Stochastic Finite Element Analysis of Thermal Deformation and Thermal Stresses of CFRP Laminated Plates*

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The effect of probabilistic fiber orientations on the response of carbon-fiber-reinforced plastic laminated plates due to thermal load is analyzed by the stochastic finite element method based on the first-order perturbation technique. The fiber orientations are taken as a two-dimensional spatial stochastic process, and the stochasticity is represented by a trigonometrical autocorrelation function. Numerical examples are carried out in regard to the rectangular CFRP laminated plates subjected to a uniform temperature rise or a constant temperature gradient through the thickness. The variance of the thermal deformation and stresses is discussed for the input statistical property of the orientations

Key Words: Structural Analysis, CFRP Laminate, Stochastic FEM, Uncertainty, Thermal Deformation

1. Introduction

Advanced composite materials such as Carbon-Fiber-Reinforced Plastics (the acronym CFRP used hereafter) have been increasingly employed as engineering materials due to their high specific strength and stiffness, thus attempts are being made to apply them to large and precise space structures, for instance, space communication antenna. One of the most important measures to meet the high surface accuracy required for the structures used in the space is to reduce the effect of the thermal deformation caused by the solar rays. It is also desired to estimate quantitatively the response variation due to the uncertainties involved in the structural system of interest⁽¹⁾.

The authors have investigated the reliability of

the composite laminated plates, taking into account the uncertain material constants and stacking parameters^{(2),(3)}. This paper deals with the variability of the thermal deformation and stresses of CFRP laminated plates by means of the perturbation-based stochastic finite element method⁽⁴⁾ in the case that the fibers undulate probabilistically as a two-dimensional stochastic process, in order to elucidate the basic property of the fluctuation of the thermal deformation. The numerical examples are concerned with rectangular CFRP/epoxy laminated plates in the elastic-plane stress state.

2. Finite Element Formulation of the Thermal Deformation of Laminated Plates

The relationship of the resultant stresses and strains of an orthotropic laminated plate with thermal deformation is expressed by Eq.(1) based on the coordinate system shown in Fig. 1,

$$\begin{Bmatrix} f \\ m \end{Bmatrix} = \begin{bmatrix} [G_1] & [G_2] \\ [G_2]^T & [G_3] \end{bmatrix} \begin{Bmatrix} \varepsilon - \varepsilon^t \\ \chi - \chi^t \end{Bmatrix} \quad (1)$$

where the superscript T indicates the transpose of a matrix and

$$\{f\} = \{f_x \ f_y \ f_{xy}\}, \text{ plane resultant stresses ;}$$

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$\{m\} = \{m_x \ m_y \ m_{xy}\}$, moments per unit length;
 $\{\varepsilon\} = \{\varepsilon_x \ \varepsilon_y \ \gamma_{xy}\}$, strains at the mid-plane;
 $\{\varepsilon^t\} = \{\varepsilon_x^t \ \varepsilon_y^t \ \varepsilon_{xy}^t\}$, thermal strains at the mid-plane;

$\{\chi\} = \{\chi_x \ \chi_y \ \chi_{xy}\}$, changes of curvature;

$\{\chi^t\} = \{\chi_x^t \ \chi_y^t \ \chi_{xy}^t\}$, thermal changes of curvature;

and $[G_1]$, $[G_2]$ and $[G_3]$ are the matrices representing the extensional stiffness, coupling stiffness and bending stiffness, respectively⁽⁵⁾. They are defined as follows for the total number of layers N .

$$\begin{aligned} [G_1] &= \sum_{i=1}^N [G]_i (z_{i+1} - z_i) \\ [G_2] &= \sum_{i=1}^N \frac{1}{2} [G]_i (z_{i+1}^2 - z_i^2) \\ [G_3] &= \sum_{i=1}^N \frac{1}{3} [G]_i (z_{i+1}^3 - z_i^3) \end{aligned} \quad (2)$$

The matrices $[G]_i$ defined with respect to the element coordinate system are expressed in the following form of Eq. (3),

$$[G]_i = [T]_i^T [G_m] [T]_i \quad (3)$$

based on the stress-strain matrix $[G_m]$ defined with respect to the LT coordinate system and the coordinate transformation matrix $[T]_i$ as given below.

$$[G_m] = \frac{1}{1 - \nu_{LT}\nu_{TL}} \begin{bmatrix} E_L & E_L\nu_{TL} & 0 \\ E_T\nu_{LT} & E_T & 0 \\ 0 & 0 & (1 - \nu_{LT}\nu_{TL})G_{LT} \end{bmatrix} \quad (4)$$

$$[T]_i = \begin{bmatrix} l^2 & m^2 & -lm \\ m^2 & l^2 & lm \\ 2lm & -2lm & l^2 - m^2 \end{bmatrix} \quad (5)$$

