An Analysis of the *T*-Shaped Crack Along the Fiber-Matrix Interface in Unidirectional Composite Materials*

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A fracture problem in unidirectionally fiber-reinforced composites containing a *T*-shaped crack at the interface is considered. The *T*-shaped crack consists of an interface crack and a crack perpendicular to the interface. A system of integral equations is formulated based on the two-dimensional elasticity and Fourier transforms. The integral equations are evaluated by a numerical method using only the Gauss-Chebyshev integral formula, which was developed by the present authors in the last paper. First, the numerical results are compared with the previous results for a *T*-shaped crack in a homogeneous material, verifying the present method to be highly satisfactory. Next, three kinds of stress intensity factors (SIFs), such as Model SIF of a perpendicular crack and Mode I and II SIFs of an interface crack, are presented for glass/epoxy, carbon/epoxy and tungsten/copper composite systems with various geometrical parameters.

Key Words: Composite Materials, Unidirectional Composites, *T*-Shaped Crack, Stress Intensity Factors, Interface, Debonding, Fracture

1. Introduction

Unidirectionally reinforced composite materials and their laminates play an important role as the structural members among various composite materials and are widely used in aerospace applications due to their high specific stiffness and excellent specific strength. Recently, they are considered to be the primary structure. Their fracture behavior, therefore, becomes very important from the safety viewpoint, while the stiffness and the maximum strength are the main issues in the secondary structure. The fracture behavior of composite materials depends on the constituent properties, geometry and the strength of the interface. The microscopic failure process and macroscopic fracture strength vary in a complicated manner^{(1)~(6)}, compared with those of conventional materials. Thus, it is necessary to investigate the

interaction between different microscopic failure processes, their relation to the macroscopic strength and the effect of the aforementioned parameters on these interactions and relations. The results of the fracture problem of unidirectionally reinforced composite materials could be a kind of data base for composites with more complicated configurations.

The crack problem between different materials has been solved theoretically for various patterns^{(7)~(15)}. Pacella and Erdogan⁽⁷⁾ analyzed the penny-shaped crack with equally spaced fibers along the concentric circle under tensile loading in the three-dimensional sense and obtained the crack arrest effect of fibers. Wang et al.⁽⁸⁾ treated the same crack under shear loading and presented the variation of the stress intensity factor along the crack-tip line. Erdogan and Ozbek⁽⁹⁾ and Kasano et al.⁽¹⁰⁾ solved axisymmetric problems of the cylindrical interface crack around the fiber in the matrix. Two-dimensional solutions are abundant in the literature including those involving the crack in an anisotropic plate^{(11)~(12)}. Each of these solutions refers to a single crack perpendicular to or along the interface. The kind of crack perpendicular to and along the interface of a bimaterial, such as a *T*-

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shaped crack, was treated by Goree and Venezia⁽¹³⁾, and Lu and Erdogan^{(14)~(15)}. However, their analyses need the cumbersome numerical evaluation of the Jacobi polynomial, and it was pointed out that numerical accuracy and convergence were not easily attainable⁽¹⁵⁾. Recently, the authors⁽¹⁶⁾ developed a simple numerical method without using the Jacobi polynomial expansion technique and presented an accurate solution to the *T*-shaped crack at the interface between two strips.

In this report, the above-described method is applied to a more realistic model, that is, a *T*-shaped crack bounded by two fibers in the matrix (see Fig. 1), and the role of the right fiber completely attached to the matrix will be highlighted. This five-strip model can take into account the fiber volume fraction which corresponds to the fiber space in this case.

Section 2 will reduce the present two-dimensional elastic problem with a *T*-shaped crack to a set of singular integral equations by means of the Airy stress function and Fourier transforms. Section 3 will show the new numerical technique in short, followed by the results in Section 4, where the numerical accuracy based on the present technique is discussed first using a bifurcation crack within the homogeneous materials⁽¹⁷⁾. Next, the stress intensity factors of both crack tips at the bimaterial interface and the effects of material constants and geometrical parameters on them are presented.

2. The Derivation of Integral Equations

A two-dimensional *T*-shaped crack model is shown in Fig. 1(a) to discuss the internal microscopic failure processes such as debonding and matrix cracking along the interface and in the matrix in unidirectionally reinforced composite materials. The model consists of five strips, and each strip is bonded perfectly except for a *T*-shaped crack of length $2b$ along the interface and of length a perpendicular to the fiber. The thickness of the adhesive layer is zero. Strips 1, 3 and 5 correspond to the matrix and strips 2 and 4 to the fiber. The width of the strip i is denoted by h_i with $h_1=h_5=\infty$ and the local rectangular coordinate system $o-x_iy_i$ is taken as shown in Fig. 1(b) in each strip. The perpendicular crack surface is subjected to a self equilibrium pressure $P_0(x)$. It is clear that the solution is symmetric with respect to the x axis and it is well known that this loading condition corresponds to tensile loading at infinity parallel to the y axis from the fracture viewpoint, that is, for the intensity of singularities at crack tips⁽¹⁵⁾.

