An Analysis of Axisymmetric Sheet Forming by the Shell Finite Element Method*
(The Case Where Gotoh’s Plastic Constitutive Equation and Fourth-Order Yield Function are Used)

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An elastic-plastic finite element formulation on the basis of the shell theory is given for the analysis of axisymmetric sheet forming, such as hydraulic bulging, stretching with a rigid punch and so forth. Gotoh's plastic constitutive equation with an approximation which is a kind of vertex-hardening theory and his fourth-order yield function for a precise expression of anisotropy are used, as well as the conventional $J_2$-flow theory. Several numerical examples are presented with the aid of a newly developed computer program in which isoparametric finite elements are used in order to conserve the continuity of the displacement and the slope across the element boundaries. Several effects of the $X$-value and the vertex, as well as the conventional $\theta$- and $n$-value, on the deformation of the shell are demonstrated.

** Key Words:** Plasticity, Shell, FEM, Axisymmetric Problem, Constitutive Equation, Sheet Forming, Anisotropy, Fourth-Order Yield Function

1. Introduction

In the past, the analysis of sheet-metal forming has been performed mainly on the basis of the 'membrane theory' because of its simplicity. However, the 'shell theory' has also been utilized recently more and more, because it can take into consideration the bending and thickness effects to a comparable extent without loss of computational merit due to 'thinness'. When the shell theory is used, the analysis cannot but be 'elastic-plastic' (and not 'rigid-plastic'), and therefore the spring-back and the residual stress after forming are also able to evaluate, which is another advantageous point of this theory. We concern ourselves here exclusively with the axisymmetric cases. Among others, Yokouchi used the Eulerian coordinates, and Nagai and Nakamachi used the coordinates embedded in the shell. When the embedded coordinates are used, the tensor algebra is inevitably applied and thus the overlooking of the variables to be involved will be certainly avoided. However, it is rather difficult for the workers unfamiliar with such special algebra to understand the formulations. In Nagai's analysis, FDM (Finite Difference Method) is applied with consideration of the shearing effect, at which point his analysis is superior to the others. However, in all these analyses, isotropy of the material is assumed, and the constitutive equation adopted is almost the classical $J_2$-flow theory $(J_2F)$.

In this paper, the anisotropy of the sheet material which is well known to play an important role in the sheet metal forming, is taken into consideration by the use of the fourth-order yield function instead of Hill's quadratic one which was previously proposed by one of the authors (Gotoh). Furthermore, to check the effect of the vertex-evolution on the subsequent loading surface on deformation, a kind of vertex-hardening constitutive equation (MG c.e.) is used, which was also previously proposed by Gotoh. The procedure of combination of the fourth-order yield function with MG c.e. is not self-evident. And MG c. e. is non-linear with respect to the equivalent strain increment. One of the main objectives of this paper is
to explain how to deal with these difficulties. The classical $J_1F$ and $J_2D$ ($J_2$-Deformation theory) as well as Hill's quadratic anisotropy\(^{(11)}\) are all included as the special cases of the formulation here. The effect of the $\bar{X}$-value (=equi-biaxial tensile yield stress/average uniaxial tensile yield stress, at the same equivalent strain) as well as those of the conventional $n$-value (=strain-hardening exponent) and $\bar{F}$-value (=average ratio of breadth strain to thickness strain in uniaxial tension) on deformation can also be studied, because the fourth-order yield function involves all these material parameters.

In this paper, we use the FEM (Finite Element Method) for the numerical analysis. The formulation of FEM is performed on the basis of the shell theory in the Eulerian coordinates, using the third-order isoparametric elements in order to conserve the continuity of the displacement and the slope across the element boundaries, though the shearing effect considered in Nagai's analysis is neglected, because a rather thinner shell is concerned with.

The computer program for numerical calculation is newly developed. The numerical examples are primarily presented for hydraulic bulging of the sheet. A few examples of punch-stretching are also presented.

