On the Elastic-Plastic Constitutive Equations in Incremental Form*

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It is emphasized that the sinusoidal response of the shear stress τ to large shear strain γ in a simple shear is not unreasonable at least in a mathematical sense (although it has long been considered unreasonable), and is dependent on the material property. Also it is found that the up-dated form of the constitutive equation is not necessarily equivalent to its original one except when the current rates of stress and strain are completely independent of the deformation state and history in a tensorial sense, which are commonly used unconditionally in the numerical deformation analysis by the incremental method. A new plastic constitutive equation is proposed which involves a new material constant ρ with $0 \leq \rho \leq 1$. It reduces to the so-called J_2 -flow theory for $\rho=0$, and resembles the deformation theory for $\rho=1$. Its stress response to simple shear is discussed with various values of ρ , and the well-known axial extension of a cylinder subjected to torsion is explained using an appropriate value of ρ . The spin to be used in the constitutive equation and the evolution equation of the internal variables are also discussed.

Key Words: Plasticity, Constitutive Equation, Large Strain, Large Deformation, Incremental Form, Spin, Simple Shear, Pure Shear, Large Torsion

1. Introduction

Since Lee⁽¹⁾ pointed out that the shear stress τ responds sinusoidally to large shear strain γ beyond 1 in a simple shear of a block of the kinematic hardening plasticity due primarily to the spin (rate of the instantaneous rigid-body rotation) used in the evolution equation of the back stress, many workers have discussed this difficulty. The spin conventionally used in the rate-type constitutive equation is the so-called continuum spin. Lee proposed another kind of spin which can be thought of as the spin of the material line element. Dafalias⁽²⁾ proposed to use the Green-Naghdi spin in the evolution equation of the back stress. This spin is also called the material spin, and was first introduced by Dienes into the stress rate. Dafalias⁽³⁾ also proposed another spin which includes an anisotropy effect. The author⁽⁴⁾ proposed a further spin related to the definition of the strain. However, a

decisive conclusion has not determined which is the most reasonable spin to be used in the constitutive equation.

In the incremental method of the numerical analysis of large elastic-plastic deformation, the up-dated Lagrangean formulation is commonly used. In this formulation, the up-dated form of the rate-type constitutive equation is unconditionally used. No confirmation on the reasonableness of this up-dated form has been given so far.

It has been well-known that when a bar is subjected to severe torsion it extends somewhat along the axis. The reason of this extension is usually attributed to the anisotropy which could be induced in the bar due to large plastic deformation. However, direct evidence of this idea has not been given yet, thus other factors could still exist.

In this paper, these problems are studied and new ideas about them are presented. A new constitutive equation governing plasticity of materials is proposed together with an appropriate spin involved in it in order to evaluate the ideas.

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2. Finite Elasticity

2.1 The original constitutive equation

Isotropic finite elasticity is expressed by the following constitutive equation (see e. g., ref. (5)) :

$$\sigma = \phi_1 I + \phi_2 B + \phi_3 B^2 \quad (1)$$

where σ = Cauchy stress tensor, I = unity tensor of second rank, $B = V^2$, V = left stretch tensor in the polar decomposition of the deformation gradient tensor F , $F = VR$, R = proper rotation tensor ($\det |R| = 1$), $F = \partial x / \partial X$, $x = X + u$, X = initial position vector, u = displacement vector, and the scalars ϕ_i ($i=1, 2, 3$) are the functions of the invariants of B , say I_B , II_B , and III_B .

2.2 The up-dated form of the constitutive equation

Equation (1) is easily converted into the following rate form :

$$\dot{\sigma} = \dot{\phi}_1 I + \dot{\phi}_2 B + \dot{\phi}_3 B^2 + \phi_2 \overset{\nabla}{B} + \phi_3 (\overset{\nabla}{B} B + B \overset{\nabla}{B}) \quad (2)$$

where $(\dot{\cdot})$ denotes the material time derivative and $(\overset{\nabla}{\cdot})$ an objective rate, namely

$$\overset{\nabla}{\sigma} = \dot{\sigma} - \omega \sigma + \sigma \omega \quad (3)$$

with a spin ω .