2.1 Basic equations

Based on the two-dimensional linear elasticity, the biharmonic equation in terms of the Airy stress

function $\phi_i(x, y)$ of the strip i is solved as follows⁽¹⁸⁾,

$$\begin{aligned} \phi_i(x, y) = & \frac{2}{\pi} \int_0^\infty \{ [(A_i + B_i x) e^{-xs} \\ & + (C_i + D_i x) e^{xs}] \cos(sy) + G_i(1 + sy) e^{-sy} \\ & \times \cos(xs) \} ds, \quad (i=1, 2, \dots, 5) \end{aligned} \quad (1)$$

by applying the Fourier transforms. Here, the subscript i denotes the strip i ($i=1, 2, \dots, 5$) and x and y are and will be used instead of x_i and y_i , respectively, if there is no confusion. After Section 2.2, x and y stand for x_3 and y_3 , respectively.

Strips 1 and 5 are semi-infinite planes, which lead to

$$A_1 = B_1 = 0, \quad C_5 = D_5 = 0 \quad (2)$$

and

$$G_1 = G_2 = G_4 = G_5 = 0 \quad (3)$$

is obtained as there is no crack within strips 1, 2, 4 and 5. The remaining seventeen unknown functions, such as $A_i(s)$, are determined from the boundary and continuity conditions. Inserting Eq. (1) into the following relations between the Airy stress function and stress $\sigma(\tau)$,

$$\sigma_{ix} = \frac{\partial^2 \phi_i}{\partial y^2}, \quad \sigma_{iy} = \frac{\partial^2 \phi_i}{\partial x^2}, \quad \tau_{ixy} = -\frac{\partial^2 \phi_i}{\partial x \partial y} \quad (4)$$

with the constitutive equations for the generalized plane-stress condition and strain $\epsilon(\gamma)$ versus displacement u relation, we can obtain the stress and displacement in the strip i as

$$\sigma_{ix} = -\frac{2}{\pi} \int_0^\infty \{ [(A_i + B_i x) e^{-xs}$$

Fig. 1 Calculation model; (a) *T*-shaped crack, (b) Local coordinates

$$+(C_i + D_i x)e^{xs}]s^2 \cos(ys) - G_i s^2(1 - \nu_i)e^{-ys} \cos(xs) \} ds \quad (5 \cdot a)$$

$$\sigma_{iy} = \frac{2}{\pi} \int_0^\infty \{ [(A_i s + B_i x s - 2B_i)e^{-xs} + (C_i s + D_i x s + 2D_i)e^{xs}]s \cdot \cos(ys) - G_i s^2(1 + \nu_i)e^{-ys} \cos(xs) \} ds \quad (5 \cdot b)$$

$$\tau_{ixy} = -\frac{2}{\pi} \int_0^\infty \{ [(A_i s + B_i x s - B_i)e^{-xs} - (C_i s + D_i x s + D_i)e^{xs}]s \times \sin(ys) + G_i y s^3 e^{-ys} \sin(xs) \} ds \quad (5 \cdot c)$$

$$u_{ix} = \frac{2}{E_i \pi} \int_0^\infty \{ [(A_i + B_i x)(1 + \nu_i)s + (1 - \nu_i)B_i]e^{-xs} - [(C_i + D_i x)(1 + \nu_i)s - (1 - \nu_i)D_i]e^{xs} \} \cos(ys) ds + \frac{2}{E_i \pi} \int_0^\infty G_i s \times [ys(1 + \nu_i) - (1 - \nu_i)]e^{-ys} \sin(xs) ds \quad (5 \cdot d)$$

$$u_{iy} = \frac{2}{E_i \pi} \int_0^\infty \{ [(A_i + B_i x)(1 + \nu_i)s - 2B_i]e^{-xs} + [(C_i + D_i x)(1 + \nu_i)s + 2D_i]e^{xs} \} \sin(ys) ds + \frac{2}{E_i \pi} \int_0^\infty G_i s [2 + ys(1 + \nu_i)]e^{-ys} \cos(xs) ds \quad (5 \cdot e)$$

$$(i=1, 2, \dots, 5)$$

where E_i and ν_i denote the Young's modulus and Poisson's ratio, respectively, and they are replaced by

$$\frac{E_i}{(1 - \nu_i^2)}, \quad \frac{\nu_i}{(1 - \nu_i)} \quad (6)$$

respectively, for the plane-strain condition.

