The Propagation Threshold of a Through Crack in a Plate Under the Mixed Mode* (Plane Bending and Cyclic Torsion)

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In order to examine the threshold condition for the propagation of a through crack under the mixed mode, plane bending tests, cyclic torsion tests, and their various combinations are carried out under the condition of the stress ratio $R = -1$ using annealed center-cracked specimens made of carbon steel. Taking account of the frictional stress acting on the crack surface under cyclic torsion, and adopting the strain-energy density criterion, the threshold stress of crack propagation under the combination of plane bending and cyclic torsion can be deduced from the individual test results of plane bending and cyclic torsion.

Key Words: Fatigue, Center-Cracked Specimen, Strain-Energy Density Criterion, Mixed Mode, Crack Propagation Threshold, Plane Bending, Cyclic Torsion

1. Introduction

Fatigue cracks initiated in machine elements or structural members are usually stressed in a mixed mode rather than in a single mode. An inclined center crack under a push-pull load, which produces the mixed-mode of modes I and II, has been mainly used for the investigations on the behavior of crack propagation in a mixed mode⁽¹⁾. However, since cracked plates in the real elements or members are often subjected to the simultaneous action of bending and torsion⁽²⁾, the examination of the propagation behavior of a through or a surface crack under such a loading condition is considered to be an important topic. Recently some experiments concerning the fracture condition in such a mixed mode have been performed⁽³⁾ but the number of systematic experiments

and investigations on fatigue crack propagation are still limited.

When a bending or a twisting moment acts on a plate, the stress gradient occurring in the plate is controlled by the plate thickness, and, except for the case of the stress ratio $R = -1$, the fatigue crack propagations in top and bottom surfaces are different from each other. Also the shape of a crack in its cross section is affected by the stress ratio. These points must be taken into account in the examination of fatigue behavior of cracked plates.

From the above viewpoint, this study examines the threshold condition for the propagation of a through crack in a plate under the combination of plane bending and cyclic torsion in the case of $R = -1$ and for a specified plate thickness.

2. Experimental Procedure

2.1 Test specimen and testing machine

A carbon steel plate SK-5 of 6 mm in thickness was used for the tests. The mechanical properties and the chemical composition are shown in Tables 1 and 2 respectively.

The material was cut in prescribed shapes and

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planed down to 2 mm in thickness, and a starter notch was machined in each plate. The vicinity of the notch was polished with emery paper # 600 and surface-finished electrolytically. The plates were then annealed at 800°C for one hour in a vacuum to remove the work hardening layer.

Figure 1(a) shows an inclined center crack specimen used for pulsating tension ($R \approx 0$) to examine the criterion adopted for the threshold condition in mixed-mode crack propagation. This specimen was cut from the plate having a mode I precrack in such a way that the angle β becomes 30°, 45° or 60°. The precrack was introduced by pulsating tension. Figure 1(b) shows a specimen used for plane bending, cyclic torsion and their various combinations.

A closed-loop electrohydraulic testing machine (60 Hz) was used for pulsating tension tests and a resonance-type fatigue testing machine (98 N-m, 33 Hz) was used for plane bending, cyclic torsion and their combinations.

2.2 Experimental procedure

A through crack with a length of 1~10 mm was made at the notch end in the specimen shown in Fig. 1(b) by plane bending. Inclined center crack specimens in Fig. 1(a) and the specimens in Fig. 1(b) were annealed at 640°C for one hour in a vacuum. Pulsating tension tests with the stress ratio $R \approx 0$ were performed on the specimens in Fig. 1(a). Plane bending, cyclic torsion and their various combinations, at which the angle ϕ between the axis of a torque bar of the testing machine and the direction of the major side of the specimen was assigned to be 30°, 45° or 60°, were performed on the cracked specimens in Fig.

1(b). The number of cycles $N = 1 \times 10^6$ was chosen to examine the crack propagation threshold. The evaluation of the crack propagation was performed by magnifying the specimen replicas 1 000 times before and after the fatigue tests, and the crack propagation of 3 μm in length at the crack tip was considered as the propagation threshold.