The spins which might be used in the constitutive equations are listed in the following :

ω_c : the continuum spin (for making the Jaumann -rate) $= (1/2)(L - L^T)$,

ω_f : the material spin or Green-Naghdi spin (for making the Dienes' rate) $= \dot{R}R^T$,

ω_v : the spin of the principal axes of V or $B = \dot{Q}Q^T$,

ω_u : the spin of the principal axes of U or $C = \dot{P}P^T$,

where L = velocity gradient tensor $= \partial v / \partial x$, v = velocity vector. The polar decomposition of F is written as $F = VR = RU$, where V = left stretch tensor and U = right stretch tensor both of which are symmetric tensors. $V = QAQ^T$ and $U = PAP^T$, in which A is the diagonal tensor composed of the principal values of V or U , and $Q = [q_1, q_2, q_3]$ and $P = [p_1, p_2, p_3]$, where q_i and p_i = the unit vectors lying along the principal axes of V (or B) and U (or C), respectively. And $Q = RP$. The stretching or Euler strain-rate D is defined equal to $(1/2)(L + L^T)$.

2.3 Discussions on the up-dated form of the constitutive equations and on τ - γ relation in simple shear

In the commonly used numerical analyses of deformation by the incremental method, the up-dated form of the constitutive equation is required. For that, the current configuration is taken as the reference configuration, and then, setting $F = U = V = P = Q =$

$R = I$, and $\dot{F} = L$, $\dot{U} = \dot{V} = D$, $\dot{Q} = \dot{P} = \dot{R} = \omega_v = \omega_u = \omega_f = \omega_c$, the constitutive equation is simplified. This procedure is done unconditionally. Therefore, Eq. (2) is readily reduced to the following simple well-known form :

$$\dot{\sigma} = \lambda (tr D) I + 2\mu D \quad (4)$$

where $(\dot{\cdot})$ denotes the so-called Jaumann rate. Usually, λ and μ are considered the material constants called Lamé's constants. In other words, the original constitutive equation (1) is reduced to the simple equation (4) in its up-dated rate type. Evidently there exists no limitation to the deformation magnitude in the three constitutive equations (1), (2) and (4). In this sense Eq. (4) differs from the classical Hookean elasticity, though it appears to be linear in terms of the strain rate D . We should note that there seems to be no logical ambiguity in derivation of Eq. (4) from Eq. (2) or the original equation (1).

However, there exists a questionable point. Equation (2) is totally equivalent to the original form (1) in the sense that if the time integration of $\dot{\sigma}$ is taken up to the current state, it will yield Eq. (1), that is, the same stress-strain relation as that expressed by Eq. (1) except the initial stress values. In contrast, Eq. (4) does not necessarily yield Eq. (1) by the time integration, because it involves no arbitrary functions, whereas Eqs. (1) and (2) involve the arbitrary functions ϕ_i (I_B, II_B, III_B), $i=1, 2$ and 3 . Therefore, the up-dated form (4) is not equivalent to its original form (1), but instead to a rather special form derived from Eq. (1).

This point is made clear by the following simple example. As is well known, Eq. (4) gives a sinusoidal relation between shear stress τ and shear strain γ in the large simple shear of a block. It gives no further relations. However, Eq. (1) gives either a monotonically increasing shear stress or a sinusoidally varying shear stress and others with shear strain depending on the choice of those arbitrary functions ϕ_i . Let us consider a simple shear of a rectangular block in which the deformation gradient tensor F is given by the following equation (see Fig. 1) :

$$F = I + \gamma l \otimes m \quad (5)$$

where $\otimes$ denotes the tensor product. The shear angle $\varphi = \tan^{-1} \gamma$. Substituting Eq. (5) into Eq. (1), we obtain the following relations (see e.g. ref. (5)) :

