The Elastic-Plastic Snapping-Through of a Curved Metal Strip Compressed Between Two Rigid Plates*  
(The Influence of the Supported End Condition on the Snap-Through)

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The quasi-static loading of a stress-free, initially curved strip, compressed by a flat, rigid plate is considered. The deformation characteristics have been analyzed using an incremental finite element technique. The process involves large displacements, and the strip finally buckles as the central region begins to move away from the plate; snap-through can occur very suddenly. The influence of the end support condition of the strip on the buckling behavior has been analyzed. The assumed end conditions were: rigidly clamped, pinned, and simply supported; as might be anticipated, buckling is not predicted for simply supported ends. It was found that at snap-through, the normalized force for the clamped end condition was larger than that for the pinned end, and the amount of plate displacement was smaller than that of the pinned end support.

Key Words: Plasticity, Contact Problem, Buckling, Snapping, Numerical Analysis, FEM

1. Introduction

In a previous paper(1), the authors performed a theoretical and experimental study of the deformation characteristics of an initially curved strip compressed by a rigid, flat plate. All the strips had been deformed to a constant radius curvature and stress relieved before testing. The experimental arrangement employed is shown schematically in Fig. 1(a). The ends of the strip were located in a notched plate, see Fig. 1(b), to simulate a pinned end condition. The span, L, and midheight, H0, were constant for all the tests. Experiments were performed on low-carbon steel and 5182-O aluminum strips of different thickness, t, but of constant width, b.

The deformation characteristics were then analyzed using an incremental finite element technique. The curved strip was divided into 24 beam-type elements and each beam element subdivided into 16 equal layers across the thickness; the curvature of the individual beam elements was ignored. The materials were assumed to be elastic-plastic, and the plastic portion of the stress-strain curve to obey an empirical power law equation of the following form: \( \sigma = \sigma_0 (1 + ce^p)^n \). The material parameters, \( \sigma_0, c, \) and \( n \) were obtained by establishing the best fit to experimental data from tensile tests.

While performing the calculations provided in ref.(1), it was recognized that it was necessary to control the

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Fig. 1 (a) Experimental arrangement  
(b) Pinned end condition for the test piece
increment of plate displacement, in order to determine accurately the region of the strip currently in contact with the plate. As described in detail, op cit, certain control parameters were used in conjunction with the finite element analysis to determine the contact zone, the yielding of an element and the onset of snap-through. The predicted load-deflection characteristics of each strip, and also its shape in the pre- and postbuckled modes, agreed well with the experimental results from the steel and aluminum test pieces.

Close agreement between theory and experiment is not routine in contact buckling problems of this type. Several articles have appeared on the deformation characteristics of shells (usually axisymmetric) and curved, thin-wall members impressed by rigid plates; a representative selection is given in refs.\(^{(12-19)}\). In general, the degree of correspondence between the predicted and observed behavior (in such cases where a comparison was possible) has not been very favorable.

The present paper is a slight extension of the work described in ref.\(^{(11)}\), the same numerical procedure is followed to assess the influence of the end support condition of the arch on the buckling behavior. A theoretical study only has been conducted, assuming either rigidly clamped, simply supported or pinned ends. For the latter condition, some of the calculations given in ref.\(^{(11)}\) are reproduced for comparison.

2. A Synopsis of the Theory Presented in ref.\(^{(11)}\)

2.1 The incremental finite element method for large displacements of an elastic-plastic arch

2.1.1 Velocity and strain rate of a beam element

The location of each element is specified by a global rectangular cartesian coordinate system, \(O-XY\). In addition, a local rectangular cartesian system, \(\sigma-xy\) is adopted as shown in Fig. 2, directed along and normal to the middle surface of a typical element, \(ij\), of length \(l\).

If \(\dot{u}, \dot{v}\) and \(\dot{\delta}\) are the velocities along the \(x\) and \(y\) directions and the rate of rotation, respectively, it follows that in the local coordinate system

\[
\begin{bmatrix}
\dot{u} \\
\dot{v}
\end{bmatrix} = [A] \begin{bmatrix}
\dot{\delta}
\end{bmatrix}
\]

(1)

where

\[
\begin{bmatrix}
\dot{\delta} \\
\dot{u}, \dot{v}, \dot{\vartheta}, \dot{\psi}
\end{bmatrix}^T
\]

(2)

and the matrix \([A]\) is a function of \(x, y\) and \(l\) and is given in ref.\(^{(11)}\).

Assuming that Kirchhoff’s hypothesis for plate bending applies to the velocities, and recognizing that the strip is thin, the relationship between the strain rate \(\dot{\varepsilon}\) and the velocity \(\{\dot{\delta}\}\) expressed in matrix form, is

\[
\dot{\varepsilon} = [B] \begin{bmatrix}
\dot{\delta}
\end{bmatrix}
\]

(3)

the matrix \([B]\) is given in ref.\(^{(11)}\), and is a function of \(x, y\) and \(l\).