2. Fundamental Equations

2.1 Constitutive equation

The constitutive equation used here is based on the first order form\(^{(9)}\) of the general MG c.e.\(^{(11)}\) with vertex-hardening effect, which is extended to involve the initial anisotropy\(^{(12)}\) expressed by the fourth-order yield function\(^{(6)(7)}\). That is, it is written down in the following simplified form:

\[
d\sigma = d\sigma^e + d\sigma^p \tag{1}
\]

\[
d\sigma^e = \dot{d}T^e/2G \tag{2}
\]

\[
d\sigma^p = (P(\Theta^*) \left[ 1 - \frac{1}{H_0} \right]) (T^*/\dot{\sigma}) d\dot{\sigma}^e \tag{3}
\]

\[
k_0 = (d\sigma^e / d\dot{\sigma}) / 3 \tag{4}
\]

\[
k_0 / H_0 = a^* - \cos \Theta^* / (1 + \cos \Theta^*) \tag{5}
\]

\[
b^* = 1 - a^* \tag{6}
\]

\[
\Theta^* = \frac{2}{3} \rho^2 ; \quad \varepsilon = \int d\dot{\sigma} \tag{7}
\]

\[
\langle P \rangle = P(P > 0) ; \quad 0 \leq P \leq 0 \tag{8}
\]

\[
\cos \Theta^* = a^* + b^* \cos \Theta^* \tag{9}
\]

\[
\theta = 3/\sqrt{2} \left[ \text{tr} (T^* T^{**}) / \sigma^e \right] \tag{10}
\]

\[
\dot{d}T^* = dT^* - d\omega T^* + T^* d\omega \tag{11}
\]

\[
dT^* = (\partial T^* / \partial \sigma) : d\sigma \tag{12}
\]

\[
\dot{d}u^p = \sqrt{\frac{3}{2}} (\text{tr}(d\sigma^p))^3/2 \tag{13}
\]

The fourth-order yield function (in-plane isotropy case):

\[
\phi^* = \sigma^* + \bar{A}_1 (\sigma^e \sigma^e + \sigma_a \sigma_a) + \bar{A}_3 \sigma^e \sigma^e + \sigma^e \tag{14}
\]

\[
\bar{A}_1 = -4 \bar{F} / (1 + \bar{F}) \tag{15}
\]

\[
\bar{A}_3 = (\bar{X})^2 - 2 \bar{A}_1 - 2 \tag{16}
\]

\[
\sigma^* = \frac{\sqrt{3}}{3} \left[ (\partial \sigma^* / \partial \sigma) \right] \left[ (\partial \sigma^* / \partial \sigma) \right]^{1/2} \tag{17}
\]

\[
\frac{\partial \sigma^*}{\partial \sigma} = A_1, \quad \frac{\partial \sigma^*}{\partial \sigma} = A_3, \quad \frac{\partial \sigma^*}{\partial \sigma} = A_3 \tag{18}
\]

\[
\begin{align*}
A_1 = - (A_1 + A_2) \\
\| \partial \sigma^* / \partial \sigma \| = (\text{tr}(\partial \sigma^* / \partial \sigma))^2 = A_1 \\
T^*/\Theta^* = k_1 \tag{19}
\end{align*}
\]

\[
T^*/\Theta^* = k_1, \quad T^*/\Theta^* = k_1 \tag{20}
\]

In the above equations, $d\sigma$ = deviatoric strain increment, $\dot{d} = \text{objective increment}$, $T^* = \text{deviatoric stress}$, $\sigma^* = \text{Cauchy stress}$, $\sigma_a^* = \text{elastic shear modulus}$, $\Theta^* = \text{half angle of the vertex cone at the loading point on the subsequent loading surface}$, $h_0 = \text{instantaneous strain-hardening coefficient}$ which is determined by Eq. (4) using the stress-strain curve obtained by uniaxial test, $h_0 = \text{instantaneous vertex-hardening coefficient}$, $\rho = \text{a new material constant which governs the vertex-evolution rate}$, $d\omega = \text{instantaneous rigid-body rotation increment}$, superscript $e = \text{elasticity symbol}$, superscript $p = \text{plasticity symbol}$, $\text{tr} = \text{trace operator such as tr} A = A_{ii}$, $T^* = \text{current stress point in the anisotropic deviatoric stress space}$.

Eqs. (14)-(20) are the expressions mostly for the axisymmetric case. During deformation, whether the plastic state of a point will continue or not is judged by the following equations.

The condition for the continuation of plastic state:

\[
0 \leq \Theta^* \leq \Theta_{\max}^* \tag{21}
\]

where $\Theta^*$ is defined by Eq. (9) which denotes the angle between $T^*$ and $\dot{d}T^*$ in the anisotropic deviatoric stress space.