On account of the symmetry with respect to the x axis, the boundary and continuity conditions can be written as follows (see Fig. 1(a)),

$$\left. \begin{aligned} u_{1x}(0, y) &= u_{2x}(-h_2, y) \\ u_{1y}(0, y) &= u_{2y}(-h_2, y) \\ \sigma_{1x}(0, y) &= \sigma_{2x}(-h_2, y) \\ \tau_{1xy}(0, y) &= \tau_{2xy}(-h_2, y) \\ (0 \leq y_{1(2)} < \infty) \end{aligned} \right\} \quad (7)$$

$$\left. \begin{aligned} u_{3x}(h_3, y) &= u_{4x}(0, y) \\ u_{3y}(h_3, y) &= u_{4y}(0, y) \\ \sigma_{3x}(h_3, y) &= \sigma_{4x}(0, y) \\ \tau_{3xy}(h_3, y) &= \tau_{4xy}(0, y) \\ (0 \leq y_{3(4)} < \infty) \end{aligned} \right\} \quad (8)$$

$$\left. \begin{aligned} u_{4x}(h_4, y) &= u_{5x}(0, y) \\ u_{4y}(h_4, y) &= u_{5y}(0, y) \\ \sigma_{4x}(h_4, y) &= \sigma_{5x}(0, y) \\ \tau_{4xy}(h_4, y) &= \tau_{5xy}(0, y) \\ (0 \leq y_{4(5)} < \infty) \end{aligned} \right\} \quad (9)$$

$$u_{2x}(0, y) = u_{3x}(0, y), \quad (b < y_{2(3)} < \infty) \quad (10)$$

$$u_{2y}(0, y) = u_{3y}(0, y), \quad (b < y_{2(3)} < \infty) \quad (11)$$

$$\sigma_{2x}(0, y) = \sigma_{3x}(0, y), \quad (0 \leq y_{2(3)} < \infty) \quad (12)$$

$$\tau_{2xy}(0, y) = \tau_{3xy}(0, y), \quad (0 \leq y_{2(3)} < \infty) \quad (13)$$

$$\sigma_{3x}(0, y) = 0, \quad (0 \leq y_3 < b) \quad (14)$$

$$\tau_{3xy}(0, y) = 0, \quad (0 \leq y_3 < b) \quad (15)$$

$$u_{3y}(x, 0) = 0, \quad (a < x_3 \leq h_3) \quad (16)$$

$$\sigma_{3y}(x, 0) = -P_0(x), \quad (0 < x_3 < a) \quad (17)$$

$$u_{1y}(x, 0) = 0, \quad (-\infty < x_1 \leq 0) \quad (18)$$

$$\tau_{1xy}(x, 0) = 0, \quad (-\infty < x_1 \leq 0) \quad (19)$$

$$u_{2y}(x, 0) = 0, \quad (-h_2 < x_2 \leq 0) \quad (20)$$

$$\tau_{2xy}(x, 0) = 0, \quad (-h_2 \leq x_2 \leq 0) \quad (21)$$

$$u_{4y}(x, 0) = 0, \quad (0 \leq x_4 \leq h_4) \quad (22)$$

$$\tau_{4xy}(x, 0) = 0, \quad (0 \leq x_4 \leq h_4) \quad (23)$$

$$u_{5y}(x, 0) = 0, \quad (0 \leq x_5 < \infty) \quad (24)$$

$$\tau_{5xy}(x, 0) = 0, \quad (0 \leq x_5 < \infty) \quad (25)$$

$$\tau_{3xy}(x, 0) = 0, \quad (0 \leq x_3 \leq h_3) \quad (26)$$

where Eqs. (7) to (9) indicate the continuity conditions between strips 1 and 2, strips 3 and 4 and strips 4 and 5, respectively. Equations (10) to (15) are related to the conditions between strips 2 and 3, and Eqs. (16) and (18) to (26) are symmetry conditions at $y=0$. Equations (18) to (26) are automatically satisfied by Eqs. (2), (3) and (5). Thus, the aforementioned seventeen unknowns, such as $A_i(s)$, could be determined by the remaining conditions (7) to (17).

2.2 Integral equations

Inserting Eq. (5) into Eqs. (7) to (13), applying the Fourier transforms to them and using the unknown functions defined by the displacement derivatives as follows⁽¹⁵⁾,

$$f_1(x) = \frac{\partial u_{3y}(x, 0)}{\partial x} \quad (27)$$

$$f_2(y) = \frac{\partial}{\partial y} [u_{3x}(0, y) - u_{2x}(0, y)] \quad (28)$$

$$f_3(y) = \frac{\partial}{\partial y} [u_{3y}(0, y) - u_{2y}(0, y)] \quad (29)$$

yields $A_{2(4,5)}$, $B_{2(4,5)}$, $C_{1(2,4)}$ and $D_{1(2,4)}$ as a function of A_3 , B_3 , C_3 , D_3 , R_i , E_i , E_{ij} etc..