2.1.2 Equilibrium between the nodal forces and stress

The nodal forces and the stress acting on an element are denoted by \(\{f\}\) and \(\sigma\), respectively, in the local coordinate system. The nodal force is made up of components \(p\) and \(q\), acting in the \(x\) and \(y\) directions, respectively, along with a bending moment \(m\); the positive directions are indicated in Fig. 2.

Upon applying the divergence theorem\(^{(10)}\) in continuum mechanics to a strip element, \(ij\), it can be shown that

\[
\{f\} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \rho \dot{u} \dot{v} \text{d}x \text{d}y
\]

(4)

where

\[
\{f\} = \{p, q, m, p_0, q_0, m_0\}^T
\]

(5)

In a similar manner to the foregoing, the nodal force \(\{F\}\) in the global coordinate system is composed of components \(P, Q\) and \(M\). The nodal force \(\{F\}\) is calculated through the transformation matrix \([T]\) as follows.

\[
\{F\} = [T] \{f\} = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \rho \dot{u} \dot{v} \text{d}x \text{d}y
\]

(6)

2.1.3 The stiffness equation in the global coordinate system

The system stiffness equation is obtained by taking the material derivative of Eq. (6), allowing for the elastic compressibility of the beam-type elements \((11,11)\). Hence,

\[
\{F\} = [T] \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \rho \dot{u} \dot{v} \text{d}x \text{d}y
\]

\[
+ \{T\} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \rho \dot{u} \dot{v} \text{d}x \text{d}y
\]

\[
+ \{T\} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \rho \dot{u} \dot{v} \text{d}x \text{d}y + \left(1 - \frac{2\nu}{E}\right) \{T\}
\]

\[
\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} [B]^T \sigma \dot{u} \dot{v} \text{d}x \text{d}y
\]

(7)

As shown in ref.\(^{(11)}\), the preceding equation can also be expressed as

\[
\{F\} = ([K_t] + [K_s] + [K_s] + [K_s]) \{\dot{\delta}\} = [K] \{\dot{\delta}\}
\]

(8)
where
\[
(F) = (\dot{P}, \dot{Q}, \dot{M}, \ddot{P}, \ddot{Q}, \ddot{M})^T
\]
\[
(D) = (U, \dot{U}, V, \dot{V}, \dot{\Theta}, \ddot{\Theta})^T
\]
(9)
(10)

2.2 Analysis of a contacting node

The contacting nodes are shown schematically in Fig. 3, and in the absence of friction the prevailing condition is \( \dot{\Theta} = 0 \) at a nodal point, but it follows that the incremental moment \( \dot{M} \) is generally not zero. Then, it is assumed that a contacting node behaves like a frictionless needle bearing, with the following boundary conditions
\[
\dot{V} = \ddot{S}, \quad \dot{P} = \dot{M} = 0 \quad \text{for} \quad Xc1 \leq X \leq Xc2 \quad (11)
\]

2.3 Control of the plate stroke

As already mentioned, it was recognized in ref.(1) that the plate displacement had to be regulated in order to detect the onset of the various events which could occur in any given increment of stroke. A number of control parameters were introduced to achieve this. As an example, the following criterion was invoked to assess whether a node was contacting the plate.

Let \( Y_0 \) specify the current position of the plate, see Fig. 3, and \( dS \) the displacement in the next increment, define \( Y_a \) and \( dV_a \) as corresponding quantities for a typical node, \( k \), in the strip. If contact occurs, then \( (Y_0 + dS) \leq (Y_a + dV_a) \) and the contact control parameter \( \eta_c \) is defined as follows:
\[
\eta_c = (Y_a - Y_0)/(dV_a - dS)
\]
(12)
The positive minimum value is denoted as \( \eta_{cm} \).

Similar parameters were introduced to test for the onset of separation of a node from the plate, the onset of yielding (either initial or subsequent) of a beam element and to limit the displacement of the node at the midspan of the strip at snap-through. The latter parameter served to avoid catastrophic buckling. The minimum of the individual control parameters was used to regulate the step size of the plate displacement\(^{11}\).

3. Calculated Results and Discussion

3.1 Strip geometry, material characteristics and end support conditions

Each strip was assumed to have the following dimensions, refer to Fig. 1 (a).