$$\left. \begin{aligned} \sigma_{(l)} &= l \cdot (\sigma l) = \phi_1 + \phi_2(1 + \gamma^2) + \phi_3(1 + 3\gamma^2 + \gamma^4) \\ \sigma_{(m)} &= m \cdot (\sigma m) = \phi_1 + \phi_2 + \phi_3(1 + \gamma^2) \\ \sigma_{(n)} &= n \cdot (\sigma n) = \phi_1 + \phi_2 + \phi_3 \\ m \cdot (\sigma n) &= n \cdot (\sigma l) = 0 \\ \tau &= l \cdot (\sigma m) = \gamma \{ \phi_2 + \phi_3(2 + \gamma^2) \} \\ \sigma_{(l)} - \sigma_{(m)} &= \gamma l \cdot (\sigma m) = \gamma \tau \end{aligned} \right\}$$

(6)

where l, m and n are the base unit vectors of the rectangular Cartesian coordinates, and $\sigma_{(l)}, \sigma_{(m)}$, and $\sigma_{(n)}$ are the physical components of the stress σ in the direction of l, m and n .

The invariants of B are given as follows:

$$I_B = II_B = 3 + \gamma^2, III_B = 1. \quad (7)$$

Therefore, ϕ_i 's are the functions of γ^2 . For simplicity, we consider the case where $\phi_1 + \phi_2 + \phi_3 = 0$, and $\sigma_{(l)} = -\sigma_{(m)}, \sigma_{(n)} = 0$. Then we obtain the following expression of the shear stress $\tau = l \cdot (\sigma m)$:

$$\tau = \phi_2 \cdot (2\gamma) / (4 + \gamma^2). \quad (8)$$

If ϕ_2 is a constant, the shear stress τ expressed by this equation increases first with shear strain γ , reaches the maximum value at $\gamma=2$ and then decreases monotonically approaching 0 asymptotically for $\gamma \rightarrow \infty$. If we assume $\phi_2 = 2 \times c[1 + (\gamma/2)^2](\gamma^2)^{(n-1)/2}$, then $\tau = c\gamma^n$ where we can assume $0 < n < 1$. That is, in this case, the shear stress increases monotonically with the shear strain following the n -th power law.

This simple example shows clearly that there could exist the material whose τ - γ relation in a simple shear is either an increasing-decreasing curve, a monotonically increasing curve or another type of curve such as a sinusoidal one, at least for finite elasticity. Then it is natural to think that there could also exist such a variety of material properties even for a nonelastic case such as plasticity. We cannot say that the sinusoidal behaviour of τ - γ curve is very peculiar which Lee, Dafalias and other many workers found and discussed with regard to a kinematically hardening plastic material.

The above discussion verifies also that Eq. (4) is only a special form of Eq. (2). λ and μ should not be considered to be constants, but instead to be the functions of the invariants of B , i.e. I_B, II_B and III_B , which will represent the current state of deformation of the body. In other words, the reduction of the rate-type constitutive equation to the up-dated form is not allowed unconditionally. The rate-type constitutive equation usually includes the deformation measures

such as B and rotation measures such as R . Therefore, except when the current constitutive relations are completely independent of the current deformation and rotation states, the simplification procedure taken in the derivation of Eq. (4) from Eq. (2) is not justified. This is true even for plasticity, at least with respect to the rotation (or spin) part.