3. Experimental Results and Considerations

3.1 Threshold condition for crack propagation under the mixed mode

Before examining the threshold condition for crack propagation under the combination of plane bending and cyclic torsion, investigations were made on the criterion which could be adopted to the threshold condition for crack propagation in the mixed mode of modes I and II.

Figure 2 shows the test results of $R \approx 0$ obtained by using the specimens in Fig. 1(a), where the values of the stress intensity factors in modes I and II were already calculated in the literature⁽⁴⁾. Several curves for the mixed mode were calculated by various criteria^{(5)~(7)}, based on the threshold values of the stress intensity factor range $\Delta K_{I,th}$ in mode I. As seen in the figure, the threshold condition in the mixed mode for the material used can be deduced with good accuracy from the strain-energy density criterion (S_{min} criterion)⁽⁵⁾.

Table 1 Mechanical properties of the material

Yield point	MPa	345
Tensile strength	MPa	712
Breaking strength	MPa	658
Elongation	%	20
Reduction of area	%	33
Young's modulus	GPa	214
Modulus of rigidity	GPa	82
Poisson's ratio	ν	0.3
1GPa=102kgf/mm ²		
1MPa=0.102kgf/mm ²		

Table 2 Chemical composition of the material

	wt. %				
	C	Si	Mn	P	S
SK5	0.86	0.21	0.36	0.007	0.006

(a) Pulsating tension specimen (b) Bending and torsion specimen

Fig. 1 Shape and dimension of specimen (thickness 2 mm)

3.2 Application of the strain-energy density criterion to the threshold condition for propagation of a through crack in a plate under the combination of bending and torsion

When a cracked plate is stressed by a bending moment, mode I crack occurs⁽⁸⁾, but under a twisting moment, combined cracks of modes II and III are observed⁽⁹⁾. Hence, under the simultaneous action of bending and twisting moment, combinations of modes I, II and III are anticipated. However, it is found that the crack propagation in this case occurs mainly in modes I and II.

Figure 3 shows the moment T_0 acting on the specimen with the Cartesian coordinates. The origin of the coordinates is at the crack tip bisecting the plate thickness. The bending and twisting moments acting on the specimen are denoted by M and T , respectively, and M^* and T^* respectively correspond to M and T per unit width of the plate. For the plate with the width b , we have:

$$\begin{aligned} M &= T_0 \sin \phi, \\ T &= T_0 \cos \phi, \\ M^* &= M/b, \quad T^* = T/b. \end{aligned} \quad (1)$$

If the Reissner plate bending theory⁽¹⁰⁾ is adopted to satisfy the boundary condition at the crack edge, the moment and shear forces near the crack tip can be expressed by the moment intensity factors K_I and K_{II} , and the shear force intensity factor K_{III} ⁽¹¹⁾, where K_I , K_{II} and K_{III} are related to the stress intensity factors k_I , k_{II} and k_{III} :

$$\begin{aligned} k_I &= (12z/h^3)K_I, \\ k_{II} &= (12z/h^3)K_{II}, \\ k_{III} &= (3/2h)[1 - (2z/h)^2]K_{III}. \end{aligned} \quad (2)$$

K_I , K_{II} and K_{III} can be expressed by the following equation⁽⁸⁾⁽⁹⁾.

$$\begin{aligned} K_I &= \Phi M^* \sqrt{a}, \quad K_{II} = \Psi T^* \sqrt{a} \\ K_{III} &= -\sqrt{10} \Omega T^* \sqrt{a} / [(1+\nu)h], \end{aligned} \quad (3)$$

where a and h are the half crack length and the plate thickness, respectively, Φ , Ψ and Ω are the functions related to a and h . By using Eqs. (2) and (3)

$$\begin{aligned} k_I &= (12z/h^3) \Phi M^* \sqrt{a}, \quad k_{II} = (12z/h^3) \Psi T^* \sqrt{a}, \\ k_{III} &= -(3/2h)[1 - (2z/h)^2] \sqrt{10} \Omega T^* \sqrt{a} / [(1+\nu)h]. \end{aligned} \quad (4)$$

Using the stress intensity factor range, the strain-energy density factor range ΔS under the combination of modes I, II and III can be expressed as follows.