- Initial radius, \( R = 921 \) mm
- Height at midspan, \( H_0 = 12.7 \) mm
- Span, \( L = 304.8 \) mm
- Width, \( b = 76.2 \) mm
- Initial thickness, \( h = 1.22 \) mm

The material was assumed to be mild steel with the following properties:
- Young's modulus, \( E = 207 \) GPa
- Yield stress, \( \sigma_y = 207 \) Mpa
- Poisson's ratio, \( \nu = 0.3 \)

Plastic portion of the stress-strain curve given by
\[
\sigma = 207(1 + 300 \delta^{0.3}) \quad \text{MPa}
\]
The three support conditions investigated were:

(a) pinned ends; (b) rigidly clamped ends; (c) simply supported ends. The boundary conditions for the end nodes for each of the three cases are,

(a) \( \ddot{U} = \ddot{V} = \dot{M} = 0 \) pinned

(b) \( \ddot{U} = \ddot{V} = \dot{\Theta} = 0 \) clamped (13)

(c) \( \ddot{V} = \ddot{P} = \dot{M} = 0 \) simply supported

3.2 Influence of the support condition on the deformation characteristics

3.2.1 Predicted load-displacement relationship and deformed shape of the arch

The influence of the end support on the load-displacement characteristics is shown in Fig. 4. The curves have been normalized by dividing the displacement \( S \) by the midheight \( H_0 \), and the plate load, \( W \) by the limited quantity \( (Dh_0L^2) \), where \( D \) is the bending stiffness, i.e., \( D = Eb^2/[12(1 - \nu^2)] \). The corresponding shape of the arch as a function of plate displacement is illustrated in Figs. 5(a) to (c). It is clear that clamping the ends provides the stiffest condition, snap-through occurs after the least amount of plate displacement but at the highest plate force. The central span of material over which snap-through takes place is also the smallest. The simply supported strip shows the least resistance to deformation and catastrophic snap-through does not occur. The calculations predict that near the end of the stroke the strip is starting to move away from the central region of the plate.

3.2.2 Calculated contact nodal force and stress distribution

Figures 6(a) to (c) depict the influence of end support conditions on the contact nodal force as a function of plate displacement. It was found convenient to represent the contact force as a nondimensional quantity by dividing the nodal load, \( Q \), by the expression \( (Dh_0L^2) \). As can be seen from Fig. 6, the

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Fig. 3 Frictionless finite element nodes in contact with the plate (needle bearing model)
distance between the contact point and the midspan of the arch is the smallest with the clamped edge support. The strip is not contacting the plate in this region.

The calculated longitudinal stress distribution across the thickness of the strip is shown in Figs. 7(a) to (c) for the three end conditions. These diagrams also illustrate how the stress varies as a function of the plate stroke, and also with the position along the length of the strip. All the stress quantities have been normalized with respect to the initial yield stress, \(\sigma_Y\), and Bauschinger effects have been ignored.

The strips with the clamped and pinned ends reveal the superposition of a bending and axial stress, as can be seen from the shift in the neutral axis across the thickness. For these two end conditions, regions of the strip were deformed plastically, in fact yielding occurred very early in the deformation process when the ends were clamped. There is also a reversal in sign of the stress when moving along the length of the strip from the edge to the center. It will also be observed that the calculations imply that regions of the strip (in the vicinity of the supports), will unload to some extent once snap-through occurs.

Very little needs to be said about the simply supported strip. A bending moment is generated to flatten the strips, and this decays to zero at the ends. With the chosen geometry, yielding does not occur.

3.2.3 The influence of certain material parameters on the deformation characteristics Calculations were performed to assess the influence of the material thickness, \(b_0\), the initial yield stress, \(\sigma_Y\), and the strain-hardening index, \(n\), on the buckling behavior. Only the pinned and clamped end conditions were considered. As shown in Fig. 8(a), increasing the thickness delays the onset of snap-through and lowers the normalized plate force at which this occurs. The actual plate load, \(W\), will naturally increase with increasing thickness. Figure 8(b) demonstrates that decreasing the initial yield stress has the same effect as increasing the strip thickness with regard to snap-through. The influence of the \(n\) value is not so strong, as is evident from Fig. 8(c).

4. Conclusions

It has been demonstrated herein that the incremental finite element technique, first proposed in ref. [11], has the flexibility to account for a range of material and process variables in the analysis of an
Contact nodal force scale \( Q/(D_b H_b/L) \)

Fig. 6 Predicted contact nodal force as a function of plate displacement

Stress scale \( 0/\sigma_0 \)

Fig. 7 Predicted longitudinal stress distribution across the thickness of the strip as a function of plate stroke and distance along the strip

initially curved strip compressed by a flat, rigid platen. Most of the major variables have been considered in the present study, which has revealed that the stiffest system is achieved when the ends of the strip are rigidly clamped.

While the gross effect of certain material and
processing parameters might have been anticipated, some rather interesting features were revealed about the evolution of the distribution of the longitudinal stress as a function of plate displacement for the three support conditions. The work of ref. (10) has already demonstrated the ability of the numerical model to reproduce experimental data with good accuracy, consequently it is felt that some confidence can be placed in the numerical predictions presented here.

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References

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