Free Vibration Analysis of Coupled External Fluid-Elastic Cylindrical Shell-Internal Fluid Systems*

Ryuji ENDO ** and Nobuyoshi TOSAKA ***

The free vibrations of an infinitely long cylindrical shell under axisymmetrical hydrodynamic pressures due to external and internal fluids are studied theoretically. The basic equations of motion for axisymmetric vibrations of such a shell are given, based on the bending theory. These equations are solved analytically by means of Fourier cosine transformation, and the general frequency equation for the coupled systems is obtained by consideration of the dynamic interaction between the shell surface and fluids. Furthermore, seven special cases are derived from the general coupled frequency equation. The general frequency equation is calculated for some kinds of mass density of the internal fluid. Hence, it is shown that the effect of the motion of the external and internal fluids on the free vibrations of a shell is notable.

Key Words: Elastic Cylindrical Shell, External Fluid, Internal Fluid, Coupled System, Coupled Vibration

1. Introduction

Recently, many investigations on fluid-shell dynamic interaction have been conducted and the subject has received considerable attention in various engineering fields⁽¹⁾. The majority of fluid-shell dynamic interaction problems, such as in storage tanks and in offshore structures has been solved numerically by means of the finite element method⁽²⁾ and boundary element method⁽³⁾.

However, the frequency equations for axisymmetric free vibrations of spherical shells in deep water have been derived theoretically by T. C. Huang and F. C. Chen⁽⁴⁾. Moreover, in our previous paper⁽⁵⁾, in order to examine the dynamic behavior of an offshore pipeline, the free axisymmetric vibration of an infinitely long cylindrical shell submerged in sea water of

infinite extent was investigated analytically. We pointed out the influence of surrounding sea water on the natural frequencies of a cylindrical shell. We have to take into consideration the effect of the internal fluid contained in a shell in order to clarify the dynamic behavior of the shell. Thus, in this paper we intend to discuss the free vibration problems of an infinitely long cylindrical shell containing fluid in deep sea water vis à vis the application of our methodology developed in previous paper⁽⁵⁾.

A mathematical model to be analyzed is presented as a coupled problem of external fluid-shell-internal fluid dynamic interaction. Following the method used in ref. 5), we formulate the fundamental coupled equations for harmonic motions in our cylindrical shell. Then if we apply the Fourier cosine transformation to our fundamental equations, general frequency equation can be derived, from which seven kinds of frequency equation can be obtained as special cases.

Numerical examples of various mass densities of the internal fluid are given and the results are compared. Through the comparisons of natural frequencies, the effects of the external fluid and the internal fluid on the motion of a shell are made clear.

* Received 21st July, 1988. Paper No. 87-0467B

** Department of Architectural Engineering, The Institute of Vocational Training, 1960 Aihara, Sagami-hara, 229, Japan

*** Department of Mathematical Engineering, Nihon University, 1-2-1 Izumi-cho, Narashino, 275, Japan

2. Formulation of the Coupled Problem

2.1 Shell equation

In an infinitely long shell of thickness h , radius a , we introduce the symmetric cylindrical coordinates (Z , r) and denote the corresponding displacement components of a point in the middle surface of a shell with u and w respectively, as shown in Fig. 1.

The equations of motion for axisymmetric vibration of such a shell are given by Flügge⁽⁶⁾,

$$u'' + \nu w' - \varepsilon w''' + \frac{a^2}{D} p_z = 0 \quad (1)$$

$$\nu u' + w + \varepsilon(w'''' - u''' + w) - \frac{a^2}{D} p_r = 0 \quad (2)$$

in which $D = Eh/(1 - \nu^2)$, $\varepsilon = (h/a)^2/12$ and $' = a(\partial/\partial z)$, where ν represents Poisson's ratio and E is Young's modulus. The load p_z in Eq. (1) represents the inertial term, given by the relation,

$$p_z = -\rho_s h \frac{\partial^2 u}{\partial t^2} \quad (3)$$

where ρ_s is the mass density of the shell and t denotes the time variable. Furthermore, the radial load p_r in Eq. (2) may be represented as the sum of inertial term and the dynamic pressures under our coupled sea water-shell-internal fluid interaction system, as follows

$$p_r = -\rho_s h \frac{\partial^2 w}{\partial t^2} + p_{we} - p_{wi} \quad (4)$$

where p_{we} and p_{wi} are the hydrodynamic pressures due to the external and internal fluids, respectively.