For Hookean elasticity, we use the following equation:

$$\sigma = \hat{D} : \epsilon^e, \quad (9)$$

where ϵ^e is the infinitesimal elastic strain. The rate type of this equation is readily obtained as follows:

$$\dot{\sigma} = \hat{D} : \dot{\epsilon}^e. \quad (10)$$

In Eqs. (9) and (10), $\hat{D}$ is a constant coefficient tensor of the fourth rank. Equation (10) is equivalent to its original form (9). It gives no oscillatory relation between τ and γ for a simple shear at a large strain level or large rotation. In this sense, Eq. (10) is never equivalent to Eq. (4). However, we should note that the original equation (9) of Eq. (10) is merely an approximation of Eq. (1) for the infinitesimal strain case, and thus Eq. (10) is also effective only for such a case, although finite rotation would be allowed for Eq. (10). The spin involved in Eq. (10) is not definite; it is not necessarily the continuum spin ω_c . If we wish to avoid the oscillatory relation between τ and γ at a large strain level of simple shear when we formulate any nonelastic (e.g. elastic-plastic) constitutive equation in the up-dated rate type, we can refer to the form of Eq. (10).

3. Large Simple Shear and Pure Shear

3.1 Definition of strain and strain-rate

Now let us introduce the following logarithmic strain to express large strain in the existence of large rotation:

$$\epsilon_v = \ln V = Q(\ln A)Q^T, \quad (11)$$

where V, A and Q are defined below Eq. (3). The objective rate of this strain with the spin ω_v , is given as follows:

$$\dot{\epsilon}_v = \dot{\epsilon}_v - \omega_v \epsilon_v + \epsilon_v \omega_v = Q(\ln A)' Q^T \quad (12)$$

$$= D + \text{sym}(F \omega_u F^{-1}), \quad (13)$$

where $\ln$ denotes the natural logarithm and $\text{sym}(A)$ the symmetric part of the tensor A . D, ω_v and ω_u are also defined below Eq. (3).

Figure 2 illustrates the shape of a block subjected to simple shear of $\gamma=1$, and pure shear at the same magnitude of the equivalent strain $\bar{\epsilon}_v (= 0.555656$ here) which is defined as:

$$\bar{\epsilon}_v = \int \dot{\bar{\epsilon}}_v dt; \quad \dot{\bar{\epsilon}}_v = [(2/3)(\dot{\epsilon}_{vij} \dot{\epsilon}_{vji})]^{1/2} \quad (14)$$

The shape designated by C' in this figure shows one designated by C after a clockwise rotation by the

Fig. 1 Simple shear of a rectangular block.

angle ϕ for the purpose of comparison with shape B of a simple shear. Shape C is that of a pure shear. From this figure, we confirm that the two types of deformation are totally different, though the simple shear is conventionally understood as "pure shear + pure rotation". If we use the same magnitude of the conventional equivalent strain defined as $\bar{\epsilon} = \int [(2/3)(D'_{ij} D'_{ji})]^{1/2} dt$ for pure shear, the difference is increased

because shapes C and C' in Fig.2 are drawn for the corresponding shear strain $\gamma = 0.962424$. This fact is not very surprising if we remember that the statement "simple shear = pure shear + pure rotation" refers to the instantaneous relation occurring in two shear deformations and not in the total deformations⁽⁵⁾. However, the extent of the difference is rather surprising.

In the numerical analysis of large elastic-plastic deformation by the incremental finite element method, the J_2 -flow theory (=the large deformation version of the classical Prandtl-Ruess' equation) is commonly used as the up-dated rate-type constitutive equation whose form is written as follows:

$$\dot{\sigma} = \bar{D}^p : D. \quad (15)$$

Here, $\dot{\sigma}$ is the Jaumann rate of Cauchy stress σ with the continuum spin ω_c . This constitutive equation gives almost the same stress history to both simple shear and pure shear in which only the shear stress τ acts with negligible other stresses for simple shear while it acts independently for pure shear. However, as seen in Fig.2, the resulting shapes are very different. As for other constitutive equations such as J_2 -deformation theory (= the large deformation version of the rate type of the classical Hencky's total strain theory⁽⁶⁾) and J_2G (or MG -) constitutive equation (which was proposed previously by the author⁽⁷⁾⁻⁽⁹⁾) and a kind of the so-called vertex-hardening plasticity theory such as the J_2 -corner theory by

Fig. 2 Deformed shapes by simple shear B , and pure shear C at the same $\bar{\epsilon}_v$. ($\psi = 24.095^\circ$, $\psi_1 = 48.190^\circ$, $\psi_2 = 20.905^\circ$)

Christoffersen and Hutchinson⁽¹⁰⁾, the stress history is different for two kinds of shear deformation, because normal stresses also exist even if they are smaller than the predominant shear stress for a simple shear.