The final form of the elastic-plastic constitutive
equation takes the following expression:

\[ \delta \sigma = \delta D \delta e - \sigma_0 \delta e \]  
(22)

where \( \delta \sigma = (\delta \sigma_x, \delta \sigma_y, \delta \sigma_z) \) and \( \delta e = (\delta e_x, \delta e_y) \) are the strain increment, \( \sigma_0 = \text{CDT}_0 \), \( C = \text{a scalar coefficient, } \delta e = \text{equivalent strain increment, and the coefficient matrix } D \text{ includes stresses, } G, H_0, \) \( \rho_0 \), and so forth. In the above equation,

\[ \delta e = \sqrt{3/2 \text{tr}(\delta e^2)} \]
(23)

Therefore, when the formulation of FEM using Eq. (22) is made, the stiffness equation to be solved becomes non-linear with respect to the nodal displacement increments. In that case, we have to use an iterative method such as the Newton-Raphson method for each incremental step of deformation. If we adopt the implicit Eulerian scheme for the time integration\(^{(33)}\), then an iteration is needed inevitably for each step and thus the drawback just mentioned will give no additive nuisance. However, for convenience of calculation, we adopt the explicit Eulerian scheme in this paper. Therefore, to avoid such a drawback, we introduce the following piecewise linear constitutive equation whose basis is Eq. (22).

When we examine closely Eqs. (1) to (3) and (22), we find that the coefficient matrix \( D \) can be thought of as the elastic coefficient matrix with the effect of the vertex hardening in plasticity. Therefore, by replacing the elastic coefficient matrix \( D^0 \) by \( D \) in the following usual elastic-plastic piecewise linear constitutive equation\(^{(48)}\), we can get simply another constitutive equation which is piecewise linear together with a vertex-hardening effect which develops in the similar manner to that in MG c.e.

\[ \delta \sigma = \left[ D^0 - D \right] \delta e \]
(24)

\[ H^* = \frac{\delta \sigma^*}{\delta e^*} \cdot \frac{1}{\sigma^*} - \frac{\delta \sigma^*}{\delta \sigma} \]

The constitutive equation thus obtained reduces to \( D^0 \) when isotropy is assumed with \( H_0 \rightarrow \infty \). As for isotropy, we can get the corresponding constitutive equation by replacing \( G, E = \text{Young's modulus and } \nu = \text{Poisson's ratio with } G^* = 1/(1/\rho + <P(\theta^*)>/<H_0), E^* \) and \( \nu^* \), respectively. The coefficient matrix \( D \) in Eq. (22) and \( E^* \) and \( \nu^* \) are given as follows:

\[ D = \frac{2}{D^0} \times \]

\[ \left[ \begin{array}{ccc} \frac{2}{G} + \frac{2}{3K} + 3P^* & \frac{1}{G} - \frac{2}{3K} - 3P^* & \frac{1}{G} - \frac{2}{3K} - 3P^* \\ \frac{2}{G} - \frac{2}{3K} - 3P^* & \frac{2}{G} + \frac{2}{3K} + 3P^* & \frac{2}{G} + \frac{2}{3K} + 3P^* \end{array} \right] \]
(25)

\[ P^* = P(\theta^*) \]

\[ D^* = (1/2) + <P^*> (C_{st} + C_{so})/(GH_0) + 3<P*/H_0>(C_{st}C_{so} - C_{st}C_{so}) + (4/3K)(1/G) \]

Therefore, when the formulation of FEM using Eq. (22) is made, the stiffness equation to be solved becomes non-linear with respect to the nodal displacement increments. In that case, we have to use an iterative method such as the Newton-Raphson method for each incremental step of deformation. If we adopt the implicit Eulerian scheme for the time integration\(^{(33)}\), then an iteration is needed inevitably for each step and thus the drawback just mentioned will give no additive nuisance. However, for convenience of calculation, we adopt the explicit Eulerian scheme in this paper. Therefore, to avoid such a drawback, we introduce the following piecewise linear constitutive equation whose basis is Eq. (22).