$$\left. \begin{aligned} A_3 s^2 &= \sum_{i=1}^7 Q_{Ai}(s) R_i(s) \\ B_3 s &= \sum_{i=1}^7 Q_{Bi}(s) R_i(s) \\ C_3 s^2 &= \sum_{i=1}^7 Q_{Ci}(s) R_i(s) \\ D_3 s &= \sum_{i=1}^7 Q_{Di}(s) R_i(s) \\ G_3(s) &= -\frac{E_3}{2s^2} \int_0^a f_1(t) \sin(ts) dt \end{aligned} \right\} \quad (30)$$

and

$$\int_0^a f_1(t) dt - \int_0^b f_3(t) dt = 0 \quad (31)$$

are obtained simultaneously⁽¹⁹⁾. Here E_{ij} 's are a function of material constants, that is,

$$\left. \begin{aligned} E_{11} &= \left[\frac{E_1}{E_2} (3 - \nu_2) - (3 - \nu_1) \right] \\ &\quad / \left[\frac{E_1}{E_2} (1 + \nu_2) + (3 - \nu_1) \right] \\ E_{12} &= \left[\frac{E_1}{E_2} (3 - \nu_2) + (1 + \nu_1) \right] \\ &\quad / \left[\frac{E_1}{E_2} (1 + \nu_2) - (1 + \nu_1) \right] \end{aligned} \right\}$$

$$\begin{aligned}
E_{13} &= \left[\frac{E_5}{E_4} (3 - \nu_4) + (1 + \nu_5) \right] \\
&\quad \left/ \left[\frac{E_5}{E_4} (1 + \nu_4) - (1 + \nu_5) \right] \right. \\
E_{14} &= \left[\frac{E_5}{E_4} (3 - \nu_4) - (3 - \nu_5) \right] \\
&\quad \left/ \left[\frac{E_5}{E_4} (1 + \nu_4) + (3 - \nu_5) \right] \right. \\
E_{15} &= (1 + \nu_4) - (1 + \nu_3) \frac{E_4}{E_3} \\
E_{16} &= (1 + \nu_4) + (3 - \nu_3) \frac{E_4}{E_3} \\
E_{17} &= -(1 + \nu_2) + (1 + \nu_3) \frac{E_2}{E_3} \\
E_{18} &= (1 + \nu_2) + (3 - \nu_3) \frac{E_2}{E_3}
\end{aligned} \quad (32)$$

and R_{ij} 's are as follows,

$$\begin{aligned}
R_j(s) &= E_3 \int_0^a f_j(t) F_j(t, s) dt \\
(j &= 1, 2, 3, 4) \\
R_5(s) &= E_3 \int_0^b f_2(t) F_5(t, s) dt \\
R_6(s) &= E_3 \int_0^a f_1(t) F_{6a}(t, s) dt \\
&\quad + E_3 \int_0^b f_3(t) F_{6b}(t, s) dt \\
R_7(s) &= E_3 \int_0^a f_1(t) F_7(t, s) dt
\end{aligned} \quad (33)$$

where

$$\begin{aligned}
F_1(t, s) &= \frac{1}{2s} [(1 - \nu_3) \sinh(ts) + h_3s \\
&\quad \times (1 + \nu_3) \sinh(ts) - ts(1 + \nu_3) \cosh(ts)] e^{-h_3s} \\
F_2(t, s) &= \frac{1}{2s} [(-2 \sinh(ts) + h_3s \\
&\quad \times (1 + \nu_3) \sinh(ts) - ts(1 + \nu_3) \cosh(ts)] e^{-h_3s} \\
F_3(t, s) &= \frac{s}{2} [(t \cosh(ts) - h_3 \sinh(ts))] e^{-h_3s} \\
F_4(t, s) &= -\frac{1}{2} [(1 - h_3s) \sinh(ts) \\
&\quad + ts \cosh(ts)] e^{-h_3s} \\
F_5(t, s) &= \frac{1}{s} \sin(ts) \\
F_{6a}(t, s) &= \frac{1}{s} \left[1 - (1 + \nu_3) \frac{ts}{2} \right] e^{-ts} \\
F_{6b}(t, s) &= -\frac{1}{s} \cos(ts) \\
F_7(t, s) &= \frac{ts}{2} e^{-ts}
\end{aligned} \quad (34)$$

Here $Q_{Ai}(s)$, ... and $Q_{Di}(s)$ are known functions determined by E_{ij} etc. ⁽¹⁹⁾