2.2 Fluid equation

Let us assume an inviscid, incompressible fluid as the internal fluid as the model for an oil or a gas contained in a shell, with sea water surrounding the shell. In the steady state problem, the velocity potential ϕ satisfies the following Laplace equation

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \phi = 0 \quad (5)$$

The hydrodynamic pressure exerted on shell surface is calculated by using the Bernoulli equation

$$p = -\rho_f \frac{\partial \phi}{\partial t} \quad (6)$$

where ρ_f is the fluid density.

Fig. 1 Geometry of an infinitely long shell

2.3 Boundary conditions

The boundary conditions of our fluid-shell-fluid coupled problems have to satisfy two constraints, namely the inherent conditions for shell continuity and the dynamic interaction at the external and internal fluid-shell interfaces. For the radial displacement of a shell and its z derivatives, we assume the following conditions;

$$w(r, z, t) \rightarrow 0 \text{ as } z \rightarrow \pm \infty \quad (7)$$

$$\begin{aligned} \frac{\partial w}{\partial z}(r, z, t) &= \frac{\partial^3 w}{\partial z^3}(r, z, t) \\ &= \frac{\partial^5 w}{\partial t^5}(r, z, t) = 0 \text{ at } z=0 \end{aligned} \quad (8)$$

The velocity potential ϕ_e of the external fluid satisfies the condition,

$$\phi_e \rightarrow 0 \text{ as } r \rightarrow \infty, z \rightarrow \infty \quad (9)$$

Also, the velocity potential ϕ_e of the external fluid and the velocity potential ϕ_i of the internal one satisfy the condition of symmetry,

$$\frac{\partial \phi_e}{\partial z} = \frac{\partial \phi_i}{\partial z} = 0 \text{ as } z=0 \quad (10)$$

In addition, on the interface boundary, the interface condition must be satisfy,

$$\frac{\partial \phi_e}{\partial r} = \frac{\partial \phi_i}{\partial r} = -\frac{\partial w}{\partial t} \text{ at } r=a \quad (11)$$

Summing up the above, we can formulate the boundary value problems of the coupled fluid-shell-fluid systems.

3. Analysis

We can write the solution of our coupled problem which is assumed to have harmonic motion, in the following forms;

$$\phi_e(r, z, t) = \Phi_e(r, z) \sin \omega t \quad (12)$$

$$\phi_i(r, z, t) = \Phi_i(r, z) \sin \omega t \quad (13)$$

$$u(z, t) = U(z) \cos \omega t \quad (14)$$

$$w(z, t) = W(z) \cos \omega t \quad (15)$$

$$p_{we}(r, z, t) = P_{we}(r, z) \cos \omega t \quad (16)$$

$$p_{wi}(r, z, t) = P_{wi}(r, z) \cos \omega t \quad (17)$$

where ω represents the natural frequency of the coupled fluid-shell-fluid system.

Substituting Eqs. (14) to (17) into fundamental Eqs. (1) and (2) of a shell, eliminating u and using the approximation $\varepsilon^2 \approx 0$, we can derive a single equation for radial displacement

$$\begin{aligned} W'''' + (2\nu + \Omega^2) W'''' + \{1 + \xi(1 - \nu^2) \\ - \xi\Omega^2\} W'' + \{\Omega^2(1 + \xi) - \Omega^4\xi\} W \\ = \frac{1 - \nu^2}{Eh} a^2 \xi \{ (P_{we}'' - P_{wi}'') + \Omega^2(P_{we} - P_{wi}) \} \end{aligned} \quad (18)$$

where Ω^2 is a nondimensional frequency parameter, in which

$$\Omega^2 = \frac{1 - \nu^2}{E} \rho_s a^2 \omega^2 \quad (19)$$

and

$$\xi = \frac{1}{\varepsilon} = 12 \left(\frac{a}{h} \right)^2 \quad (20)$$

To solve our problem with a region of infinite extent, we introduce the Fourier cosine transformation of Φ and W , as follows

$$\bar{\Phi}(r, k) = \int_0^\infty \Phi(r, z) \cos(kz) dz \quad (21)$$

$$\bar{W}(r, k) = \int_0^\infty W(z) \cos(kz) dz \quad (22)$$

Using Eq.(12) or (13) and relation (21), the Laplace Eq.(5) is transformed into

$$\frac{d^2 \bar{\Phi}}{dr^2} + \frac{1}{r} \frac{d\bar{\Phi}}{dr} - k^2 \bar{\Phi} = 0 \quad (23)$$