In the above, E_L , E_T , G_{LT} and ν_{LT} stand for the elastic constants of a layer, and l and m indicate the direction cosines of the fiber orientation θ_i of the i -th layer. The thermal strains $\{\varepsilon^t\}$ at the mid-plane and thermal changes of curvature $\{\chi^t\}$ are given by those of the i -th layer with respect to the LT coordinate system in Eqs. (6) and (7).

$$\{\varepsilon^t\} = [T]_i^{-1} \{\varepsilon^t\}_i \quad (6)$$

$$\{\chi^t\} = [T]_i^{-1} \{\chi^t\}_i \quad (7)$$

The thermal strains and changes of curvature of the

i -th layer are expressed by using the temperature difference to a certain reference temperature ΔT , the temperature gradient in the through-the-thickness direction τ , and the coefficients of thermal expansion α_L and α_T in the L and T directions, as given below.

$$\{\varepsilon^t\}_i = \{\Delta T \alpha_L \ \Delta T \alpha_T \ 0\} \quad (8)$$

$$\{\chi^t\}_i = \{\tau \alpha_L \ \tau \alpha_T \ 0\} \quad (9)$$

The origin of the z -coordinate is taken at the mid-plane of the laminated plate.

The stiffness equation for the overall system can be formulated easily on the basis of the principle of the minimum potential energy and is summarized in the form of Eq. (10) by using the stiffness matrix $[K]$, the unknown nodal displacement vector $\{u\}$ and the thermal load vector $\{F^t\}$.

$$[K] \{u\} = \{F^t\} \quad (10)$$

Only the thermal load is taken into account in this analysis, while the effect of any external mechanical load is discarded.

3. Stiffness Fluctuation Due to Small Undulation of Fibers

The rectangular laminated plates are analyzed, in which only the fiber orientations fluctuate probabilistically in the plate plane. It is assumed that the fiber orientations are a spatial stochastic process in the plane, but remain uniform in an element. Another assumption is made that the uncertain fiber orientations are not correlated between the layers. The uncertain fiber orientation θ_j in an element in a layer is expressed by using the mean value $\bar{\theta}_j$ and a small probabilistic variable ξ_j whose mean is zero, as given below.

$$\theta_j = \bar{\theta}_j + \xi_j \quad (11)$$

Then the fluctuation of the stiffness matrix can be approximated by the first-order Taylor series expansion with respect to the probabilistic variables in the following form⁽⁴⁾.

$$[K] = [\bar{K}] + \sum_{j=1}^M [K_j'] \xi_j \quad (12)$$

The first-order rate of change $[K_j']$ is evaluated by means of the differentiation of the stiffness matrix with respect to ξ_j at the deterministic term. The upper bar and superscript I indicate the deterministic term and the first-order rate of change, respectively. M means the total number of the probabilistic variables determined as the product of the total number of the elements and that of the layers. In a similar manner, the fluctuation of the thermal load vector is approximated by Eq. (13).

$$\{F^t\} = \{\bar{F}^t\} + \sum_{j=1}^M \{F_j^t\} \xi_j \quad (13)$$

Fig. 1 Coordinate systems of laminated plate

4. Evaluation of Response Fluctuation

The fluctuation of the nodal displacements due to the fluctuations of the stiffness matrix of Eq.(12) and the thermal load vector of Eq.(13) is assumed in the same form of the Taylor series expansion as given below.

$$\{u\} = \{\bar{u}\} + \sum_{j=1}^M \{u_j'\} \xi_j \quad (14)$$

Substituting Eqs.(12), (13) and (14) into Eq.(10) and applying the first-order perturbation technique to the result, we have the deterministic term and the first-order rate of change of the displacements in the following form.

$$\{\bar{u}\} = [\bar{K}]^{-1} \{\bar{F}\} \quad (15)$$

$$\{u_j'\} = [\bar{K}]^{-1} \{F_j'\} - [K_j'] \{\bar{u}\} \quad (16)$$

The fluctuation of the in-ply stresses can be evaluated by using the stress-strain matrix which takes into account the fluctuation of the fiber orientation, after the derivation of the strains from the displacements based on the strain-nodal displacement matrix. In this way, the first-order approximation of the fluctuation of the stresses on the lower surface of the i -th layer with respect to the XY coordinate system is summarized as Eq.(17),

$$\{\sigma\}_i = \{\bar{\sigma}\}_i + \sum_{j=1}^M \{\sigma_j'\}_i \xi_j \quad (17)$$