Next, Eq.(30) is inserted into Eqs.(14) to (17) using Eq.(5), which shows that Eq.(16) is satisfied and Eqs.(14), (15) and (17) are converted to a set of integral equations with unknown functions $f_i(t)$:

$$\frac{1}{\pi} \int_0^a \frac{f_1(t)}{t-x} dt + \frac{1}{\pi} \int_0^a f_1(t) K_{11}(t, x) dt$$

$$\begin{aligned}
&+ \frac{1}{\pi} \int_0^b [f_2(t) K_{12}(t, x) + f_3(t) K_{13}(t, x)] dt \\
&= -\frac{2}{E_3} P_0(x) \quad (0 < x < a)
\end{aligned} \quad (35)$$

$$\begin{aligned}
&\frac{1}{\pi} \int_0^b \frac{f_2(t)}{t-y} dt + \lambda f_3(y) + \frac{1}{\pi} \int_0^a f_1(t) K_{21}(t, y) dt \\
&+ \frac{1}{\pi} \int_0^b [f_2(t) K_{22}(t, y) + f_3(t) K_{23}(t, y)] dt = 0 \\
&(0 \leq y < b)
\end{aligned} \quad (36)$$

$$\begin{aligned}
&\frac{1}{\pi} \int_0^b \frac{f_3(t)}{t-y} dt - \lambda f_2(y) + \frac{1}{\pi} \int_0^a f_1(t) K_{31}(t, y) dt \\
&+ \frac{1}{\pi} \int_0^b [f_2(t) K_{32}(t, y) + f_3(t) K_{33}(t, y)] dt = 0 \\
&(0 \leq y < b)
\end{aligned} \quad (37)$$

where

$$\lambda = \frac{4 + E_{17} - E_{18}}{4 + E_{17} + E_{18}} \quad (38)$$

The kernels K_{ij} are known functions containing generalized Cauchy singular functions K_{ij}^s :

$$\begin{aligned}
K_{11}^s &= \frac{4E_{17}}{E_{18}} \frac{x^2}{(t+x)^3} - \frac{6E_{17}}{E_{18}} \frac{x}{(t+x)^2} \\
&+ \left(\frac{8E_2}{E_3E_{18}} - 1 - 2a_{11} \frac{E_2}{E_3} \right) \frac{1}{t+x'} \\
K_{12}^s &= -\frac{8E_2}{E_3E_{18}} \frac{x^2t}{(t^2+x^2)^2} \\
&+ \left(\frac{8E_2}{E_3E_{18}} - 2a_{11} \frac{E_2}{E_3} \right) \frac{t}{t^2+x^2} \\
K_{13}^s &= \frac{8E_2}{E_3E_{18}} \frac{t^2x}{(t^2+x^2)^2} + 2a_{11} \frac{E_2}{E_3} \frac{x}{t^2+x^2} \\
K_{21}^s &= -\frac{8}{a_{11}E_{18}} \frac{y^2t}{(t^2+y^2)^2} + \frac{2t}{t^2+y^2} \\
K_{22}^s &= \frac{1}{(t+y)}, \quad K_{23}^s = 0 \\
K_{31}^s &= -\frac{8}{a_{11}E_{18}} \frac{t^2y}{(t^2+y^2)^2} + \frac{2y}{t^2+y^2} \\
K_{32}^s &= 0, \quad K_{33}^s = \frac{-1}{(t+y)}
\end{aligned} \quad (39)$$

and ordinary parts K_{ij}^0 which were described in detail in Ref. [19].

Integral equations (31) and (35) to (37) are formally the same as those in Refs. [14] and [16]. The functions of A_i , E_{ij} , R_j , F_j etc., in those references can be obtained as a special case of the present solution ⁽¹⁹⁾.

2.3 Singularity of f_i and stress intensity factor

The stress intensity factors k_i at crack tips are defined by

$$k_1(a) = \lim_{x \rightarrow a^+} \sqrt{2(x-a)} \sigma_{3y}(x, 0) \quad (40)$$

$$\begin{aligned}
k_1(b) + ik_2(b) &= \lim_{y \rightarrow b^+} \sqrt{2}(y-b)^{1/2} [\sigma_{3x}(0, y) \\
&+ i\tau_{3xy}(0, y)]
\end{aligned} \quad (41)$$

which results in

$$k_1(a) = -\frac{E_3}{\sqrt{2}} g_1(a) \quad (42)$$

$$\left. \begin{aligned} k_1(b) + ik_2(b) &= -\frac{a_{11}E_2}{\sqrt{2}}\sqrt{1-\lambda^2}g_2(b) \\ a_{11} &= \frac{1}{E_{18}} + \frac{1}{4+E_{17}} \end{aligned} \right\} \quad (43)$$

where⁽²⁰⁾

$$\left. \begin{aligned} f_1(t) &= \frac{g_1(t)}{(a-t)^{\beta_1}} \\ f_2(t) + if_3(t) &= \frac{g_2(t)}{(b-t)^{\beta_2}} \end{aligned} \right\} \quad (44)$$

$$\left. \begin{aligned} \beta_1 &= \frac{1}{2} \\ \beta_2 &= \frac{1}{2} + i\omega, \quad \omega = \frac{1}{2\pi} \ln\left(\frac{1+\lambda}{1-\lambda}\right) \end{aligned} \right\} \quad (45)$$

The function f_1 has the singularity of order 1/2 at the perpendicular crack tip, and f_2 and f_3 have the oscillatory singularity of order 1/2 + $i\omega$ at the interface crack tip.