$$\Delta S = a_{11} \Delta k_I^2 + 2a_{12} \Delta k_I \Delta k_{II} + a_{22} \Delta k_{II}^2 + a_{33} \Delta k_{III}^2. \quad (5)$$

For the plate bending problem, a_{11} , a_{12} , a_{22} and a_{33} are given as follows.

$$\begin{aligned} a_{11} &= (1 + \cos \theta)[3 - \nu - (1 + \nu) \cos \theta] / 8E, \\ a_{12} &= \sin \theta [(1 + \nu) \cos \theta - (1 - \nu)] / 4E, \\ a_{22} &= [4(1 - \cos \theta) \\ &\quad + (1 + \nu)(3 \cos \theta - 1)(1 + \cos \theta)] / 8E, \\ a_{33} &= (1 + \nu) / 2E. \end{aligned} \quad (6)$$

If the moment range is denoted by ΔT_0 , Eq. (5) becomes:

$$\begin{aligned} \Delta S &= \frac{2a}{E} \left[\frac{3\Phi \Delta T_0}{bh^2} \right]^2 \left\{ \left(\frac{z}{h} \right)^2 F(\theta, \phi) \right. \\ &\quad \left. + \left[1 - \left(\frac{2z}{h} \right)^2 \right]^2 G(\phi) \right\}, \end{aligned} \quad (7)$$

with the functions $F(\theta, \phi)$ and $G(\phi)$ being defined as:

$$\begin{aligned} F(\theta, \phi) &= b_{11} \sin^2 \phi + 2b_{12} \lambda \sin \phi \cos \phi + b_{22} \lambda^2 \cos^2 \phi, \\ G(\phi) &= 5\gamma^2 \cos^2 \phi / [8(1 + \nu)], \\ b_{ij} &= 8Ea_{ij} (i, j = 1, 2), \quad \lambda = \Psi/\Phi, \quad \gamma = \Omega/\Phi. \end{aligned} \quad (8)$$

Since the stress intensity factors vary with the position z , it is necessary to know this position where crack propagation first occurs. According to the $S_{\min}$ criterion, crack propagation occurs when $\Delta S_{\min}$ becomes the critical value S_c peculiar to the material. The relation between z and the allowable moment range ΔT_0 that a cracked plate can sustain without crack propagation at the position z is given as follows.

Fig. 2 Threshold conditions for crack propagation in the mixed mode

Fig. 3 The coordinates and the moment acting on the specimen

$$\Delta T_0 = \frac{b\sqrt{h/a}(Eh^3\Delta S_{\min})^{1/2}}{3\sqrt{2}\Phi\{(z/h)^2F(\theta, \phi)_{\min} + [1 - (2z/h)^2]G(\phi)\}^{1/2}}, \quad (9)$$

where $F(\theta, \phi)_{\min}$ satisfies the following equation:

$$\partial F/\partial \theta = 0, \quad \partial^2 F/\partial^2 \theta > 0. \quad (10)$$

In Eq. (9), $\Delta S_{\min}$ is equal to S_c and both $F(\theta, \phi)_{\min}$ and $G(\phi)$ are independent of z . Since the minimum value of ΔT_0 can be obtained at $z = \pm h/2$, the crack propagation always initiates in the surface layer of the plate irrespective of the ϕ value, and a crack propagates in the direction in which the minimum value of ΔS is attained⁽¹¹⁾. Therefore, in this case, it is obvious that $K_{III} = 0$ from Eq. (4). With this understanding, in the next section we will examine the crack propagation at the specimen surface and consider the threshold condi-

tion for mixed-mode crack propagation.

3.3 Threshold stress for crack propagation under the combination of plane bending and cyclic torsion

The threshold stress amplitude curves of crack propagation for $R = -1$ are shown in Figs. 4(a) and (b). $\Delta\sigma$ and $\Delta\tau$ are the maximum bending stress amplitude and the maximum shearing stress amplitude occurring in the rectangular cross section by ΔM^* and ΔT^* respectively. Namely,

$$\begin{aligned} \Delta\sigma &= \Delta M^*/(h^2/6), \\ \Delta\tau &= \Delta T^*/(\xi h^2), \end{aligned} \quad (11)$$

where ξ equals 0.327 for the specimen adopted in this test. Since $\Delta\tau$ takes a constant value except for the area near both ends of the plate corresponding to 5% of the plate width⁽¹²⁾, it is rational to adopt $\Delta\tau$ as a nominal stress. In the figure, calculated values of ΔS for $\phi = 0^\circ$ and 90° are shown, which were obtained using a threshold stress of $2a = 1.5$ mm in mode I and numerical values of Φ and Ψ . It is found that the calculation for $\phi = 90^\circ$ almost agrees with the test result but that for $\phi = 0^\circ$ is lower than the test results.