Figure 3 illustrates the τ - γ curve of a block subjected to simple shear superposed by a pure rotation for the J_2 -flow theory and J_2G , where the imposed deformation is as follows:

$$\left. \begin{matrix} v_x = \dot{\gamma}y \\ v_y = 0 \end{matrix} \right\} \text{ (simple shear)} + \left. \begin{matrix} v_x = \alpha \dot{\gamma}y \\ v_y = -\alpha \dot{\gamma}x \end{matrix} \right\} \text{ (pure rotation)}. \quad (16)$$

The curve for the J_2 -flow theory is coincident with the stress-strain curve reformed to the relation between τ and γ by the replacement of $\bar{\sigma}$ by $\tau = \bar{\sigma}/\sqrt{3}$, and $\bar{\epsilon}$ by $\gamma = \sqrt{3}\bar{\epsilon}$ which is assumed here to obey the n -th power law (i.e. $\bar{\sigma} = c\bar{\epsilon}^n$; $c = 591 \text{ MPa}$, $n = 0.333$). Therefore, it is a monotonically increasing curve. The J_2 -flow theory expressed by Eq.(15) is similar to the rate-type elastic constitutive equation expressed by Eq. (4) (and not to Eq. (10)). Therefore, as in Eq. (4), it is expected that the τ - γ relation becomes oscillatory for the large value of γ , in contrast to the curve in Fig. 3. The reason for this is explained in the following.

J_2 -flow theory is composed of the elasticity equation (4) (again, not Eq. (10)) and the plasticity equation, Reuss' equation. Equation (4) will induce sinusoidally shear stress if the rotation of the material element is large, even if the elastic part of the strain is small. This is equally true for Eq. (15), i.e., J_2 -flow theory. Here, we are considering a simple shear in the range of a very large shear strain in which in a usual sense the element rotation is large. Therefore, the curve of the J_2 -flow theory in Fig.3 is surprising, though this type of monotonic increasing curve has been considered to be natural as a work-hardening material.

Fig. 3 Effect of pure rotation superposed on simple shear on τ - γ curve (α in Eq.(16) in the text)

The plastic part of the J_2 -flow theory, i.e., Reuss' equation, reduces to the following simple relation for a simple shear :

$$\dot{\tau}/\dot{\gamma}^p = H^p/3 \quad (17)$$

where the superfix p indicates plasticity, and $H^p = d\bar{\sigma}/d\bar{\epsilon}^p$ = the slope of the stress-plastic strain curve. Namely, indifferent to the magnitude of shear strain, there exists no stress components other than the shear stress τ , and no strain components other than the shear strain γ . We note in this sense that no element rotation affecting (at least) the stress state is supposed. If the spin is assumed to exist, it can only affect the existing shear stress to yield other stress components. This supposition guarantees Eq.(17) to hold, and thus the monotonically increasing curve of τ - γ in Fig. 3 appears. No rotation (or spin) in a simple shear is considered peculiar because of the conventional statement "simple shear = pure shear + pure rotation". However, as described above, the conventional plasticity theory assumes no rotation (or spin) in the material element subjected to unlimited simple shear. Hopefully, Fig. 4, which will illustrate schematically a material element sliding parallel to the base plane under the action of shear force as in the laminar flow of a viscous fluid, illustrates this. In this case, it is not difficult to understand that there exists no rotation of the material element and thus the existing stress does not suffer from any rotation. This kind of deformation could be assumed to occur in the single crystal, and in the polycrystalline materials such as metals with some idealization or approximation. This point is the same as that discussed in the previous paper⁽⁴⁾ in which a new spin was proposed which coincides with the continuum spin ω_c for a pure rotation and becomes zero for simple shear as well as for pure shear.