3. Numerical Evaluation of Integral Equation

Once the integral equations (31) and (35) to (37) are solved, the stress intensity factors can be easily calculated from Eqs. (42) and (43). The new method proposed in ref.[16] is used, that is, only the Gauss-Chebyshev integral formula is applied without the Jacobi polynomial expansion. The numerical processes are rewritten here in short.

First, the integral equations are normalized to have the integration domain of (-1,1) by the linear transformations:

$$\left. \begin{aligned} t &= \frac{a}{2}(t_1+1), \quad x = \frac{a}{2}(s_1+1) \\ (0 < t, x < a; -1 < t_1, s_1 < 1) \\ t &= \frac{b}{2}(t_2+1), \quad y = \frac{b}{2}(s_2+1) \\ (0 \leq t, y < b; -1 \leq t_2, s_2 < 1) \end{aligned} \right\} \quad (46)$$

Normalized functions are denoted by the superscript * and have the same functional value.

The function f_1^* is set as follows⁽¹⁴⁾,

$$\left. \begin{aligned} f_1^*(t_1) &= \frac{g_1^*(t_1)}{(1-t_1^2)^{1/2}} \left\{ \equiv f_1\left[\frac{a}{2}(t_1+1)\right] \right\} \\ g_1^*(-1) &= 0 \end{aligned} \right\} \quad (47)$$

The functions f_2^* and f_3^* are transformed to⁽¹⁶⁾

$$\left. \begin{aligned} f_2^*(t_2) &= \frac{1}{(1-t_2^2)^{1/2}}(g_{2R}^*(t_2)\cos\theta - g_{2I}^*(t_2)\sin\theta) \\ f_3^*(t_2) &= \frac{1}{(1-t_2^2)^{1/2}}(g_{2R}^*(t_2)\sin\theta + g_{2I}^*(t_2)\cos\theta) \\ \theta &= \omega \ln\left[\frac{2}{b(1-t_2)}\right] \\ \left\{ f_2^*(t_2) + if_3^*(t_2) \right\} &\equiv f_2\left[\frac{b}{2}(t_2+1)\right] + if_3\left[\frac{b}{2}(t_2+1)\right] \end{aligned} \right\} \quad (48)$$

with the real and imaginary parts, g_{2R}^* and g_{2I}^* , of the complex g_2^* which is obtained after a little modification to g_2^* (see Eq.(44)).

Substituting f_i^* from Eqs.(47) and (48) into the normalized integral equations and using the Gauss-

Chebyshev integral formula, one can arrive at a set of linear algebraic equations:

$$\begin{aligned} &\frac{1}{N_1} \sum_{i=1}^{N_1} g_1^*(t_{1i}) \left[\frac{1}{t_{1i}-s_{1j}} + \frac{a}{2} K_{11}^* \right] \\ &+ \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} [g_{2R}^*(t_{2i})(K_{12}^* \cos\theta_i \\ &+ K_{13}^* \sin\theta_i) + g_{2I}^*(t_{2i})K_{13}^* \cos\theta_i \\ &- K_{12}^* \sin\theta_i] = -\frac{2}{E_3} P_0^*(s_{1j}), \\ &(j=1, 2, \dots, N_1-1) \end{aligned} \quad (49)$$

$$\begin{aligned} &\frac{1}{N_2} \sum_{i=1}^{N_2} \frac{(1+t_{2i})^{1/2}}{t_{2i}-s_{2j}} [g_{2R}^*(t_{2i})\cos\theta_i \\ &- g_{2I}^*(t_{2i})\sin\theta_i] + \frac{\lambda}{(1-s_{2j})^{1/2}} \\ &\times [g_{2R}^*(s_{2j})\sin\theta_j + g_{2I}^*(s_{2j})\cos\theta_j] \\ &+ \frac{a}{2N_1} \sum_{i=1}^{N_1} g_1^*(t_{1i})K_{21}^* + \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} \\ &\times [g_{2R}^*(t_{2i}) \times (K_{22}^* \cos\theta_i + K_{23}^* \sin\theta_i) \\ &+ g_{2I}^*(t_{2i})(K_{23}^* \cos\theta_i - K_{22}^* \sin\theta_i)] = 0, \\ &(j=1, 2, \dots, N_2-1) \end{aligned} \quad (50)$$

$$\begin{aligned} &\frac{1}{N_2} \sum_{i=1}^{N_2} \frac{(1+t_{2i})^{1/2}}{t_{2i}-s_{2j}} [g_{2R}^*(t_{2i})\sin\theta_i + g_{2I}^*(t_{2i})\cos\theta_i] \\ &- \frac{\lambda}{(1-s_{2j})^{1/2}} [g_{2R}^*(s_{2j})\cos\theta_j - g_{2I}^*(s_{2j})\sin\theta_j] \\ &+ \frac{a}{2N_1} \sum_{i=1}^{N_1} g_1^*(t_{1i})K_{31}^* + \frac{b}{2N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} \\ &\times [g_{2R}^*(t_{2i})(K_{32}^* \cos\theta_i + K_{33}^* \sin\theta_i \\ &+ g_{2I}^*(t_{2i})(K_{33}^* \cos\theta_i - K_{32}^* \sin\theta_i)] = 0, \\ &(j=1, 2, \dots, N_2-1) \end{aligned} \quad (51)$$