One example of the crack propagation from a precrack which occurred under a higher stress amplitude than the threshold stress is shown in Fig. 5. Since the precrack was made by plane bending ($\phi = 90^\circ$), the precrack from the notch initiates at the plate surface, and since the periphery of the precrack in its cross section has a curvature, the length of the precrack becomes maximum at the plate surface. Besides, even if ϕ is different from 90° , the length of the crack propagated from the precrack becomes maximum at the plate surface (Fig. 5). This fact seems to suggest that the initiation of crack propagation from the precrack is in the surface layer of the plate as mentioned before.

3.3.1 Estimation of the threshold stress of crack propagation As is stated above, K_{III} is thought to be zero at the beginning of crack propagation. Therefore, using the numerical solution⁽⁸⁾⁽⁹⁾ for

Fig. 4 Threshold stress amplitude curve of crack propagation

Fig. 5 One example of the state of crack propagation ($\phi = 30^\circ$)

the functions Φ and Ψ in Eq. (3), the angle θ which makes the ΔS in Eq. (7) minimum at $z = \pm h/2$ was calculated numerically, using Eq. (9). Then using the value thus obtained and the test result for $\phi = 90^\circ$, the threshold stress amplitude for crack propagation in other angles ϕ was calculated by the $S_{\min}$ criterion. The results are shown in Fig. 6. The calculated value for each ϕ is lower than the experimental value; the smaller the angle ϕ , the larger is the difference between them. This means that the estimation of the threshold condition by the $S_{\min}$ criterion is irrelevant when the bending and twisting moment act simultaneously on the plate with a through crack. However, it was already confirmed that the test results of an inclined center crack in the mixed mode of modes I and II with the stress ratio $R \approx 0$ can be explained by the $S_{\min}$ criterion. Therefore the reason why the above result is obtained should be discussed.

Since an inclined center crack maintains its opening through stress cyclic in the test with $R \approx 0$, it is thought that frictional force hardly acts on the crack surface in this case. On the other hand, if a round bar with a micro surface crack is stressed under the combination of bending and twisting moments, frictional stress due to crack closure must be taken into account for the estimation of the threshold stress of crack propagation⁽¹³⁾⁽¹⁴⁾. Furthermore, unlike a round bar, warping occurs in the plate with a rectangular cross section due to the twisting moment, and a precrack zigzags as shown in Fig. 7. Such warping is considered as one of the main causes of crack closure, and, together with the roughness of crack surfaces,

Fig. 6 Calculated values based on the test result for angle $\phi = 90^\circ$ and experimental values of the threshold stress in crack propagation

hinders a free displacement in mode II. Thus, frictional force acts on the crack surface in the mixed mode. Also, due to the curved periphery of a precrack, the real values of Φ and Ψ are themselves different from the numerical solutions. Such a frictional force and a curved periphery of a crack certainly produce the difference between the experimental threshold value and the calculated one. However, it is evident from Eqs. (1), (4) and (7) that the ratio of Δk_{II} to Δk_I increases with the decreases of ϕ , and Δk_{II} itself decreases due to the frictional force. It is therefore concluded that the increase of the difference between the experimental value and the calculated one as ϕ decreases is mainly due to the influence of frictional stress which obstructs the crack propagation.

3.3.2 Estimation of the threshold stress for crack propagation taking account of the frictional force In this section, using the test results of $\phi = 0^\circ$ and 90° and taking account of both the curvature of the crack periphery and frictional force, we estimate the threshold stress for crack propagation in the cases of $\phi = 30^\circ, 45^\circ$ and 60° .