With reference to Fig. 3, it is seen that in accordance with the above discussion, when a pure rotation is superposed on the simple shear (i.e., $\alpha \neq 0$), the curve of τ - γ is influenced by the rotation even for J_2 -flow theory, showing the lower curve for the larger value of the spin or α , or showing even a curve with a

Fig. 4 Idealized simple shear with no effect of rotation on stresses

down-slope after a peak point. In this case, the pure rotation is forced externally, and thus even the J_2 -flow theory cannot be free from the effect of the rotation.

The J_2G (MG-) constitutive equation gives a rising-sinking curve as already reported in the previous papers⁽⁷⁾⁻⁽⁹⁾ even for a simple shear with no superposed rotation. This type of τ - γ curve is also expected for other constitutive equations, such as J_2 -deformation theory and J_2 -corner theory, because they all elicit normal stress components other than the shear stress for simple shear. Of course, if a spin other than the continuum spin is used in these constitutive equations, they might elicit a monotonically increasing τ - γ curve, because the material spin ω_r , the spin ω_v and ω_u , or the spin proposed by the author⁽⁴⁾ all have a lesser effect on the stress components than the continuum spin ω_c . In other words, the plastic constitutive equations including the conventional one with the kinematic hardening other than J_2 -flow theory are more sensitive to pure rotation than is J_2 -flow theory.

We might say that the down-slope tendency of τ - γ curve is physically unreasonable, even though it cannot be rejected by the mathematical logics. If it is true, though its trueness is not definite at present, then we have to choose the appropriate spin carefully.

4. Plasticity, A New Proposal

4.1 A new definition of strain-rate

We introduce the following new strain rate with a new material parameter ρ in reference to Eq. (13) :

$$\overset{\nabla}{\epsilon}_v^* = \mathbf{D} + \rho \cdot \text{sym}(\mathbf{F}\omega_u\mathbf{F}^{-1}) ; \rho = 0 \sim 1, \quad (18)$$

where ρ could be a material constant, or a scalar function of the equivalent strain $\bar{\epsilon}_v^*$ defined in the same manner as $\bar{\epsilon}_v$ in Eq. (14), or a scalar function of the invariants of ϵ_v , and it characterizes the extent to which the current constitutive equation depends on the current deformation state. If ρ is set equal to 0, $\overset{\nabla}{\epsilon}_v^*$ reduces to the commonly used Euler strain-rate $\mathbf{D}$, i.e., stretching. If ρ is set equal to 1, $\overset{\nabla}{\epsilon}_v^*$ is coincident with $\overset{\nabla}{\epsilon}_v$ in Eq.(12) or (13) which is coaxial with the total strain ϵ_v itself. Let us concern ourselves with rigid-plasticity for a while.

For a Mises material, we may put as follows :

$$\overset{\nabla}{\epsilon}_v^* = \sigma' \dot{\lambda} ; \dot{\lambda} > 0 \text{ (for plastic loading)}, \quad (19)$$

which is coincident with Reuss' equation for $\rho = 0$, while it resembles Hencky's deformation theory for $\rho = 1$ because the resulting total strain remains coaxial with the stress. We should note that the incompressibility is expressed by

$$\text{tr} \overset{\nabla}{\epsilon}_v^* = \text{tr} \overset{\nabla}{\epsilon}_v = \text{tr} \mathbf{D} = 0.$$

The strain-rate $\dot{\epsilon}_v^*$ defined above may be used in other types of constitutive equations, such as the J 2 G (or MG-) constitutive equation and J_2 -corner theory, for improvement.