$$\begin{aligned} &\frac{a}{N_1} \sum_{i=1}^{N_1} g_1^*(t_{1i}) - \frac{b}{N_2} \sum_{i=1}^{N_2} (1+t_{2i})^{1/2} [g_{2R}^*(t_{2i})\sin\theta_i \\ &+ g_{2I}^*(t_{2i})\cos\theta_i] = 0 \end{aligned} \quad (52)$$

with the unknowns, $g_1^*(t_{1i})$, $g_{2R}^*(t_{2i}(s_{2j}))$ and $g_{2I}^*(t_{2i}(s_{2j}))$, where t and s with subscripts of two indices are given by⁽²¹⁾

$$\left. \begin{aligned} t_{ki} &= \cos\left(\frac{2i-1}{2N_k}\pi\right) \\ s_{kj} &= \cos\left(\frac{j\pi}{N_k}\right) \\ (i &= 1, 2, \dots, N_k; k=1, 2) \\ (j &= 1, 2, \dots, N_k-1; k=1, 2) \end{aligned} \right\} \quad (53)$$

Moreover, $g_{2R}^*(s_{2j})$ and $g_{2I}^*(s_{2j})$ in Eqs.(50) and (51) are approximated by

$$\left. \begin{aligned} g_{2R}^*(s_{2j}) &= \frac{1}{2} [g_{2R}^*(t_{2j}) + g_{2R}^*(t_{2j+1})], \\ g_{2I}^*(s_{2j}) &= \frac{1}{2} [g_{2I}^*(t_{2j}) + g_{2I}^*(t_{2j+1})] \end{aligned} \right\} \quad (54)$$

and an auxiliary condition

$$\sigma_{2x}(0, 0) = 0 \quad (55)$$

is used. Equations (49) to (52) together with Eq.(55) and the second equation in Eq.(47) are N_1+2N_2 algebraic equations with N_1+2N_2 unknowns of $g_1^*(t_{1i})$, $g_{2R}^*(t_{2i})$ and $g_{2I}^*(t_{2i})$. Equation (55) corresponds to Eq. (50) with $s_{2j} = -1$.

The stress intensity factors (42) and (43) are written as

$$k_1(a) = -\frac{E_3}{\sqrt{2}} \frac{\sqrt{a}}{2} g_1^*(1) \quad (56)$$

$$k_1(b) + ik_2(b) = -\frac{a_{11}E_2}{\sqrt{2}} \sqrt{1-\lambda^2} \frac{\sqrt{b}}{\sqrt{2}} \times [g_{2R}^*(1) + ig_{2I}^*(1)] \quad (57)$$

using Eqs.(44) to (48).

4. Numerical Results

Inserting the material constants, loading condition and geometrical parameters into the algebraic equations in Section 3, one can obtain $g_1^*(t_{1i})$, $g_{2R}^*(t_{2i})$ and $g_{2I}^*(t_{2i})$. The stress intensity factors(SIFs) can then be obtained from Eqs.(56) and (57). In this paper the plane-strain condition and a constant pressure P_0 are assumed together with $E_1=E_3=E_5=E_m$, $E_2=E_4=E_f$, $\nu_1=\nu_3=\nu_5=\nu_m$ and $\nu_2=\nu_4=\nu_f$, where the subscripts f and m denote the fiber and matrix, respectively.

First, the present results are compared with the previous solutions for a bifurcation crack in the homogeneous material. The Mode I SIF $k_1'(a)$ at the perpen-

dicular crack tip, and the Mode I and II SIFs, $k_1'(b)$ and $k_2'(b)$, at the interface crack tip are shown in Figs. 2(a) and 2(b), respectively, as a function of the half-crack length ratio b/a with the parameter of Young's modulus ratio E_f/E_m . Here, the superscript ' indicates a non-dimensional SIF by $P_0(a/2)^{1/2}$ or $P_0(b/2)^{1/2}$. Dotted curves⁽¹⁷⁾ were obtained using the body force technique with the perturbation method. Present results for $E_f/E_m=0.9$ and 1.1 bound the dotted curve from both sides and the ones for $E_f/E_m=0.99$ and 1.01 practically coincide with the dotted curve in this figure, which shows good accuracy of the present results.

Figures 3 to 5 present the SIF for typical kinds of composite materials: glass/epoxy (GFRP) in the solid line, carbon/epoxy (CFRP) in the chained line and tungsten/copper(W/Cu) in the dotted line. Their material constants are shown in Table 1.