Since the periphery of a precrack in this test has a curvature, functions Φ and Ψ for such a crack will differ slightly from those of numerical solutions. Also, the frictional force due to a twisting moment decreases the effective displacement and stresses in the vicinity of a crack tip. This causes a drop of the value of function Ψ obtained by the numerical solution which does not consider the frictional force. Concerning this point, although the frictional force in cyclic torsion may be different from that in the combination of the plane bending and cyclic torsion⁽¹⁴⁾, it was assumed that the degree of drop in function Ψ due to frictional force in cyclic torsion can be applied to other cases.

If the curvature and frictional force are taken into account, the use of the $S_{\min}$ criterion to the threshold condition for mixed mode crack propagation is possible. Hence, denoting the minimum values of a_{11} and a_{22} as $a_{11\min}$ and $a_{22\min}$ respectively and using Eqs.

Fig. 7 The occurrence of precrack

Fig. 8 Calculated values of threshold stress for crack propagation using the test results for angles $\phi=0^\circ$ and 90°

(5) and (7), it follows that :

$$a_{11\min}\Delta k_I^2 = a_{22\min}\Delta k_{II}^2, \quad (12)$$

provided that functions Φ^* and Ψ^* in k_I and k_{II} are different from the numerical solutions. Though these values are unknown, denoting the threshold stress amplitude for $\phi=90^\circ$ and 0° as $\Delta\sigma_{th}$ and $\Delta\tau_{th}$ respectively, and using Eqs. (2), (3), (6), (11) and (12), the relation between Φ^* and Ψ^* can be deduced :

$$\left(\frac{\Psi^*}{\Phi^*}\right)^2 = \lambda^2 = \frac{a_{11\min}(\Delta\sigma_{th})^2}{a_{22\min}(6\xi\Delta\tau_{th})^2}. \quad (13)$$

Therefore, using the threshold stress amplitudes for $\phi=90^\circ$ and 0° in Fig. 4 and Eq. (13), the value of λ can be calculated for the optional crack length taking the effects of the curvature and frictional force into consideration.

Denoting the threshold stress under the combination of bending and torsion as $\Delta\tau_{th}^*$ and using Eqs. (1), (6), (7) and (11), it follows that :

$$\begin{aligned} (\xi\Delta\tau_{th}^*)^2 F(\theta, \phi)_{\min} / \cos^2 \phi \\ = (\Delta\sigma_{th})^2 (1-\nu)/9. \end{aligned} \quad (14)$$

Using the λ value in Eq. (13) and determining θ which satisfies the condition of making $F(\theta, \phi)$ in Eq. (9) a minimum, $F(\theta, \phi)_{\min}$ for the optional crack length can be calculated by Eq. (8), and $\Delta\tau_{th}^*$ for each ϕ in the mixed mode can be obtained by Eq. (14).

Test results and calculated curves by Eq. (14) are shown in Fig. 8. As seen in the figure, using Eq. (13) which takes into account the curvature of a crack and frictional force and Eq. (14), the threshold stresses for crack propagation under the combinations of plane bending and cyclic torsion can be predicted with sufficient accuracy.

4. Conclusions

A through precrack was made in a plate of carbon steel by plane bending, and plane bending tests, cyclic torsion tests, and their various combinations were carried out, in order to examine the threshold condition for the crack propagation under mixed mode. The results obtained are summarized as follows.

(1) The strain-energy density criterion can be applied successfully to the threshold condition for the propagation of an inclined center crack under the combination of modes I and II with the stress ratio $R \approx 0$.

(2) If the $S_{\min}$ criterion is used to estimate the threshold stresses for crack propagation under the various combinations of bending and torsion, the values obtained by using the test results in plane bending and the stress intensity factors in modes I and II based on numerical solutions are lower than the experimental ones. This is mainly due to the frictional force acting on a crack surface under the action of a twisting moment on the plate.

(3) In the application of the $S_{\min}$ criterion, it is necessary to find the relation between the correction factor of the stress intensity factor in mode I and that in mode II by taking the effect of the curvature and the frictional force into account. This relation can be determined from the test results of plane bending and cyclic torsion.

(4) Using the above relation combined with the $S_{\min}$ criterion, the threshold stress of crack propagation under the combination of bending and torsion can be deduced with good accuracy from the plane bending and cyclic torsion test results.

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