where

$$\{\bar{\sigma}\}_i = [\bar{G}]_i \{ [B_\epsilon] + z_i [B_\kappa] \} \{\bar{u}\} - [G']_i \{ \{\epsilon^t\} + z_i \{\chi^t\} \} \quad (18)$$

$$\{\sigma_j'\}_i = [G_j']_i \{ [B_\epsilon] + z_i [B_\kappa] \} \{\bar{u}\} + [\bar{G}]_i \{ [B_\epsilon] + z_i [B_\kappa] \} \{u_j'\} - [G_j']_i \{ \{\epsilon^t\} + z_i \{\chi^t\} \} \quad (19)$$

and $[B_\epsilon]$ and $[B_\kappa]$ are the parts of the strain-nodal displacement matrix related to the mid-plane strains and changes of curvature. The matrix $[G']_i$ is given by Eq.(20).

$$[G']_i = [T] [G_m] \quad (20)$$

5. Evaluation of Response Statistics

Based on the formulation stated in Section 4, the fluctuation of any displacement or stress X can be obtained by means of the first-order approximation in the following form.

$$X = \bar{X} + \sum_{j=1}^M X_j' \xi_j \quad (21)$$

It turns out that the expectation and variance of X are given by Eqs.(22) and (23) on the basis of the first-order approximation method for the calculation with the aid of the second moment $E[\xi_j \xi_k]$ of the probabilistic variables ξ_j and ξ_k ⁽⁶⁾.

$$E[X] = \bar{X} \quad (22)$$

$$\text{Var}[X] = \sum_{j=1}^M \sum_{k=1}^M X_j' X_k' E[\xi_j \xi_k] \quad (23)$$

The objective of the analysis is to evaluate the effect of the fiber orientations in the case that they are

the spatial stochastic process in the XY plane. It is reasonable to assume that the fiber orientations fluctuate as a spatial stochastic process with the periodicity caused by the way of the fabrication of the laminated plate. The probabilistic variation of ξ_j is therefore assumed in the form of Eq.(24), corresponding to the probabilistic undulation of the fibers illustrated in Fig. 2.

$$\xi_j = P \cdot \cos\left(\frac{2\pi L}{W_L} + \phi_L\right) \cos\left(\frac{2\pi T}{W_T} + \phi_T\right) \quad (24)$$

In the above, ϕ_L and ϕ_T denote the independent probabilistic variables which are distributed uniformly within the range of 0 to 2π , and P is the probabilistic variable whose mean is zero and variance is $\text{Var}[P]$, independent of ϕ_L and ϕ_T . W_L and W_T are the wavelengths of the fiber undulation in the L and T directions, respectively. According to the assumption that ξ_j is uniform in an element, the autocorrelation function $R(\lambda_L, \lambda_T)$ of ξ_j is given by Eq.(25), where λ_L and λ_T indicate the component lengths of the distance between the element centroids in the L and T directions.

$$R(\lambda_L, \lambda_T) = \frac{1}{4} \text{Var}[P] \cos\left(\frac{2\pi\lambda_L}{W_L}\right) \cos\left(\frac{2\pi\lambda_T}{W_T}\right) \quad (25)$$

The second moment in Eq.(23) can be derived from the above auto-correlation function.

6. Numerical Examples

The numerical examples show the variability of the thermal deformation and stresses of the rectangular laminated plates subjected to uniform temperature distribution (Case 1) and to the temperature gradient in the through-the-thickness direction (Case 2) in Fig. 3. Figure 4 illustrates the finite element division of the plate. The finite element used is the isoparametric, quadratic square element with eight nodes⁽⁷⁾. The boundary condition of the four edges is free, and the in-plane displacements and deflection at the center are fixed. The rigid body rotation of the plate is constrained.

Fig. 2 Undulation of fibers

The material constants used in the analysis are $E_L=222$ GPa, $E_T=5.7$ GPa, $G_{LT}=4.2$ GPa, $\nu_{LT}=0.263$, $\alpha_L=-8.0 \times 10^{-7}$ $1/^\circ\text{C}$ and $\alpha_T=3.5 \times 10^{-5}$ $1/^\circ\text{C}$ for the carbon/epoxy composite. Two-ply and three-ply laminates are subjected to the analysis. The expectation of the fiber orientations of the layers is listed in Table 1. The thickness of all the layers is 0.1 mm.

Figure 5 shows the comparison between the deterministic solutions (blank circles) and the perturbation solution (solid line) of the deflection at the corner-point C of the two-ply laminate in the case that the fiber orientation in the element m near the center is varied under the temperature condition of Case 1. This figure verifies the perturbation solution within the orientation range from -3 degrees to $+3$ degrees.