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\[ \left[ \begin{array}{ccc} \frac{2}{G} + \frac{2}{3K} + 3P^* & \frac{1}{G} - \frac{2}{3K} - 3P^* & \frac{1}{G} - \frac{2}{3K} - 3P^* \\ \frac{2}{G} - \frac{2}{3K} - 3P^* & \frac{2}{G} + \frac{2}{3K} + 3P^* & \frac{2}{G} + \frac{2}{3K} + 3P^* \end{array} \right] \]
(25)

\[ P^* = P(\theta^*) \]

\[ D^* = (1/2) + <P^*> (C_{st} + C_{so})/(GH_0) + 3<P*/H_0>(C_{st}C_{so} - C_{st}C_{so}) + (4/3K)(1/G) \]
\[
\delta \Sigma = \begin{bmatrix}
\frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \frac{\partial N_3}{\partial \xi} & \frac{\partial N_4}{\partial \xi} & \frac{\partial M_1}{\partial \xi} & \frac{\partial M_2}{\partial \xi} & \frac{\partial M_3}{\partial \xi} & \frac{\partial M_4}{\partial \xi}
\end{bmatrix}
\int \sigma_0(1 + x\eta) \, d\eta
data_0(x_1 + x_2) \, d\eta
\int \sigma_0(1 + x\eta) \, d\eta
data_0(x_1 + x_2) \, d\eta
\int \sigma_0(1 + x\eta) \, d\eta
data_0(x_1 + x_2) \, d\eta
\int \sigma_0(1 + x\eta) \, d\eta
data_0(x_1 + x_2) \, d\eta
\int \sigma_0(1 + x\eta) \, d\eta
\]

(27)

The integrations in Eq. (27) are taken from \(-t/2\) to \(+t/2\), where \(t\) = current thickness at the point under consideration.

The curvature increments in Eq. (26) are defined only for \(\eta = 0\), whereas \(\delta \xi_8\) and \(\delta \xi_9\) are the linear functions as follows:

\[
\frac{\partial \xi_8}{\partial \xi_0} = \frac{\partial \xi_8}{\partial \xi_0} + \text{g} \left( \frac{\partial \xi_8}{\partial \xi_0} - \xi_0 \frac{\partial \xi_8}{\partial \xi_0} \right)
\frac{\partial \xi_9}{\partial \xi_0} = \frac{\partial \xi_9}{\partial \xi_0} + \text{g} \left( \frac{\partial \xi_9}{\partial \xi_0} - \xi_0 \frac{\partial \xi_9}{\partial \xi_0} \right)
\]

(28)

In Eq. (27), \(d\eta = \text{exp}(\xi_1) \, d\eta_0\), \(\eta_0\) = initial coordinate in thickness direction, \(\xi_0 = \) thickness strain, and \(\delta \xi_0 = \left( \delta \xi_0 + \delta \xi_9 \right) \left( K - \left( \delta \xi_8 + \delta \xi_9 \right) \right)\) due to the incompressibility of plastic deformation, where \(K\) = elastic bulk modulus. Using these relations together with the constitutive equation (24), we can lead the following relation between \(\delta \xi_0\) and \(\delta \xi_8\):

\[
\delta \xi_8 = D \delta \xi_0
\]

(29)

where the coefficient \(D\) is a \((4 \times 4)\)-matrix which involves the integrations of \(\sigma_0, \sigma_0, y_0, y_0\) and so on through thickness and, to be noted, which is not generally symmetric. For the elastic elements, the expression of \(D\) is rather simple.

If we employ Eq. (22) instead of Eq. (24) as the constitutive equation, we obtain the following relation between \(\delta \xi_0\) and \(\delta \xi_8\):

\[
\delta \xi_8 = D' \delta \xi_0 - \delta \mu
\]

(30)

where \(\delta \mu\) is an integration including \(\sigma_0\) and \(\delta \xi_9\).

Now the shell finite element adopted in this paper is the third order isoparametric element such that it makes the slope as well as the displacement at the nodal point continuous along the element boundary. That is, we introduce the following shape functions of a parameter \(\xi\) which varies from \(-1\) to \(1\) with a particular element:

\[
N_i = (1/4)(-\xi_i, \xi_i + 3\xi_i, \xi_i + 2)
N_i' = (1/4)(1 + \xi_i, \xi_i) \left( 1 - \xi_i \right) \times \left( \pm 1 \right)
\]

(31)

where the subscript \(i=1\) or \(2\) indicating the two endpoints of the element, respectively. Namely, \(\xi_1 = -1\) and \(\xi_2 = 1\). The double signs \(\pm\) should be + for \(i=1\) and \(-\) for \(i=2\), respectively. Making use of Eq. (31), the displacement increment distribution within one element is given by the following equation:

\[
\delta \xi_0 = \sum \left( N_i \delta u_i + N_i' \left( \delta u_i d\xi \right) \right)
\delta \xi_0' = \sum \left( N_i \delta u_i + N_i' \left( \delta u_i d\xi \right) \right)
\]

(32)

where \(\sum\) is the summation notation over \(i=1\) and \(2\).

