the Gauss-Chebyshev integration formula

was proposed to evaluate the integral equations for the interaction problem between the cracks perpendicular to and on the interface of two bonded strips, that is, the T-shaped crack problem. Firstly, the problem was formulated as a system of singular integral equations by means of the Airy stress function and Fourier transforms. Secondly, the oscillatory part of the singularity at the interface crack tip was separated from the weight function of numerical integrals so that the Gauss-Chebyshev integration formula could be applied to the numerical integration to obtain a set of linear algebraic equations. This new method reduced the analytical and computational efforts considerably compared with the previous method using both the Jacobi polynomial expansion and Gauss-Chebyshev integration formula.

(2) The present method was applied to the aforementioned T-shaped crack problem. The comparison between the present and previous results and those with the finite element method clearly showed that the present new method yielded stress intensity factors and a crack opening displacement along the crack surface accurately with a reasonable amount of computational efforts.

(3) The effects of material constants and geometrical parameters on the stress intensity factors and the crack opening displacement were studied with various practical material constants. All of these nondimensional results are related to the debonding or delamination along the interface in composite materials. The results shown in Figs. 4 and 5 are especially interesting. That is, the stress intensity factor $k'_1(a)$ at the perpendicular crack tip increases with the half

crack length ratio b/a increasing for small b/a , and converges to a constant value for $b/a > 5$. The stress intensity factors $k_I(b)$ at the interface crack tip have large positive values for small b/a , and converge to zero for $b/a > 5$, which implies that the interface crack tends to propagate at small b/a , and stop at large b/a . The possible combinations of b and a , obtained using certain criteria, and the effects of other parameters on them are important in the investigation of the fracture toughness of composite materials.

(4) Since the present numerical method is simple and reduces both the analytical and computational efforts, it could be easily applied to a more complicated problem such as the one with adhesive layers. For the present study, this method is applied to the problem of a T-shaped crack at the interface bounded by two fibers, however this could also correspond to the transverse crack in a 0 degree layer with the delamination between 0 degree and 90 degree layers in laminates.

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