The input temperature distribution causes such a displacement field that the displacements are zero at the center and increase towards the edges symmetrically in regard to the center. It turns out that even small undulation of the fibers greatly influences the fluctuation of the deflection at the corner-point C. The standard deviation of the probabilistic variable P in Eq.(24) is taken equal to 3 degrees, and P , φ_L and φ_T are assumed to be uncorrelated for the evaluation of the influence.

Figure 6 shows the effect of the number of the elements on the calculation of the deflection at the point C. The abscissa of the figure is taken as the ratio of the plate length l to the undulation wavelength W_L of the fibers in the L direction, that is, the wave number of the fibers in the longitudinal direction of the rectangular plate. It is seen in the figure that the

Fig. 3 Input temperature distribution

Fig. 4 Finite element division of rectangular plate

standard deviation calculated by the coarse element division diverges with the decrease of the undulation wavelength. This implies that at least two elements are required in the wave direction in order to simulate the effect of a wave.

Figure 7 depicts the effect of the wavelength W_L on the deflection fluctuation in regard to three cases of the wavelength W_T in the T direction: equal to infinity, W_L and $l/4$. The calculation parameters are the same as in Fig. 6. The case of the infinite W_T corresponds to the undulation of the adjacent fibers with full correlation, and the degree of correlation decreases with W_T . $W_T=l/4$ is the shortest wavelength which can be dealt with by the element division

Table 1 Expectations of fiber orientations

Layer	1	2	3
2-ply laminate	0°	0°	—
3-ply laminate	0°	120°	60°

Fig. 5 Comparison of deterministic solution of deflection

Fig. 6 Effect of element division on standard deviation of deflection

of six by twelve employed in the following analyses. The circled numbers in the figure indicate the samples of the fiber undulation shown in the upper right part of the figure.

The effect of the fiber undulation on the deflection fluctuation is analyzed for a few cases of different

temperature and stacking conditions. Figures 8, 9 and 10 show the standard deviations of the response of the two-ply laminate under the temperature condition of Case 2 and the three-ply laminate in Case 1 and Case 2, respectively. It is seen as a common tendency in these figures that the deflection fluctuation decreases

Fig. 7 Effect of undulation wavelength on standard deviation of deflection (Two-ply laminate under temperature condition of Case 1)

Fig. 10 Effect of undulation wavelength on standard deviation of deflection (Three-ply laminate under temperature condition of Case 2)

Fig. 8 Effect of undulation wavelength on standard deviation of deflection (Two-ply laminate under temperature condition of Case 2)

Fig. 9 Effect of undulation wavelength on standard deviation of deflection (Three-ply laminate under temperature condition of Case 1)

Fig. 11 Distribution of coefficient of variation of σ_x on the lower surface of each layer (%)

Fig. 12 Effect of undulation wavelength on coefficient of variation of σ_x (Three-ply laminate under temperature condition of Case 2)

with the wavelength W_L . This means that small undulation of the fibers has little effect on the macroscopic deformation of the plate.

The distribution of the fluctuation of the in-ply stresses is also investigated. Figure 11 illustrates the contour of the coefficient of variation of the lower surface stress σ_x of each layer of the three-ply laminate under the condition of $W_L = W_T = 400$ mm in Case 2. The expectation of the stress is constant in the layer, being equal to 1.11 MPa, 0.33 MPa and 0.37 MPa for the first, second and third layers, respectively. The coefficient of variation of σ_x increases towards the edges. It cannot necessarily be said that the coefficient of variation is large in the layer of large expectation. This is an important finding to be considered, when examining the strength of the laminated plates.

The effect of W_L on the stress σ_x in each layer is shown in Fig. 12. The same calculation parameters are taken as that of Fig. 11, except the value of W_L changed. The stress fluctuation is not decreased with the wavelength as markedly as that of the deflection fluctuation, giving rise to a more considerable effect of a short wavelength on the stress fluctuation than on the deflection fluctuation.

7. Concluding Remarks

From viewpoint of basic elucidation of the ther-

mal deformation of CFRP laminated plates, the uncertainty of the fiber orientations in each layer is dealt with as a two-dimensional stochastic process, and the variability of the thermal deflection and stresses due to the fiber undulation is investigated. The result is summarized as follows.

1. One wavelength should be divided into two finite elements in the case that the probabilistic variation of the fiber orientations is incorporated in the finite element analysis.

2. The fluctuation of the deflection is large when the undulation wavelength is as long as the plate length, and is decreased with the wavelength.

3. The undulation of the fibers influences the fluctuation of the local stresses more than it influences the macroscopic deflection, and the degree of the influence is not diminished even in the case of a short undulation wavelength.

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