4.2 Discussions of large simple shear or torsion of a circular cylinder

Now let us consider simple shear again. For this type of deformation, Eq. (18) gives the following components:

$$\left. \begin{aligned} \dot{\epsilon}_{v11}^* &= \rho \dot{\gamma} / (4 + \gamma^2) \\ \dot{\epsilon}_{v22}^* &= -\dot{\epsilon}_{v11}^* \\ \dot{\epsilon}_{v12}^* &= (1/2) \dot{\gamma} \{4 + (1 - \rho) \gamma^2\} / (4 + \gamma^2) \end{aligned} \right\} \quad (20)$$

$$\dot{\epsilon}_v^* = (\dot{\gamma} / \sqrt{3}) [4\rho^2 \gamma^2 + \{16 + 8(1 - \rho) \gamma^2 + (1 - \rho)^2 \gamma^4\}^{1/2} / (4 + \gamma^2)] \quad (21)$$

For a plane strain state, the plastic constitutive equation (19) gives the following stress components:

$$\begin{aligned} \bar{\sigma}_v^* &= [(3/2)(\sigma'_{ij}\sigma'_{ji})]^{1/2} = \tau \cdot (\sqrt{3}/2) \\ &\times [(16\rho^2 \gamma^2 / (4 + (1 - \rho) \gamma^2)^2 + 4)]^{1/2} \\ -\sigma_z &= -\sigma_{22} = \sigma_{11} = \bar{\sigma}_v^* \\ &\times (2/\sqrt{3}) \rho \gamma / [16 + 4\{1 + (1 - \rho) \gamma^2 + (1 - \rho)^2 \gamma^4\}^{1/2}] \end{aligned} \quad (22)$$

Using these formulae, we obtain the numericals for $e_z = 0$ in Table 1 for a rigid-perfectly plastic body with $\bar{\sigma}_v^* = Y_0 = 492.66$ MPa. Note that for these columns, e_z is kept at 0 during simple shear, where e_z is the axial extension, and the Table is drawn for the shear strain $\gamma = 1$. We can see from this Table that the larger compressive axial stress σ_z is induced for a larger value of ρ by the axial constraint, up to a half of the shear stress. In contrast, the shear stress τ is smaller for larger ρ . This reflects the sensitivity to the rigid-body rotation, or in other words, the degree of dependence on the current state of deformation.

On the other hand, when we subject the material element to simple shear deformation without any

constraint in the z -direction (i.e., $\sigma_z = 0$), we easily find the following expression of the axial extension strain e_z :

$$e_z(\%) = \{[1 + (\gamma/2)^2]^{(\rho/4)} - 1\} \times 100. \quad (24)$$

Using this formula, e_z is calculated and recorded in the column of $\sigma_z = 0$ in Table 1. This is also illustrated in Fig. 5 together with the corresponding experimental curve by Takahashi et al.⁽¹¹⁾ In this figure, the adopted value of ρ is 0.4 in order to best fit the experimental curve.

Takahashi et al.⁽¹¹⁾ obtained the dashed curve shown in Fig. 5 by the torsion experiment of an aluminium tube. Such axial extension of a tube under torsion has been well known for a long time, and is usually explained as an effect of anisotropy induced by a large plastic torsional deformation. Of course, this explanation is understood not to be decisive but just a presumption. Figure 5 demonstrates another possible explanation. The theoretical curve in this figure is highly coincident with the dashed one. This solid curve is derived from Eq. (18) and (19) with $\rho = 0.4$, as mentioned above. Therefore, we can say that the cause of the axial extension of a tube under torsion is possibly the dependence of the current material plastic property on the current state of deformation, though such dependence is usually neglected completely. If we use this value of $\rho (=0.4)$ for simple shear under the constraint of $e_z = 0$, we find from Table 1 that the axial constraint stress $|\sigma_z|$ amounts to more than 17% of the shear stress which is no longer negligible, and the shear stress itself is reduced by more than 1.5% when compared with that for $\rho = 0$, where we consider the strain level of $\gamma = 1$.