Figure 3 shows the effect of perpendicular crack length a on the SIF. The SIF $k_1'(a)$ decreases and the absolute values of $k_1'(b)$ and $k_2'(b)$ increase gradually with a increasing. The reinforcing effect of fiber 4 is clear in $k_1'(a)$.

The variation of the SIF as a function of the half-crack length ratio b/a for $a=0.5h_3$ is shown in Fig. 4(a) and the one for $a=0.95h_3$ is in Fig. 4(b). The SIF $k_1'(a)$ increases monotonically and $k_2'(b)$ decreases with b/a increasing. Both curves converge to a certain value and zero for $b/a > 5$. The absolute value of $k_1'(b)$ increases first, shows the maximum and converges to zero for $b/a > 5$. The fiber 4 with high values of E_f/E_m is effective in arresting the propagation of the perpendicular crack near itself (see $k_1'(a)$ in Fig. 4(b)).

Fig. 2 Stress intensity factors of a T-shaped crack in the homogeneous material⁽¹⁷⁾ and along the bimaterial interface; (a) $k_1'(a) \equiv k_1(a)/(P_0\sqrt{a/2})$, (b) $k_1'(b) \equiv k_1(b)/(P_0\sqrt{b/2})$

Fig. 3 Effect of perpendicular crack length a on the stress intensity factors; $k_1'(a) \equiv k_1(a)/(P_0\sqrt{a/2})$, $k_2'(b) \equiv k_2(b)/(P_0\sqrt{b/2})$

Figure 5 shows the effect of the fiber volume fraction on the SIF for various half-interface crack length b using the ratio of fiber width h_2 to fiber space h_3 . The SIF $k'_1(a)$ and $k'_2(b)$ first decrease slightly and converge to certain values with h_2/h_3 , that is, the fiber volume fraction, increasing. The SIF $k'_1(b)$ is almost constant although it shows a somewhat complicated

tendency when $b/h_3=0.1$. The values for $b/h_3=5$ are not presented in Fig. 5(b).

Figure 6 rearranges the effects of E_f/E_m which were shown in Figs. 2 to 5. The SIF $k'_1(a)$ decreases with E_f/E_m increasing and increases with b increasing. There is no meaningful effect of E_f/E_m on $k'_1(b)$ and $k'_2(b)$.

5. Conclusions

This paper analyzed a T -shaped crack elastically, which was bounded by two fibers and was located at the interface in unidirectionally reinforced composite materials. First, the problem was formulated as a system of singular integral equations by means of the Airy stress function and Fourier transforms. Next, a method proposed by the present authors in the preceding paper was applied to evaluate the integral equations numerically without using the Jacobi polynomial⁽¹⁶⁾. Stress intensity factors of a T -shaped crack, that is, Mode I at the perpendicular crack tip $k'_1(a)$ and Mode I and II at the interface crack tip, $k'_1(b)$ and $k'_2(b)$, were obtained. Good accuracy of the present

Table 1 Young's modulus E (GPa) and Poisson's ratio ν

	Fiber			Matrix	
	Glass	Carbon	Tungsten	Epoxy	Copper
E	69.0	232.0	362.0	3.4	130.0
ν	0.22	0.28	0.30	0.36	0.39

Fig. 4 Effect of half-interface crack length b on the stress intensity factors; $k'_1(a) \equiv k_1(a)/(P_0\sqrt{a/2})$, $k'_2(b) \equiv k_2(b)/(P_0\sqrt{b/2})$

Fig. 5 Effect of fiber width to fiber space ratio h_2/h_3 on the stress intensity factors; (a) $k'_1(a) \equiv k_1(a)/(P_0\sqrt{a/2})$, (b) $k'_2(b) \equiv k_2(b)/(P_0\sqrt{b/2})$

Fig. 6 Effect of Young's modulus ratio E_f/E_m on the stress intensity factors; (a) $k'_1(a) \equiv k_1(a)/(P_0\sqrt{a/2})$, (b) $k'_i(b) \equiv k_i(b)/(P_0\sqrt{b/2})$

results was demonstrated by comparing with a previous solution⁽¹⁷⁾ for a bifurcation crack in the homogeneous material. The effects of perpendicular crack length a , half-interface crack length b , fiber width to fiber space ratio h_2/h_3 and Young's modulus ratio E_f/E_m were described, such as:

(1) the value of $k'_1(a)$ decreases due to the approaching fiber and the absolute values of $k'_i(b)$ decrease with a increasing,

(2) the value of $k'_1(a)$ increases and the absolute values of $k'_2(b)$ almost decrease with b increasing; they converge to a certain value and zero, respectively, for $b/a > 5$,

(3) the effect of h_3 on $k'_1(a)$ and $k'_i(b)$ is small for constant h_2 , a/h_3 and b/h_3 ,

(4) the value of $k'_1(a)$ decreases with E_f/E_m increasing, although $k'_i(b)$ is nearly constant.

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