Applying Eq. (32) to Eq. (26), we obtain the following equation which expresses \(\delta E\) as the function of the vector of the nodal displacement increments \(\delta\) with respect to a particular element:

\[
\delta E^* = B \delta^*
\]

(33)

where

\[
\delta^* = \left( \delta u_1, \delta u_1, (\delta u_1)_t, (\delta u_2)_t, \right)
\delta u_2, \delta u_2, (\delta u_2)_t, (\delta u_2)_t \right)
\]

and \((, )\) means the differentiation with respect to \(,\) and the subscripts 1 and 2 denote the nodes. The superscript \(e\) here indicates that the quantities with it belong to the element under consideration. \(B\) is a \((4 \times 8)\)-matrix and a function of the variables \(\xi, r\) and so on.

2.3 Formulation of shell finite element method

The virtual work principle with respect to an element is written as follows:

\[
\int \left( \sigma_0(1 + x\eta) \delta u + \sigma_0(1 + x\eta) \delta \xi_8 \delta \xi_8 \right) \, dV^*
\]

(34)

where \(F^*\) is the element nodal force vector. Taking the time derivative of this equation and introducing various equations deduced above into it, after a lengthy manipulation of equations, we arrive at the following type of the element stiffness equation with respect to the element nodal displacement increments and force increments.

\[
k^{**} \delta^* = \delta f^{**} + \delta p^*
\]

(35)

\[
k^{**} = k' - (k_v + k_n)
\]

\[
k' = k_l + k_n + k_{nl} + k_{nv}
\]

\[
h_l = \int B^t D B r_i d\xi
\]

\[
h_n = \int B^t \Sigma r_i d\xi
\]

\[
h_{nl} = \int B^t \Sigma \beta r_i d\xi
\]

\[
h_{nv} = \int B^t \Sigma \beta r_i d\xi
\]

\[
k_v = (\text{correction matrix for external forces})
\]

\[
k_{nv} = (\text{correction matrix for hydrostatic pressure})
\]

\[
\delta f^{**} = (\text{nodal force increments vector})
\]

\[
\delta p^* = (\text{nodal force increments vector})
\]

due to hydrostatic pressure increment

\[
r = \sum \left( N_i \xi_1 + N_i' \left( d\xi d\xi \xi \right) \right)
\]

(36)

\[
z = \sum \left( N_i \xi_2 + N_i' \left( d\xi d\xi \xi \right) \right)
\]

(37)

In the above equations, the integrations are taken from \(\xi = -1\) to \(1\), the summations \(\Sigma\) are taken over \(i = 1\) and \(2\), and \(t = d\xi d\xi = (r^t + \xi^t)^t r_i = d\xi d\xi i\). \(r_i, \xi_2 = d\xi d\xi : k_l, k_n\) correspond to the so-called initial stress matrix. \(\delta f^*\) and \(k_{nv}\) exist only when the hydrostatic pressure \(p\) acts as the external force, and \(\delta p^*\) is proportional to \(\delta p^*\) and \(k_{nv}\) to \(p\). Furthermore, in Eq. (36), \(U_r, \beta\) and \(b\) have the following meanings, respectively.
\[ \begin{align*}
\partial u(l) &= U, \delta^* \\
\delta B^t &= \beta^* \\
\delta e(l) &= b_\delta^*
\end{align*} \] (38)

If we employ Eq. (22) as the constitutive equation instead of Eq. (24), we obtain the following type of the element stiffness equation:

\[ h^{**} \delta^* - \int_0^l B^t \mu r l d\xi = \delta f^{**} + \delta p^* \] (39)

This equation and thus the global stiffness equation constructed from it for all elements are nonlinear with respect to the nodal displacement increments \( \delta \) to be solved. Therefore, we have to use an iterative method to solve the global stiffness equation. In what follows, Eq. (24) and thus Eq. (35) are used.