Eq.(23) is known as the modified Bessel's differential equation of the order zero. Thus, the solution can be obtained analytically with the unknowns A and B . Taken into consideration with the boundedness of the velocity potential in r , the solution for the external and internal fluids then reduces to the following expressions for each potential

$$\bar{\Phi}_e(r, k) = AK_0(kr) \quad (24)$$

$$\bar{\Phi}_i(r, k) = BI_0(kr) \quad (25)$$

where K_0 and I_0 are modified Bessel functions of the order zero. Further, we can transform the equation governing the radial displacement W of a shell as follows.

Through consideration of relation (22), Eq.(18) transforms to

$$\begin{aligned} & \left[-k^6 + k^4(2\nu + \Omega^2) - k^2(1 + \xi(1 - \nu^2)) \right. \\ & \quad \left. - \xi\Omega^2 \right] + \{ \Omega^2(1 + \xi) - \Omega^4\xi \} \bar{W} \\ & = \frac{1 - \nu^2}{Eh} a^2 \xi (\Omega^2 - k^2) (\bar{P}_{we} - \bar{P}_{wi}) \end{aligned} \quad (26)$$

In Eq.(26), the hydrodynamic pressure terms $\bar{P}_{we}$ and $\bar{P}_{wi}$ which appear on the right hand side must be given as the expression with a transformed radial displacement $\bar{W}$.

Carrying out the Fourier cosine transformation on interaction (11) with Eqs.(12), (13) and (15), we can determine the unknowns A and B in our solutions (24) and (25) respectively. Substituting these results into Eqs.(24) and (25), we can obtain the transformed velocity potentials $\bar{\Phi}_e$ and $\bar{\Phi}_i$. Now, taking advantage of the Fourier transformation of Bernolli Eq.(5), transformed hydrodynamic pressures act on the middle surface of a shell according to the following relations

$$\bar{P}_{we} = -\rho_{fe} \omega^2 \frac{K_0(ka)}{\frac{dK_0}{dr}(ka)} \quad (27)$$

$$\bar{P}_{wi} = -\rho_{fi} \omega^2 \frac{I_0(ka)}{\frac{dI_0}{dr}(ka)} \bar{W} \quad (28)$$

where ρ_{fe} and ρ_{fi} are the densities of the external and internal fluids, respectively.

4. Frequency Equation

The hydrodynamic pressures $\bar{P}_{we}$ and $\bar{P}_{wi}$ were found to be representative in terms of the transformed radial displacement $\bar{W}$ in the previous section. Thus, the coupled problem of fluid-shell-fluid dynamic interaction of Eqs. (1) to (11) reduces to a free vibration problem in which the natural frequencies play an important part.

We substitute relations (27) and (28) into (26) in order to obtain the frequency equation, and derive the homogeneous equation. Fixing the coefficients of this homogeneous equation at zero results in the frequency equation

$$\begin{aligned} & \Omega^4 \left\{ \xi - \frac{1}{\rho_{sh}} \xi \left(\rho_{fe} \frac{K_0(ka)}{\frac{dK_0}{dr}(ka)} - \rho_{fi} \frac{I_0(ka)}{\frac{dI_0}{dr}(ka)} \right) \right\} - \Omega^2 \left\{ k^4 \right. \\ & \quad \left. + k^2 \xi + 1 + \xi \right. \\ & \quad \left. + \frac{1}{\rho_{sh}} k^2 \xi \left(\rho_{fi} \frac{I_0(ka)}{\frac{dI_0}{dr}(ka)} - \rho_{fe} \frac{K_0(ka)}{\frac{dK_0}{dr}(ka)} \right) \right\} \\ & \quad + k^6 - 2\nu k^4 + k^2 + k^2 \xi (1 - \nu^2) = 0 \end{aligned} \quad (29)$$

For each value of the integer k , the frequency Eq.(29) gives two individual roots in Ω^2 . Equation (29) is the general frequency equation for the free vibration of an infinitely long cylindrical shell containing fluid in deep sea water, because the shell complied with the bending theory under hydrodynamic pressures. Moreover, seven frequency equations for each special case can be derived from Eq. (29).