For a pure shear, Eq. (19) is coincident with the classical Reuss' equation, because $\omega_u = 0$ for this case. Therefore, the shear stress exists alone throughout the deformation history of pure shear, which is

Table 1 The Effects of ρ on stresses and axial extension

ρ	$e_z = 0$			$\sigma_z = 0$
	$-\sigma_z$	τ	$-\sigma_z/\tau$	$e_z(\%)$
0.0	0.0	284.44	0.0	0.0
0.1	11.60	284.20	0.0408	0.559
0.2	23.62	283.45	0.0833	1.122
0.4	48.74	280.23	0.1739	2.257
0.5	61.70	277.66	0.2222	2.829
0.6	74.84	274.42	0.2724	3.404
0.8	101.26	265.80	0.3810	4.564
1.0	127.21	254.40	0.5	5.737

(Data at $\gamma = 1$; stress in MPa)

Fig. 5 Theoretical axial extension of a tube under torsion evaluated by Eq. (24) with $\rho = 0.4$ in the text, and corresponding experimental data by Takahashi et al.⁽¹¹⁾

different from the case of the simple shear discussed above. Therefore, the difference of the deformed shapes seen in Fig. 2 for simple shear and pure shear appears substantially natural for this type of constitutive equation.

4.3 A New elastic-plastic constitutive equation

The new elastic-plastic constitutive equation corresponding to Eq. (19) is readily formulated by mixing up the elastic part and Eq. (19), thus we have

$$\begin{aligned} \dot{\sigma} &= \dot{\sigma} - \omega^* \sigma + \sigma \omega^* = \hat{D} : \dot{\epsilon}_v^* \}, \\ \omega^* &= (1 - \rho) \omega_c + \rho \omega_v \end{aligned} \quad (25)$$

where the coefficient tensor $\hat{D}$ can be coincident with that used in the J_2 -flow theory, and the true stress-logarithmic strain curve obtained by the uniaxial tensile test can be used as the strain-hardening characteristics. The spin ω^* in Eq. (25) is a linear combination of the continuum spin ω_c and the spin ω_v according to the mixing ratio of D and $\dot{\epsilon}_v^*$ as will be understood from the definition equation of the strain-rate $\dot{\epsilon}_v^*$, Eq. (18).

If the kinematic hardening law is taken into consideration, the spin ω_v instead of ω^* will be recommended in the evolution equation of the back stress, because the internal variables such as the back stress are directly associated with the material inelastic deformation state. The constitutive equation expressed by Eq. (25) gives either an oscillatory relation or monotonically increasing one between τ and γ for simple shear at a large strain level according to the value of ρ , even for a kinematically hardening material.

5. Conclusions

Controversial problems which are related to the spin to be used in the objective rates of various quantities such as stress, strain and internal variables, are intrinsically discussed in order to clarify the ambiguity about them. The oscillatory response of the shear stress τ to the shear strain γ in a large simple shear which has been commonly understood to be peculiar, is now clearly explained to be due to the material sensitivity to the rigid-body rotation or spin and is not necessarily peculiar, though it can be avoided by an appropriate choice of the spin. It is emphasized that the simplifying procedure commonly used in making the up-dated form of the rate-type constitutive equation is not always justified, because when the material property is dependent on the current state of deformation in a tensorial sense, the procedure

can lead to an incorrect form of the up-dated rate-type constitutive equation. It is clearly proved why the J_2 -flow theory gives the monotonically increasing shear stress τ with the shear strain in simple shear, though its apparent form is coincident with that of the up-dated rate-type of the finite elasticity which is now known to yield a sinusoidal relation between τ and γ . A new type of strain rate is proposed and used to explain the well-known axial extension of a tube subjected to severe torsion. A new elastic-plastic constitutive equation of the rate-type is proposed by the use of the new strain rate, together with a newly defined spin.

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