The procedure to obtain the global stiffness matrix or equation from Eq. (35) follows that in the usual FEM. Additionally, when the analysis of forming processes using a rigid tool is concerned with, the condition for a material point to move along the tool surface and the non-linear frictional condition should be taken into consideration. To satisfy the former condition, the tool surface is approximated by a piecewise linear shape. The Coulomb friction law is adopted as the latter condition.

Finally, we obtain the following global stiffness equation (a system of linear equations) with respect to the unknown nodal displacement increment vector \( \delta^* \):

\[ K_{\delta^*} = \delta F \] (40)

where \( K \) is the non-symmetric global stiffness matrix. We solve Eq. (40) by the skyline method.

We develop a new computer program for the numerical analysis of sheet-metal formings on the basis of the equations thus deduced. The electric computer used is the FACOM M-360 at the Gifu University Computation Center.

3. Numerical Examples and Discussions

3.1 Analysis of hydraulic bulging of circular metal sheet

Numerical analysis is mainly performed for hydraulic bulging of a circular commercial aluminium-killed steel sheet whose strain-hardening property is expressed by the following Swift-type law:

\[ \sigma^* = C \left( \varepsilon + \varepsilon_0 \right)^n \] (41)

where \( C = 483.5 \) MPa, \( \varepsilon_0 = 0.008125 \), \( n = 0.232 \). The experimentally determined other material constants are as follows: \( E \) (Young's modulus) = 196.2 GPa, \( \nu \) (Poisson's ratio) = 0.29, \( \sigma_0 \) (initial tensile yield stress) = 161.9 MPa, \( \bar{\rho} = 1.56 \), \( \bar{X} = 1.28 \) and \( \rho = 0.922 \). The initial thickness \( t_0 = 0.830 \) mm. The tool design dimensions are as follows: radius of clamped position = 29 mm, die profile radius = 8 mm, and radius of die cavity = 21 mm.

To examine the effects of the material properties, the calculations are done also for various values of \( n \), \( \bar{\rho} \) and \( \bar{X} \); other than those mentioned above. The potential theory \((\rho=0)\) as well as MG c.e. is used for comparison.

The integrations involved in the formulation are numerically evaluated by the Simpson method with five points for the through-thickness direction, and by the Gauss method with three points for the meridian \((\xi-)\) direction. These numbers of integration points are determined by trial and error considering the accuracy and computation time. The increment of the hydraulic pressure for one step is determined to be \( \delta p = 0.1962 \) MPa (=0.002 kgf/mm\(^2\)) by the similar manner.

In Fig. 2, various stages of deformation are illustrated for the middle surface of the shell. The mark \( \triangledown \) in the figure denotes the position of the clamping bead. The sticking condition is applied to the material points on the die surface. Each deformed shape except in the vicinity of the die is, as can be seen in the figure, close to a sphere. The hydraulic pressure at each stage is equal to 1.5692 MPa multiplied by the integer which indicates the order of the illustrated shape. (That is, each deformed shape is at every 80 steps of the calculation.)

In Fig. 3, the complicated variations of the \( \sigma^* \)-distribution at earlier stages of deformation are illustrated. On the pressure side, \( \sigma^* \) is negative due to bending around the central portion at the earliest stage of deformation, which turns out to be positive on the whole surface with deformation. On the other hand, on the outer free surface, \( \sigma^* \) is constantly positive around the central portion throughout the deformation history, whereas it is negative due to bending at the position close to the clamping bead. In Fig. 4,

![Fig. 2 Stages of deformation for hydraulic bulging of a sheet (Al-killed steel)](image-url)
the distributions of $\sigma_s$ and $\varepsilon_s$ at the intermediate stages of deformation are illustrated. From this figure, it is found that the membrane states of stress and strain distributions prevail over almost whole area except in the vicinity of the clamping bead after the peak height of the bulged sheet (i.e., the bulge height) exceeds 10 mm. Let us make a simple numerical examination. The membrane theory gives us the formula of $\sigma_s$ on the spherically bulged sheet as

$$\sigma_s = \frac{\rho^*}{p/2t}$$

where $\rho^*$ = radius of curvature. From Fig. 4, we read $\sigma_s = 49.0 \times 9.81$ MPa at the center, and $p = 2.0 \times 9.81$ MPa. We have the corresponding value of $\varepsilon_s$ as to be 0.20, then $t = t_0 \exp(\varepsilon_s) = 0.68$ mm. These values give $\rho^* = 33.3$ mm. From Fig. 2, we can read almost the same value of $\rho^*$, which verifies simply the accuracy of the calculation to be sufficient enough.