Case A: External fluid-shell coupled system (bending theory)⁽⁵⁾

If we assume the mass density of the internal fluid to be $\rho_{fi}=0$ in Eq. (29), we arrive directly at

$$\begin{aligned} & \Omega^4 \left\{ \xi - \frac{\rho_f}{\rho_{sh}} \xi \frac{K_0(ka)}{\frac{dK_0}{dr}(ka)} \right\} \\ & - \Omega^2 \left\{ k^4 + k^2 \xi + 1 + \xi - \frac{\rho_f}{\rho_{sh}} k^2 \xi \frac{K_0(ka)}{\frac{dK_0}{dr}(ka)} \right\} \\ & + k^6 - 2k^4\nu + k^2 + k^2 \xi (1 - \nu^2) = 0 \end{aligned} \quad (30)$$

Case B: Internal fluid-shell coupled system (bending theory)

In this case, if we assume the mass density of the external fluid to be $\rho_{fe}=0$ in Eq. (29), we obtain the following equation

$$\begin{aligned} & \Omega^4 \left\{ \xi + \frac{\rho_{fi}}{\rho_{sh}} \xi \frac{I_0(ka)}{\frac{dI_0}{dr}(ka)} \right\} \\ & - \Omega^2 \left\{ k^4 + k^2 \xi + 1 + \xi + k^2 \xi \frac{\rho_{fi}}{\rho_{sh}} \frac{I_0(ka)}{\frac{dI_0}{dr}(ka)} \right\} \\ & + k^6 - 2\nu k^4 + k^2 + k^2 \xi (1 - \nu^2) = 0 \end{aligned} \quad (31)$$

Case C: Vibration in a vacuum (bending theory)⁽⁵⁾

Substituting the mass densities of both the external and internal fluids, $\rho_{fe}=0$ and $\rho_{fi}=0$ into Eq. (29), we obtain

$$\Omega^4 \xi - \Omega^2 (k^4 + k^2 \xi + 1 + \xi) + k^6 - 2k^4 \nu + k^2 + k^2 \xi (1 - \nu^2) = 0 \quad (32)$$

Case D: External Fluid-shell-internal fluid coupled system (membrane theory)

For membrane theory, multiplication of Eq. (29) by $1/\xi$ and the introduction of the limit procedure $1/\xi \rightarrow 0$ result in

$$\Omega^4 - \Omega^2 k^2 = 0 \quad (33)$$

Case E: External fluid-shell coupled system (membrane theory)

This frequency equation yields the same result as Case D.

Case F: Internal fluid-shell coupled system (membrane theory)

Multiplying Eq. (31) by $1/\xi$ and taking the limit procedure in Case B $1/\xi \rightarrow 0$ gives

$$\Omega^4 + \Omega^2 k^2 = 0 \quad (34)$$

This equation has no real number solution except zero.

Case G: Vibration in a vacuum (membrane theory)

Multiplying Eq. (32) in Case C by $1/\xi$ and apply-

Table 1 Material properties used in calculations

Models	Internal fluids	External fluid	Shell material
MODEL1	Gasoline $\rho_{fi} = 0.66 \times 10^{-6}$ (kg·sec ² /cm ⁴)	Water $\rho_{fe} = 1.02 \times 10^{-6}$ (kg·sec ² /cm ⁴)	Steel $\rho_s = 8.00 \times 10^{-6}$ (kg·sec ² /cm ⁴) $\nu = 0.3$
MODEL2	Water		
MODEL3	Glycerine $\rho_{fi} = 1.26 \times 10^{-6}$ (kg·sec ² /cm ⁴)		
MODEL4	Vacuum	Vacuum	
MODEL5	Water		
MODEL6	Vacuum

Fig. 2 Natural frequency for lower branch Ω_i^2 VS h/a for $\nu=0.3$

Table 2 Natural frequency parameters for $\nu=0.3$, $h/a=0.03$ (bending theory)

Modes	MODEL2		MODEL4	
	Ω_u^2	Ω_l^2	Ω_u^2	Ω_l^2
1	1.120	0.519	1.179	0.606
2	4.085	0.695	4.098	0.781
3	9.083	0.763	9.091	0.829
4	16.084	0.809	16.090	0.863
5	25.085	0.856	25.090	0.902
6	36.086	0.917	36.090	0.958
7	49.087	1.005	49.091	1.044
8	64.088	1.133	64.091	1.171
9	81.090	1.314	81.092	1.355
10	100.091	1.566	100.093	1.609

Modes	MODEL5		MODEL6	
	Ω_u^2	Ω_l^2	Ω_u^2	Ω_l^2
1	1.174	0.600	1.300	0.700
2	4.098	0.778	4.116	0.885
3	9.091	0.828	9.101	0.906
4	16.090	0.862	16.096	0.923
5	25.089	0.901	25.095	0.952
6	36.090	0.958	36.094	1.003
7	49.091	1.044	49.094	1.086
8	64.091	1.171	64.095	1.213
9	81.092	1.354	81.095	1.397
10	100.093	1.608	100.096	1.654

Table 3 Natural frequency parameters (membrane theory)

Modes	MODEL2		MODEL6	
	Ω_u^2	Ω_l^2	Ω_u^2	Ω_l^2
1	1.000	0.000	1.300	0.700
2	4.000	0.000	4.116	0.884
3	9.000	0.000	9.100	0.900
4	16.000	0.000	16.095	0.905
5	25.000	0.000	25.093	0.907
6	36.000	0.000	36.092	0.908
7	49.000	0.000	49.092	0.908
8	64.000	0.000	64.091	0.909
9	81.000	0.000	81.091	0.909
10	100.000	0.000	100.096	0.909

ing the limit procedure $1/\xi \rightarrow 0$ yields

$$\Omega^4 - \Omega^2(k^2 + 1) + k^2(1 - \nu^2) = 0 \quad (35)$$

5. Numerical Results

Numerical results for the natural frequencies of each system are shown in this section.

The values of the material properties for each case of coupled shell models used in the calculations are shown in Table 1.

The numerical examples of natural frequencies based on bending theory and membrane theory for the special cases are shown in Table 2 and 3. In addition, the lower branches Ω_i^2 for each case of steel shells containing gasoline, water or glycerine in water are plotted as functions of h/a for each mode number $k=1, 2$ and 3 in Fig. 2.

6. Conclusion

Frequency equations of axisymmetric vibrations for an infinitely long cylindrical shell containing certain fluids in deep sea water are derived analytically based on bending theory and some numerical results for the natural frequencies are calculated numerically. The comparison of the corresponding results in frequency equations for seven special cases are given in

Tables 2 and 3 and in Fig. 2. The natural frequencies of each case of steel shells containing gasoline, water or glycerine in water are lower than the natural frequencies of the coupled shell- water systems for each mode number.

References

- (1) Tosaka, N., The Formulation for the Dynamic water wave-elastic shell interaction, Trans. Arch. Ins. Jpn., (in Japanese), No. 314 (1982), p. 176.
- (2) Tosaka, N. and Kakuda, K., Numerical Approximation of Coupled Fluid-Structure Systems, Finite Element Methods in Flow Problems, (1982), p. 803, Univ. of Tokyo Press.
- (3) Tosaka, N. and Kakuda, K., Numerical Method for Water Wave- Structure Dynamic Interaction Problems, Ocean Space Utilization'85, Vol. 1 (1985), p. 345, Springer-Verlag.
- (4) Huang, T. C. and Chen, F. C. Free Axisymmetrical Vibration of Underwater Spherical Shells, Developments in Mechanics, 8, Proc. of 14th Midwest. Mech. Conf., (1974), p. 3.
- (5) Endo, R. and Tosaka, N., Free Vibration Analysis of the Elastic Cylindrical Shell in Water, Theoret. Appl. Mech., Vol. 31 (1981), p. 303.
- (6) Flügge, W., Stresses in Shells, (2nd ed.), (1973), p. 215, Springer-Verlag.