Table 1 is the summary of the calculation conditions adopted here and the consumed computer times. The numerical examples described thus far are for the condition of No. 2 in this table. As can be found in Table 1, the CPU time becomes quadruple when the number of the nodal points is increased from 30 to 100, whereas the accuracy of the results is found to remain almost unchanged.

The calculation by the quadruple precision needs the CPU time 6.41 times as much as that by the double precision, giving no drastically good results. As this table shows, the calculation is done for the sheet with the initial thickness of 0.3 mm as well as 0.83. It shows that the membrane state mentioned above appears earlier than that in the case of the 0.83 mm-thick sheet. The spring-back and the residual stresses after unloading are also evaluated for one case, though they are not reported here.

In Figs. 5(a) - (d), the distributions of the thickness strain are illustrated at the same bulge height, each of which shows the effect of $n$, $\tilde{r}$, $\tilde{X}$ and the vertex-hardening, respectively, on $\varepsilon_s$. In Fig. 5(a), we see that the strain is more apt to concentrate at the central portion of the sheet for smaller $n$-value. In Fig. 5(b), we see that the larger $\tilde{X}$-value suppresses the strain concentration at the center. In Fig. 5(c), we see that the influence of the $\tilde{r}$-value is not very significant. And, in Fig. 5(d), we see that the vertex-hardening (in MG c.e.) promotes the strain concent-

Fig. 3 Variations of $\sigma_s$-distribution at early stage of deformation; (a) pressure side, (b) outer free surface (Al-killed steel; $h = $ bulge height at the top)

Fig. 4 Distributions of $\varepsilon_s$ and $\sigma_s$ at intermediate stage of deformation

JSME International Journal
Table 1 Various calculation conditions and consumed CPU time

<table>
<thead>
<tr>
<th>No.</th>
<th>C.E.</th>
<th>ε-mm</th>
<th>No. of N.P.</th>
<th>n</th>
<th>$\bar{X}$</th>
<th>R</th>
<th>Ac</th>
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C.E. = Const. Eq., N.P. = Nodal Points, Ac = Accuracy
CPU = (sec/step), PO = Potential Theory,
DP = Double Precision, QP = Quadruple Precision.

Fig. 5 Effects of various factors on distribution of thickness strain:
(a) $n$-value, (b) $X$-value, (c) $\bar{X}$-value (d) vertex evolution

Figure 7 illustrates the stages of deformation. The mark ▼ in this figure denotes the material point which is the end of the area in contact with the punch surface. First yielding takes place at the clamped point on the pressure side for hydraulic bulging, while it does at the center on the outer free surface for punch stretching. In Fig. 8, the variation of $\sigma_x$-distributions along the surface of the punch side is illustrated. The counterpart along the opposite surface is very different from this figure. As can be seen in this figure, the stress variation is very remarkable. Therefore, the deformation increment for each step should be controlled within a sufficiently small amount (say, $\Delta h \leq 0.01$ mm, where $\Delta h =$ increment of the punch stroke). Figure 8 shows that $\sigma_x$ is negative at the central portion of the sheet at the early stage of deformation, turning out to be positive in a wider area with being stretched by the punch.

4. Conclusions

For the purpose of making it more informative
and instructive to calculate numerically the axisymmetric sheet-metal forming, a new formulation of the shell elastic-plastic FEM is developed by introducing Gotoh's vertex-hardening plastic constitutive equation (MG c.e.) and his fourth-order anisotropic yield function, and by employing the third-order isoparametric finite element which makes both the displacement increment and the slope continuous across the element boundaries. Furthermore, the computer program for the numerical calculation is newly developed by which the bulging of the sheet by hydraulic pressure is mainly analyzed to give several numerical examples. Complex variations of the stress distributions (especially at the early stage of deformation), the effects of $n$, $r$ and $X$-values and the vertex-evolution (in MG c.e.) on the strain distributions, the relations between the bulge height and the pressure under various conditions, and so forth are discussed. A few examples of the numerical calculation of stretching by a hemispherical punch are also presented, which guarantees a further development of this kind of numerical analysis to be done successfully in future.

References


